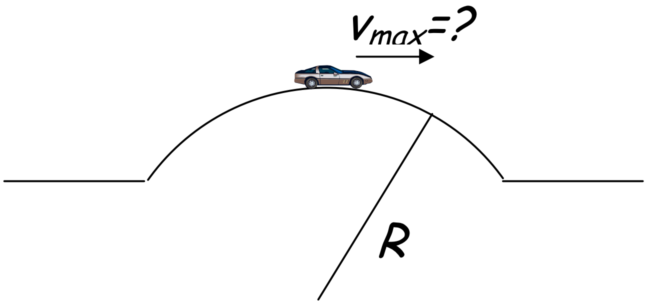
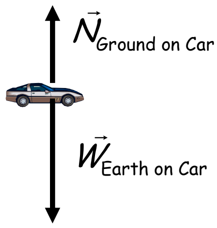
Circular Hill

A car is driving over a semi-circular hill of radius R. What is the fastest the car can drive over the top of the hill without its tires lifting off of the ground?



(1) Comprehend the Problem

If the car drives over a circular hill, it travels in a circle. Apparently, if the car travels too fast it cannot remain on a circular path that follows the hill. One or more of the forces responsible for the car's staying on the circle must be limited in some way. Let's use a free body diagram to see if we can determine what forces are responsible for the car's circular motion.



The net force on the car must point to the center of the circle (because it's in circular motion). As the car travels faster and faster, the hill needs to provide less and less normal force to keep the car from penetrating it. Eventually the car could be going fast enough that the normal force is zero… if it went any faster that this critical speed, the weight won't be enough to keep the car going in a circle coinciding with the hill and the wheels will lift off of the ground.

Since the car travels in a circle, we'll almost certainly use the relationship between centripetal acceleration and tangential speed.

(2) Represent the Problem in Formal Terms (Describe the Physics)

We have been given or identified the following quantities:

m = the mass of the car

v = the speed of the car

R = the radius of the circle the car's following (i.e. the hill's radius)

$\vec{N}$ = The normal force from the ground on the car = $\begin{cases} \text{magnitude } N \\ \text{direction up} \end{cases}$

$\vec{W}$ = The weight from the earth on the car = $\begin{cases} \text{magnitude } mg \\ \text{direction down} \end{cases}$

We also know the following relationships that might be useful:

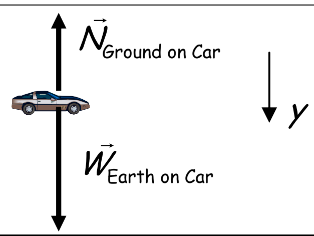
$$\vec{F}_{net} = M\vec{a} \qquad \text{(Newton's Second Law)}$$

$$a_{Car} = \frac{v_{car}^2}{R} \text{ toward circle center} \qquad \text{(Car's in uniform circular motion)}$$

(3) Plan the Solution

We can apply Newton's Second Law to the car at the top of the hill to relate the forces on the car to its acceleration. We know the form of the car's acceleration since it's in circular motion. We can set the normal force equal to zero; this should happen when the car is going as fast as it can and still stay on the circle.

(4) Execute the Solution

We'll use the free body diagram to write Newton's Second Law for the car at the top of the hill. We choose downward as positive because we know that the car is accelerating downward (i.e. toward the center of the circle)	
Apply Newton's Second Law in the y-direction. Circular motion implies $a=v^2/R$ toward the center of the circle.	$F_{net,y} = ma_y$ $N_y + W_y = m\left(\frac{v_{Car}^2}{R}\right)$ $(-N) + mg = \frac{mv_{Car}^2}{R}$
At the maximum speed, the normal force will be zero. The entire downward force toward the center is supplied by the car's weight alone.	$(0) + \cancel{m}g = \frac{\cancel{m}v_{max}^2}{R}$ $v_{max} = \boxed{\sqrt{gR}}$

(5) Interpret and Evaluate the Solution

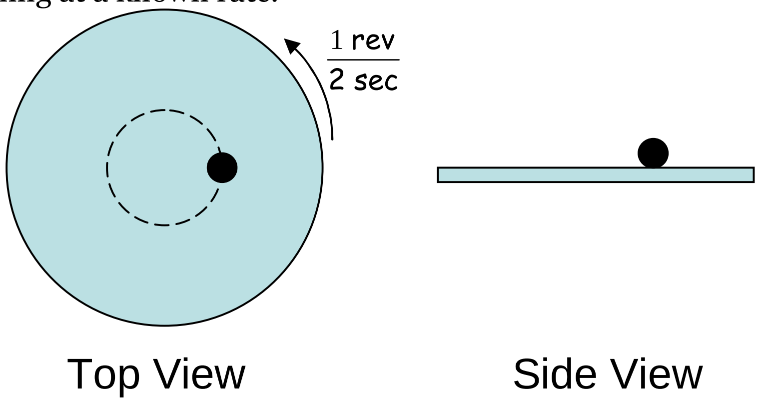
As a quick check, let's say the hill had a radius of around 100 m. The fastest speed the car could go would then be $\sqrt{\left(9.81\frac{m}{s^2}\right)(100\ m)} = 31\frac{\cancel{m}}{\cancel{s}} \times \frac{1\ km}{1000\ \cancel{m}} \times \frac{3600\ \cancel{s}}{1\ hr} = 112\frac{km}{hr}$ (about 70 miles per hour). This seems at least reasonable for the speed of a car lifting off the top of a steep hill.

Playground Slipping

Some children are playing with a large rock on the playground. They keep putting it on a merry-go-round spinning at 1 revolution every 2 seconds. They're trying to find out how far away from the center they can place the rock before it slides to the outside. You look up the coefficient of static friction for rock on metal and find it's around 0.3. What is the farthest distance from the center of the merry-go-round that the rock can sit and still not slide?

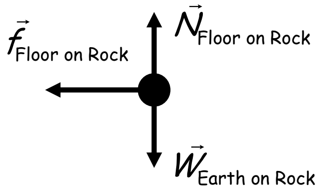
(1) Comprehend the Problem

Let's start with a sketch to see what's going on. The rock is sitting on a merry-go-round (a flat, spinning disk) spinning at a known rate.



We know that the rock is travelling in a circle. The question is how large of a radius can the circle have before the rock must slip? Since it's travelling in a circle at constant speed, we know that the acceleration of the rock is related to its velocity ($a = v^2/r$). In addition, we know how long it takes the rock to make a complete revolution (2 seconds).

We need to know what force produces the rock's acceleration toward the center of the circle. Let's figure this out by drawing the free body diagram for the rock (we'll use the "side view").



There are three forces on the rock, the normal force from the "floor" of the merry-go-round, the weight from the earth, and the static friction force from the floor of the merry-go-round. Since the static friction is the only force in the horizontal direction, we know it must point toward the center of the circle in order to pull the rock toward the center (as demanded by the rock's center-pointing acceleration).

The static friction force has a maximum strength it can provide, given by $f_{\text{static, max}} = \mu_s N$. As we move the rock outward, the static friction force required to keep it going around the circle increases. There will apparently be a maximum radius at which the static friction force hits this maximum value. If the rock is placed any farther away, it will slip.

(2) Represent the Problem in Formal Terms (Describe the Physics)

We are given or have identified the following quantities as useful:

T = the time it takes the merry-go-round to make one revolution

m = the mass of the rock

$\vec{N}$ = the normal force from the floor on the rock = $\begin{cases} \text{magnitude } N \\ \text{direction up} \end{cases}$

$\vec{W}$ = the weight force from the earth on the rock = $\begin{cases} \text{magnitude } mg \\ \text{direction down} \end{cases}$

$\vec{f}_s$ = the static frictional force from the floor on the rock = $\begin{cases} \text{magnitude } f_s \\ \text{direction toward center} \end{cases}$

r = the distance from the circle center to the rock (the radius)

We also have the following relationships that might be important.

$$\vec{F}_{\text{net}} = M\vec{a} \qquad \text{Newton's Second Law}$$

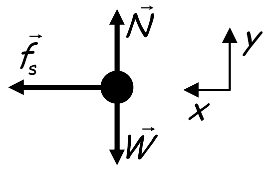
$$f_{\text{static, max}} = \mu_s N \qquad \text{static friction maximum}$$

$$a_c = \frac{v^2}{r} \qquad \text{centripetal acceleration}$$

(3) Plan the Solution

Since it's in uniform circular motion, we know the rock's acceleration in terms of its velocity and radius. We can use Newton's Second Law to relate this acceleration to the forces on the rock. By inserting the maximum value for the static frictional force, we should be able to find the largest radius at which the rock won't slide.

(4) Execute the Solution

Draw the free body diagram for the rock. We choose toward the center of the circle as positive, as this is the direction the rock actually accelerates.	
Apply Newton's Second Law in the x-direction. Since the rock is in uniform circular motion, we know how its acceleration is related to v and r.	$F_{\text{net},x} = ma_x$ $f_{s,x} + N_x + W_x = ma_x$ $f_s + 0 + 0 = m\left(\frac{v^2}{r}\right)$ $f_s = m\left(\frac{v^2}{r}\right)$

Since we don't know the rock's velocity, we need to write it in terms of the radius r and the revolution time T (a.k.a. the period). The rock travels one circumference in time T.	$v = \frac{\text{distance}}{\text{time}} = \frac{2\pi r}{T}$
Replace the velocity in our Newton's Second Law x-equation with this expression for v. Now we have the radius in terms of things we know (period T and mass m) and the static friction force.	$f_s = m\frac{v^2}{r}$ $f_s = \frac{m}{r}\left(\frac{2\pi r}{T}\right)^2 = 4\pi^2 \frac{mr^{\not 2}}{\not r T^2}$ $f_s = 4\pi^2 \frac{mr}{T^2}$ $r = \frac{T^2}{4\pi^2 m} f_s$
The maximum radius is where the static friction force is at its maximum.	$r_{max} = \frac{T^2}{4\pi^2 m} f_{s,max}$ $= \frac{T^2}{4\pi^2 m}(\mu_s N)$
We need the normal force from the floor on the rock. We can find this from applying Newton's Second Law in the y-direction. We know the acceleration of the rock in the y-direction is zero.	$F_{net,y} = ma_y$ $f_{s,y} + N_y + W_y = m(0)$ $0 + N + (-mg) = 0$ $N = mg$
We can insert this normal force value into the expression for the maximum radius to get our answer. (Note that the mass cancels out)	$r_{max} = \frac{T^2}{4\pi^2 \not m}\mu_s(\not m g)$ $r_{max} = \frac{T^2 \mu_s g}{4\pi^2}$
Now we can insert the numerical values from the problem statement.	$r_{max} = \frac{(2\ \not s)^2 (0.3)\left(9.81 \frac{m}{\not s^2}\right)}{4(3.1416)^2} = 0.30\ \not m \times \frac{100\ \text{cm}}{1\ \not m} = \boxed{30\ \text{cm}}$

The rock will slide if it is placed more than 30 cm (about 1 ft) from the center of the merry-go-round.

If we place the rock anywhere closer to the center than this, it will not slide on the merry-go-round. The rock would then just travel around the center in a circle with the rest of the merry-go-round.