Algebra Review

This review contains some explanations about simultaneous linear equations and quadratic equations.

I. Simultaneous equations

In Physics 101 you will need algebra such as that illustrated by the following question: there is a roller R on a slope being pulled up by a weight m which is going down as shown in the figure:

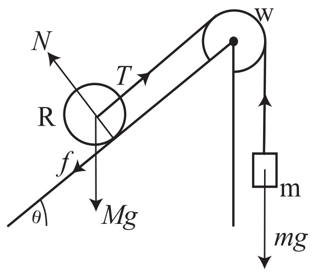

Figure 1: A roller climbing up a slope

At this point you need not understand the physics of the problem.

In this figure, R is a cylindrical roller of mass M and radius R, w is a free rotating massless wheel, and m is a block of mass m. The roller and the mass is connected by a massless string. T is the tension in the string, f is the friction and g is the acceleration of gravity (9.8 m/s^2). θ is the angle the slope makes with the horizontal. We assume that the roller rolls without any slip.

You will be able to set up equations relating T, f, etc., and the acceleration a of the roller with the aid of the fundamental principles of mechanics in a couple of months. The resultant equations read

$$ma = -T + mg, \tag{1}$$

$$Ma = T - f - Mg\sin\theta, \tag{2}$$

$$\frac{1}{2}Ma = f \tag{3}$$

Solve this set of equations for a.

The simplest way is to add (1), (2) and (3) (i.e., add all the right hand sides and left hand sides separately and equate them):

$$\begin{array}{rrcl} & ma & = & -T + mg \\ & Ma & = & T - f - Mg\sin\theta \\ +) & \frac{1}{2}Ma & = & f \\ \hline & \left(m + \frac{3}{2}M\right)a & = & mg - Mg\sin\theta. \end{array} \tag{4}$$

Therefore,

$$a = (mg - Mg\sin\theta)/\left(m + \frac{3}{2}M\right). \tag{5}$$

In the actual situation, you must be able to set up these equations. To this end the basic strategy is to pretend that you know all the quantities and to write down the relations among them without considering how to solve the problem.

In general, you must be able to solve simultaneous equations for x and y as unknown of the following form (assuming $AD \neq BC$):

$$\begin{aligned} Ax + By &= S, &(6)\\ Cx + Dy &= T, &(7) \end{aligned}$$

Here, A, B, C, D, S and T are numbers. We will write down the solutions x and y later.

We often encounter simpler equations as

$$\begin{aligned} x + 2y &= 3, &(8)\\ x + 3y &= 4. &(9) \end{aligned}$$

Look at the structure of the equations before jumping into any actual calculation and choose the least laborious way. In this case, the easiest way is to subtract (8) from (9):

$$\begin{array}{rrcl} & x + 3y & = & 4 \\ -) & x + 2y & = & 3 \\ \hline & y & = & 1 \end{array} \qquad (10)$$

That is, $y = 1$. Consequently, $x = 3 - 2y = 1$. After obtaining the solution, you should check it by substituting the obtained values of x ad y into the original equation not used in getting x: $x + 3y = 1 + 3 = 4$. Correct.

Alternatively, you could solve x from (8) as $x = 3 - 2y$ and substitute this into (9) to obtain $(3 - 2y) + 3y = 4$, that is, $y = 1$. The rest is the same as before.

To solve

$$\begin{aligned} 2x + 3y &= 4, &(11)\\ x - y &= 1, &(12) \end{aligned}$$

a wise way may be to subtract $2\times$ (12) from (11):

$$\begin{array}{rrrcl rrrcl} & & 2x + 3y & = & 4 & \rightarrow & & 2x + 3y & = & 4 \\ -) & 2\times & x - y & = & 1 & \rightarrow & -) & 2x - 2y & = & 2 \\ \hline & & & & & & & 5y & = & 2 \end{array} \qquad (13)$$

That is, $y = 2/5$. Consequently, $x = 1 + y = 7/5$. Check: $2x + 3y = 14/5 + 6/5 = 4$, correct.

Alternatively, you could get x from (12) as $x = 1 + y$ and substitute this into (11) to obtain $2(1 + y) + 3y = 4$ or $5y = 2$, that is $y = 2/5$. The rest is the same as before.

To solve

$$\begin{aligned} 2x + 3y &= 4, &(14)\\ 3x - 4y &= 1, &(15) \end{aligned}$$

a convenient way may be to subtract $2\times$ (15) from $3\times$ (14):

$$\begin{array}{rrrcl rrrcl} & 3\times & 2x + 3y & = & 4 & \rightarrow & & 6x + 9y & = & 12 \\ -) & 2\times & 3x - 4y & = & 1 & \rightarrow & -) & 6x - 8y & = & 2 \\ \hline & & & & & & & 17y & = & 10 \end{array} \qquad (16)$$

That is, $y = 10/17$. Consequently, $x = (4 - 3y)/2 = 19/17$.

With the same technique we can solve the general set of equations (1) and (2) as

$$\begin{array}{rrrcll} & D\times & Ax + By = S & \rightarrow & & ADx + BDy = DS \\ -) & B\times & Cx + Dy = T & \rightarrow & -) & BCx + BDy = BT \\ \hline & & & & & (AD - BC)x = DS - BT \end{array} \tag{17}$$

That is,

$$x = \frac{DS - BT}{AD - BC}. \tag{18}$$

Consequently,

$$y = \frac{AT - CS}{AD - BC}. \tag{19}$$

Linear Algebra is the math branch studying such equations systematically. If you have not learned Linear Algebra (matrices and determinants, vector spaces, etc) yet, you should learn it as soon as possible. It is much more important than elementary physics.

Let us solve some practice examples.

(1)

$$x + 8y = 4, \tag{20}$$
$$x + 5y = 2. \tag{21}$$

[$x = -4/3$, $y = 2/3$]

(2)

$$3x - 4y = 5, \tag{22}$$
$$x + 4y = 3. \tag{23}$$

[$x = 2$, $y = 1/4$]

(3)

$$2x - 2y = 5, \tag{24}$$
$$x + 3y = 3. \tag{25}$$

[$x = 21/8$, $y = 1/8$]

(4)

$$
\begin{aligned}
3x - 2y &= 5, & (26)\\
-7x + y &= 8. & (27)
\end{aligned}
$$

$[x = -21/11, y = -59/11]$

(5)

$$
\begin{aligned}
3x - 4y &= 5, & (28)\\
2x + 3y &= 7. & (29)
\end{aligned}
$$

$[x = 43/17, y = 11/17]$

(6) There are some cranes and porpoises. The total number of legs is 100, and the total number of organisms is 60. How many porpoises are there?

(7) There are some cranes and tortoises. The total number of legs is 100, and the total number of organisms is 40. How many tortoises are there?

(8) An apple is 70 cents and a banana is 30 cents. You bought 30 fruits and paid 17 dollars. How many apples did you buy?

(6) 50 cranes and 10 porpoises; $2x + 0 \times y = 100$, $x + y = 60$
(7) 30 cranes and 10 torpoises; $2x + 4y = 100$, $x + y = 40$
(8) 20 apples; $70x + 30y = 1700$, $x + y = 30$

II. Quadratic equations

In Physics 101 you will need the formua for the roots of a quadratic equation ($a \neq 0$):

$$ax^2 + by + c = 0. \tag{30}$$

For example, if you wish to know when a ball thrown from a tower of height h with an initial velocity $\boldsymbol{v}_0$ reaches the ground, you must solve

$$0 = H + v_{0y}t - \frac{1}{2}gt^2 \tag{31}$$

for t, where g is the acceleration of gravity (9.8 m/s^2).

The formula for the roots of (30) is given by

$$x_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{32}$$

You should learn this by heart (you should be able to derive this as well; this is demonstrated after examples).

Let us do some practice examples (the solutions are on the next page). (Simpler examples should be solved directly without the formula or with the use of factorization: $(x + \alpha)(x + \beta) = x^2 + (\alpha + \beta)x + \alpha\beta$).

(1) $x^2 - 3 = 0$.

(2) $x^2 - 3x + 2 = 0$.

(3) $x^2 + x - 6 = 0$.

(4) $2x^2 + 5x - 3 = 0$.

(5) $6x^2 - x - 1 = 0$.

(6) $x^2 + 2x - 3 = 0$.

(7) $2x^2 + 3x - 5 = 0$.

(8) $-3x^2 - 2x + 5 = 0$.

(9) $4.9x^2 - 3.2x - 11 = 0$.

(10) $-1.3x^2 + 5.1x - 2.2 = 0$.

(1) $x = \pm\sqrt{3} = \pm 1.73\cdots$.
(2) $= (x-1)(x-2)$, so $x = 1$ or 2.
(3) $= (x-2)(x+3)$, so $x = 2$ or -3.
(4) $= (2x-1)(x+3)$, so $x = 1/2$ or -3.
(5) $= (2x-1)(3x+1)$, so $x = 1/2$ or $-1/3$.
(6) $= (x-1)(x+3)$, so $x = 1$ or -3.
(7) $= (2x+5)(x-1)$, so $x = -2.5$ or 1. In this case, using the formula for the root may be easier (but do not use the calculator).
(8) $= -(x-1)(3x+5)$, so $x = 1$ or $-5/3$, In this case, using the formula for the root may be easier (but do not use the calculator).
(9) $x = (3.2 \pm \sqrt{3.2^2 + 4 \times 11 \times 4.9})/9.8 = 1.857\cdots$ or $-1.20\cdots$
(10) $x = (-5.1 \pm \sqrt{5.1^2 - 4 \times 2.2 \times 1.3})/(-2.6) = 0.5\cdots$ or $-3.42\cdots$.

You should learn (32) by heart, but you should be able to *derive* the formula by yourself. This is much more important than physics; at this elementary level you must not use the formulas you cannot understand. Let us demonstrate the formula; this you must be able to do by yourself.

$$0 = ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x\right] + c. \tag{33}$$

Now, we need a small trick called the *completion of square*. Note that (since $(x+B)^2 = x^2 + 2Bx + B^2$)

$$x^2 + Ax = x^2 + Ax + \frac{A^2}{4} - \frac{A^2}{4} = \left(x + \frac{A}{2}\right)^2 - \frac{A^2}{4}. \tag{34}$$

We utilize this relation:

$$x^2 + \frac{b}{a}x = \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}. \tag{35}$$

Therefore,

$$0 = a\left[x^2 + \frac{b}{a}x\right] + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right] + c \tag{36}$$

$$= a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}. \tag{37}$$

This implies that

$$a\left[\left(x + \frac{b}{2a}\right)^2\right] - \frac{b^2 - 4ac}{4a} = 0 \tag{38}$$

or

$$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0. \tag{39}$$

Therefore,

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}. \tag{40}$$

Thus, we finally arrived at the desired formula:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{41}$$