Rudiments of Vectors

A **vector** $\boldsymbol{v}$ is an object that has a direction and a magnitude, satisfying the following scalar multiplication and addition rules (Fig.1). Velocity, force, acceleration, displacement, etc., are examples of vectors. The magnitude (length) of vector $\boldsymbol{v}$ is written as $|\boldsymbol{v}|$.
(1) **Scalar multiplication** $\boldsymbol{v} \to$ 'some number'$\times\boldsymbol{v}$ does not change the direction of the vector, if this number is a positive number. Let us write this number as α (alpha). If $\alpha < 0$, then the direction is flipped. The magnitude satisfies $|\alpha\boldsymbol{v}| = |\alpha|\,|\boldsymbol{v}|$.
(2) **Addition** of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ to make $\boldsymbol{a}+\boldsymbol{b}$ is to graft the arrow corresponding to $\boldsymbol{b}$ at the head of the arrow corresponding to $\boldsymbol{a}$.[1]
∗ Draw $2.3\boldsymbol{v}$ and $-0.5\boldsymbol{v}$ in the figure. Also draw $\boldsymbol{c}+\boldsymbol{d}$.
∗ Draw two arrows as you like them and construct their sum.

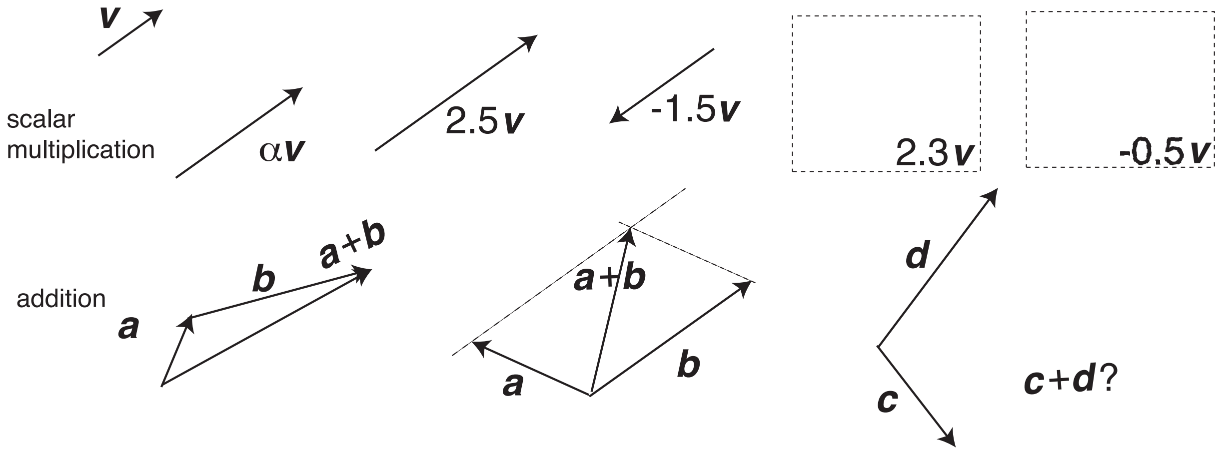

Figure 1: Vector: scalar multiplication and addition. Here $\alpha > 0$ is assumed. Complete the figures.

Addition does not care about the order of vectors being added: $\boldsymbol{a}+\boldsymbol{b} = \boldsymbol{b}+\boldsymbol{a}$. You must be able to explain this rule graphically.
If you have three or more vectors, you can add them in any order: the results are the same: for example $(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c} = \boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c}) = \boldsymbol{b}+(\boldsymbol{a}+\boldsymbol{c})$, etc.
You can exchange the order of addition and multiplication: $\alpha(\boldsymbol{a}+\boldsymbol{b}) = \alpha\boldsymbol{a}+\alpha\boldsymbol{b}$. This is a kind of the distributive law as you encountered in the high school algebra classes.

[1] or to draw the diagonal segment connecting the vertices of the parallelopiped constructed with these two vectors as in Fig.1.

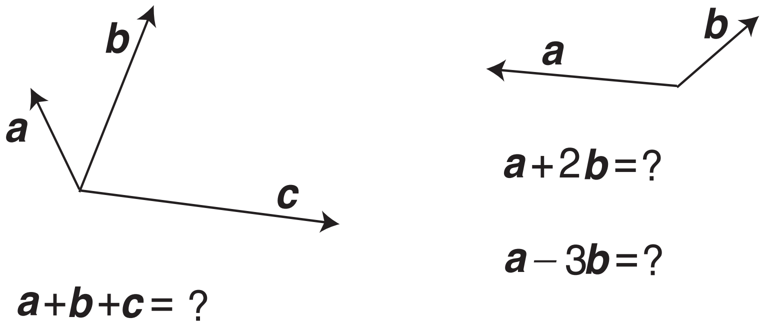

Figure 2: Exercise. Construct the vectors asked in the figure.

Subtraction of vector $\boldsymbol{b}$ from $\boldsymbol{a}$ is adding $-\boldsymbol{b}$ to $\boldsymbol{a}$ (Fig.3):

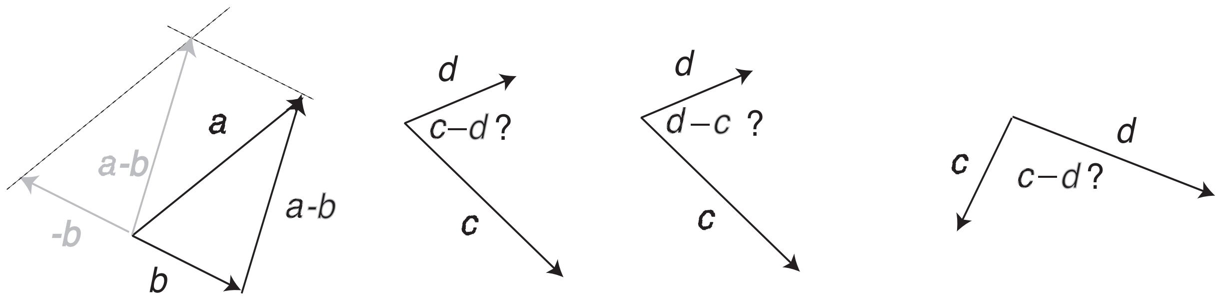

Figure 3: Subtraction of vectors: You may draw vectors at any place convenient, parallelly translating them in space. Although $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$, it is far better to draw the vector (arrow) pointing the head of vector $\boldsymbol{a}$ from that of vector $\boldsymbol{b}$, if the arrows denoting vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ start from the same point. Draw vectors wanted in the figure.

Remember that vector $\boldsymbol{a}-\boldsymbol{b}$ connects two heads of the arrows corresponding to the two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$. Its direction can be determined easily by checking your result: $(\boldsymbol{a}-\boldsymbol{b})+\boldsymbol{b} = \boldsymbol{a}$.
$*$ Draw two vectors of your choice, and construct their differences.

You can apply the construction of the difference of two vectors to obtain the **relative velocity**. Suppose you are moving with velocity $\boldsymbol{w}$ *relative to* ground and your friend is moving with velocity $\boldsymbol{u}$ *relative to* ground. The apparent velocity (= relative velocity) of your friend seen *from you* is $\boldsymbol{u} - \boldsymbol{w}$. (Notice that you are not moving relative to yourself: $\boldsymbol{w} - \boldsymbol{w} = 0$!)

Decomposition of vectors.
If two distinct directions are given, we can decompose a given vector $\boldsymbol{v}$ as a sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ that are respectively pointing the given distinct directions (Fig.4).

You can apply this to obtain normal forces on the incline, for example (Fig.5):

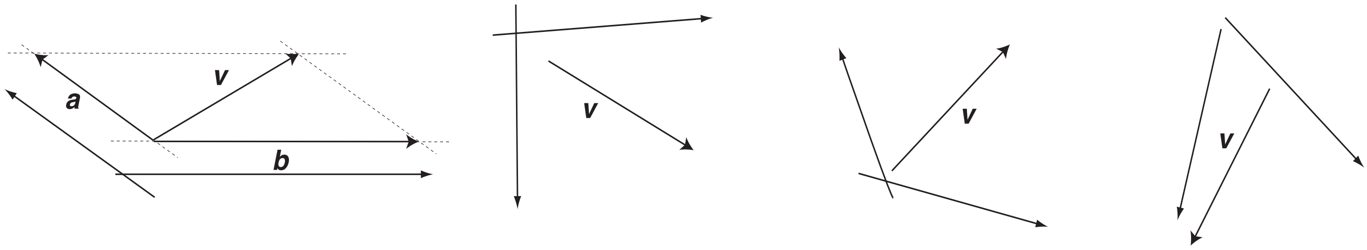

Figure 4: Decomposition of a vector $\boldsymbol{v}$ into two directions as $\boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b}$. Complete the right three figures, constructing '$\boldsymbol{a}$' and '$\boldsymbol{b}$.'

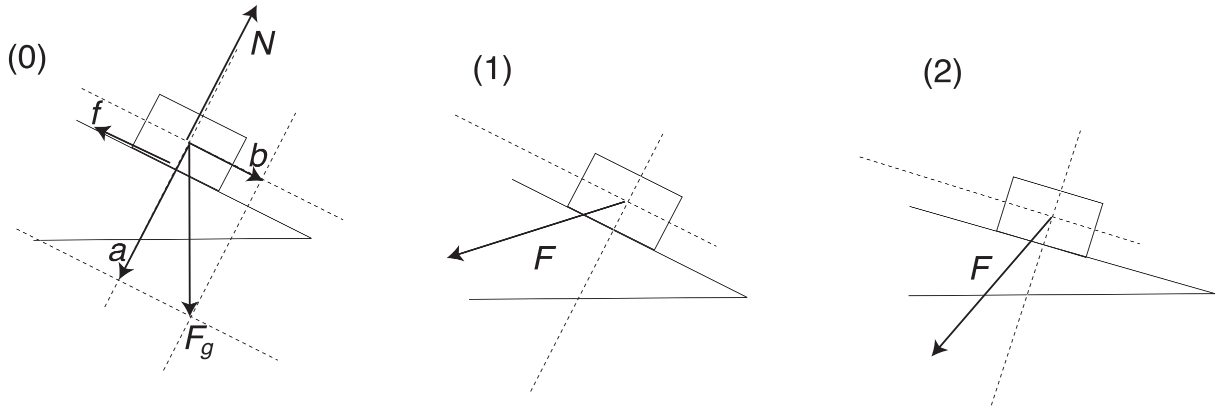

Figure 5: Decomposition of force into two directions. (0) There is a gravity F_g always acting vertically downward. Assuming that the block does not slide down the incline, the frictional and normal force acting on the block may be constructed graphically as shown. (1) Suppose you push the block as shown in the figure. Find its components parallel and perpendicular to the surface of the incline. (2) Suppose the force due to gravity + your pushing is $\boldsymbol{F}$ as shown in the figure. Assuming that the block is stationary, find the frictional and normal forces.

Component expression of vectors

If two distinct directions are given as coordinate directions, you can decompose a given vector as a sum of vectors pointing in these two coordinate directions, applying the decomposition rule of the vector explained above. Therefore, if you specify the magnitudes (with $\pm$ signs) of these component vectors, you can uniquely specify the original vector (Fig.6):

The multiplication and addition rules can be expressed in terms of components as follows:

$$\alpha(a_x, a_y) = (\alpha a_x, \alpha a_y), \tag{1}$$
$$(a_x, a_y) + (b_x, b_y) = (a_x + b_x, a_y + b_y). \tag{2}$$

$*$ Let $\boldsymbol{a} = (2, 4)$ and $\boldsymbol{b} = (-1, 5)$. Compute $\boldsymbol{a} + \boldsymbol{b}$, $\boldsymbol{a} - \boldsymbol{b}$, $2\boldsymbol{a} - \boldsymbol{b}$ and $-2\boldsymbol{a} + 5\boldsymbol{b}$.

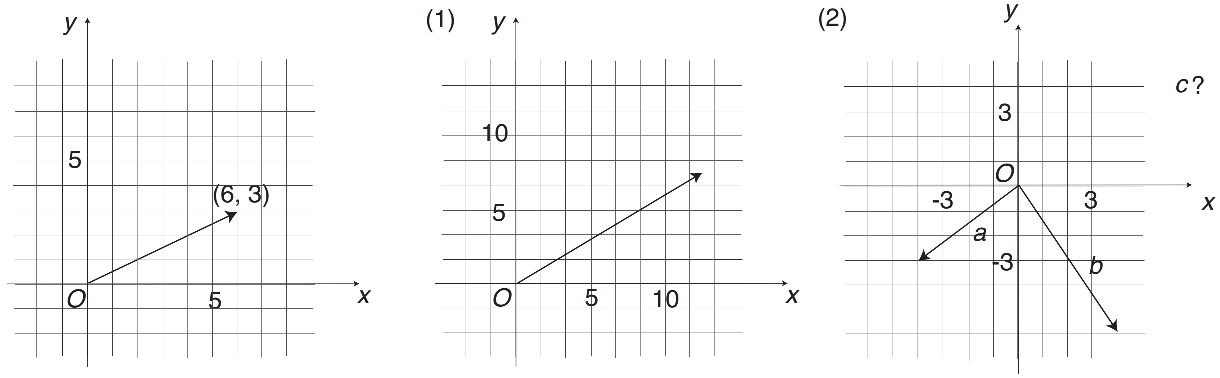

Figure 6: Coordinate expression of vectors: (1) Find the x and y components. (2) For $\boldsymbol{a}$ and $\boldsymbol{b}$, find components. Draw a vector $\boldsymbol{c}$ given by $(-4, 2)$.

The magnitude of the vector may be obtained with the aid of the Pythagoras theorem:

$$|(a_x, a_y)| = \sqrt{a_x^2 + a_y^2}. \tag{3}$$

∗ Find the magnitudes of the following vectors: $(2, 5)$, $(-2, 5)$, $(3, 4)$.
∗ See Fig.7. Illustrate the addition of vectors.

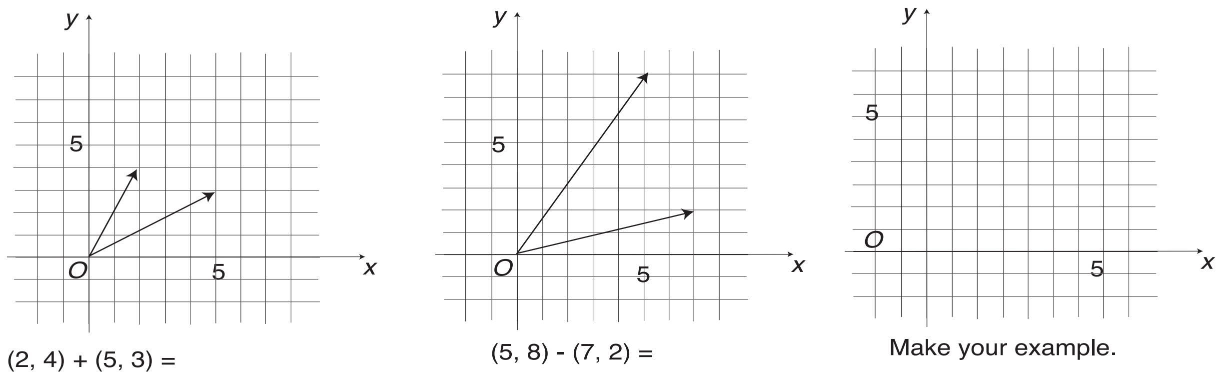

Figure 7: Draw the results of addition examples.

The component expression of a vector depends on the choice of the coordinates, so *the same* vector can be represented by many different pair of numbers (Fig.8). Needless to say, the magnitude of the vector may be computed from any of such expressions.

Trigonometry rudiments

The relation between angles relevant to inclines is illustrated in Fig.9:

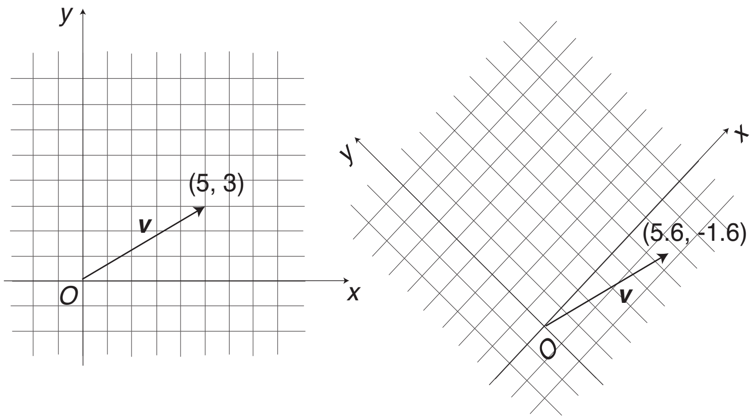

Figure 8: The component expression of a vector $\boldsymbol{v}$ is its description relative to a given coordinate system. Therefore, the same vector can have many different expressions as illustrated here.

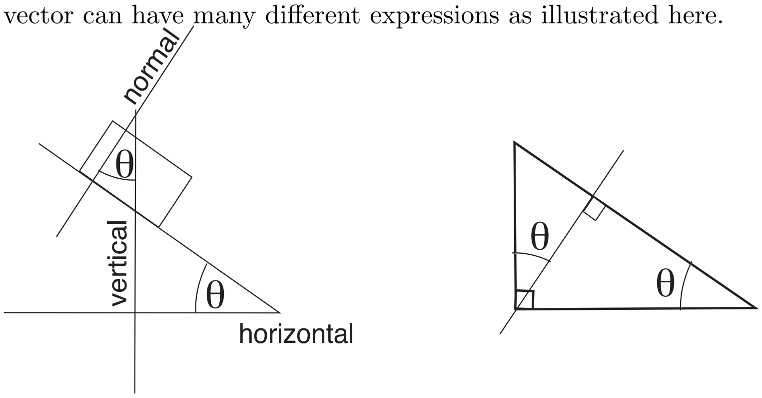

Figure 9: The angle θ of the slope and the angle that the normal direction makes with the perpendicular direction are the same. You must be able to demonstrate the statement with the aid of the figure on the right.

If you cannot immediately tell whether the angle you are interested in is θ or $90° - \theta$, consider the limiting case of $\theta \to 0$ as illustrated in Fig.10. When the slope becomes very shallow, is the angle you are interested in vanishing? Then it is θ.

If your memory of trig functions is not sure, use the mnemonic figure (Fig.11). Represetative values of trig functions should be memorized:
$*$ Write down the following values (exact values as $\sin 45° = 1/\sqrt{2}$): $\sin 0°$, $\sin 90°$, $\sin 45°$, $\cos 45°$, $\tan 45°$, $\sin 30°$, $\tan 60°$, etc.

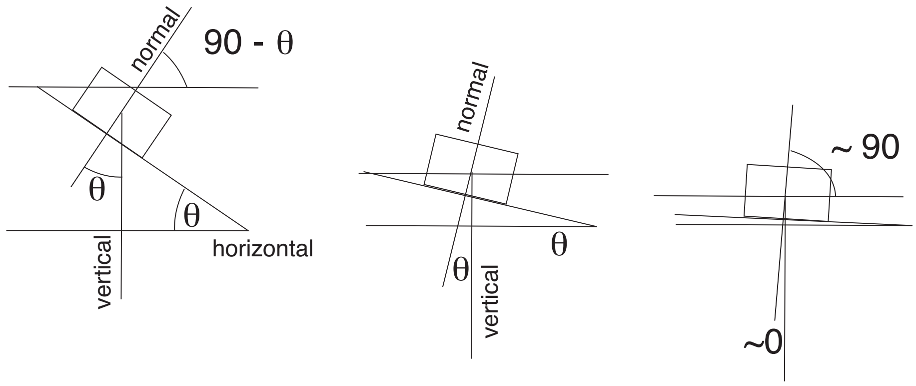

Figure 10: If you are not sure which angle is θ (the slope angle of the incline), imagine you change the slope. If the slope is almost horizontal, what happens to the angle you wish to know?

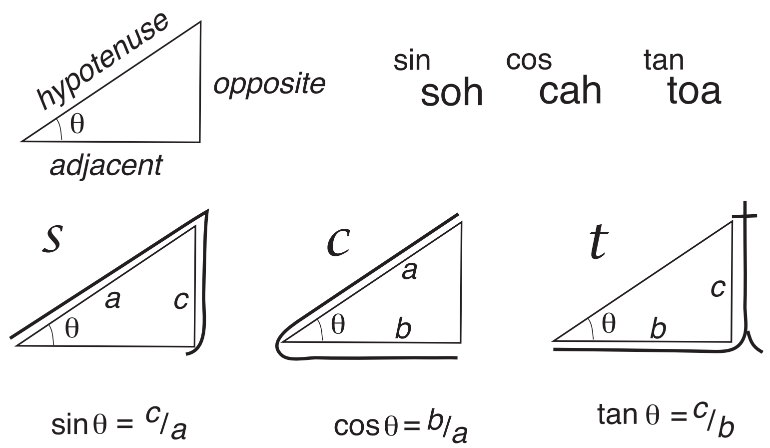

Figure 11: Sine, cosine, and tangent may be memorized with their initial letters.