2. Relative and Circular Motion

A) Overview

We will begin with a discussion of relative motion in one dimension. We will describe this motion in terms of displacement and velocity vectors which will allow us to generalize our results to two and three dimensions.

We will then discuss reference frames that are accelerating and will find that the description of phenomena in these frames can be somewhat confusing. For the remainder of the course, we will restrict ourselves to the description of phenomena in inertial reference frames, those frames that are not accelerating.

Finally, we will consider a specific motion, namely uniform circular motion and find that it can be described in terms of a centripetal acceleration that depends on the angular velocity and the radius of the motion.

B) Relative Motion in One Dimension

In the last unit we discovered that two-dimensional projectile motion can be described as the superposition of two one-dimensional motions: namely, motion at constant acceleration in the vertical direction and motion at constant velocity in the horizontal direction. We then discussed the description of such a motion from two different reference frames. In particular, we saw that the motion of a ball thrown straight up on a train moving at constant velocity would be described as one-dimensional free fall by an observer on the train, but would be described as a two-dimensional projectile motion by an observer on the ground. In this unit, we will develop a general equation that relates the description of a single motion in different reference frames.

We will start with a one-dimensional case. Suppose you are standing at a train station as a train passes by traveling east at a constant 30 m/s. Your friend Mike is on the train and is walking toward the back of the train at 1 m/s in the reference frame of the train. What is Mike's velocity in the reference frame of the station?

Intuitively, you probably realize the answer is 29 m/s, but it will prove useful to develop, in this one-dimensional example, a general procedure that can be easily generalized to the more non-intuitive cases that involve two or three dimensions.

We start by drawing the displacement vectors of Mike in the two reference frames we have here: the train and the ground as shown in Figure 3.1. At any time the displacement of Mike with respect to the ground is just equal to the displacement of Mike with respect to the train plus the displacement of the train's origin with respect to the ground's origin.

$$\vec{r}_{Mike,Ground} = \vec{r}_{Mike,Train} + \vec{r}_{Train,Ground}$$

If we now differentiate this equation with respect to time, we obtain the equation we want: an equation that relates the velocities in the two frames. In particular, we see that
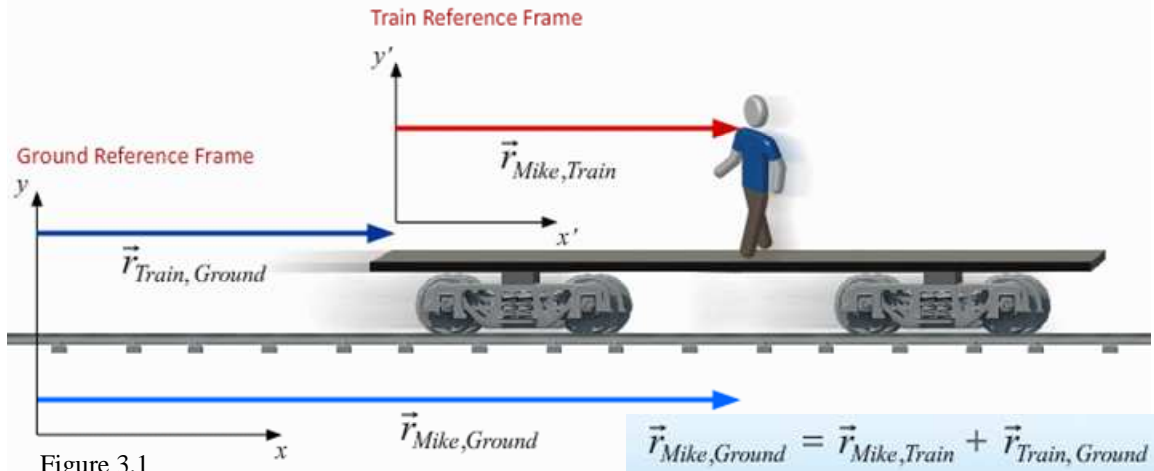
Figure 3.1
The displacement vectors of Mike in both the Train and Ground reference frames as well as the displacement vector of the origin of the Tranin freference frame in the Ground reference frame.

the velocity of Mike with respect to the ground is equal to the velocity of Mike with respect to the train plus the velocity of the train with respect to the ground.

$$\vec{v}_{Mike,Ground} = \vec{v}_{Mike,Train} + \vec{v}_{Train,Ground}$$

Consequently, if we take east to be positive, then we see that Mike's velocity with respect to the ground is +29 m/s. Had Mike been walking towards the front of the train, his velocity would have been + 31 m/s.

We hope this result seems reasonable and intuitive to you. Perhaps you're even wondering why we went to the trouble introducing a formalism to obtain a very intuitive result. I think the answer will be come clear in the next section when we discuss relative motion in two and three dimensions.

C) Relative Motion in Two Dimensions

We now want to consider relative motion in more than one dimension. How do we go about solving this problem? Well, the formalism that we introduced in the last section is all we need to solve this problem. The key is that displacements, our starting point in the preceding section, are *vectors*! Therefore, when we differentiate the displacement equation, we obtained an equation that relates the velocities as vectors which holds in two and three dimensions as well!
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Figure 3.2 shows Mike, in his motorboat, crossing a river. The river flows eastward at a speed of 2 m/s relative to the shore. Mike is heading north (relative to the water) and moves with a speed of 5 m/s (relative to the
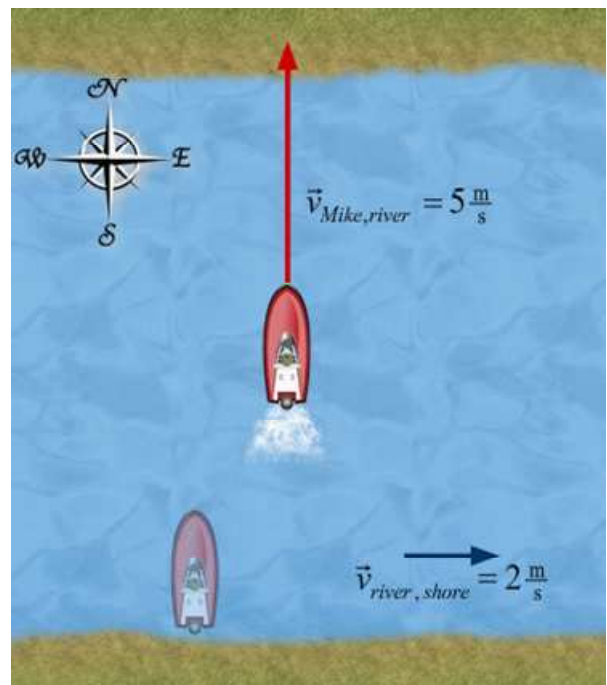

Figure 3.2
Mike heads due North moving at 5 m/s with respect to the river that itself flows due East at 2 m/s with respect to the shore.

water). If you are standing on the shore, how do you describe his motion? How fast is he moving and in what direction?

To answer this question, we just apply our equation that relates the velocity vectors. Namely, Mike's velocity with respect to the shore is just the vector sum of his velocity with respect to the water (5m/s due north) plus the velocity of the river with respect to the shore (2 m/s due east).

$$\vec{v}_{Mike,Shore} = \vec{v}_{Mike,River} + \vec{v}_{River,Shore}$$

Since these velocities are orthogonal, we can evaluate the vector sum using the Pythagorean theorem as shown in Figure 3.3.. Namely, the magnitude of Mike's velocity with respect to the shore is just length of the hypotenuse of the triangle. We can also obtain his direction from this right triangle by noting that $\tan\theta = 2/5$, where θ is the angle (east of north) that Mike moves as seen by an observer on the shore.

This example illustrates all you have to know about relative motion. Writing down the displacement equation and differentiating it yields the relative velocity equation which can be applied in any relative motion problem.

As a note to the curious, though, this derivation does eventually break down once we consider speeds close to the speed of light. Namely, one of the postulates of special relativity states that the speed of light is the same in all reference frames. In order for the speed of light to be the same in all reference frames, time *cannot* be the same in all reference frames. Time is no longer a river; observers in reference frames that are in relative motion must report different measurements for the time intervals between the same two events. Consequently, our derivation of the simple relation between relative velocities will not work in special relativity. The expression that describes how velocities add in special relativity is given by:

$$\vec{v}_{A,C} = \frac{(\vec{v}_{A,B} + \vec{v}_{B,C})}{1 + (v_{A,B}v_{B,C}/c^2)}$$

where c is the speed of light. Clearly, in the limit that the speeds $v_{A,B}$ and $v_{B,C}$ are small compared to that of light, this expression reduces to the form that we have just derived in the framework of Newtonian mechanics. Therefore, for our purposes in this course, we will simply use the Newtonian result for the addition of relative velocities.
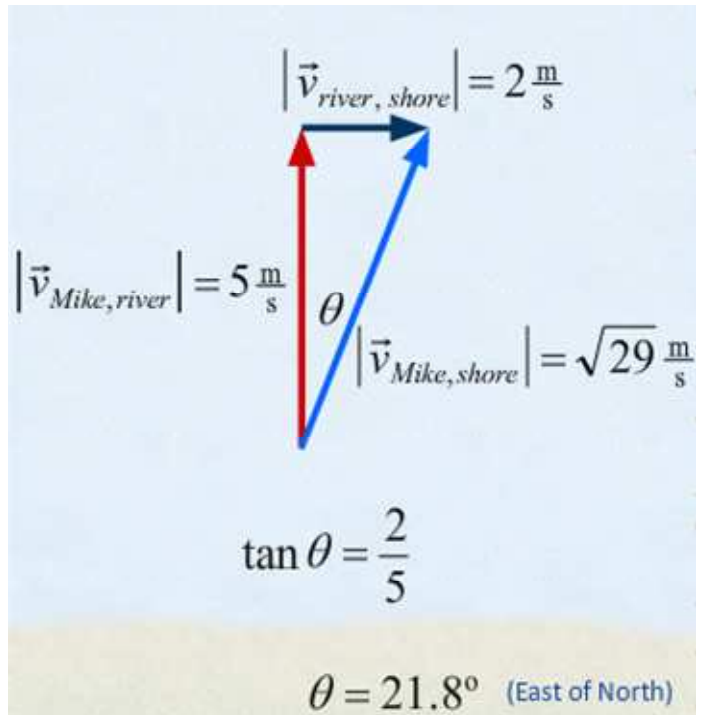
$$\vec{v}_{A,C} = (\vec{v}_{A,B} + \vec{v}_{B,C})$$

$$\left|\vec{v}_{river,shore}\right| = 2\tfrac{m}{s}$$

$$\left|\vec{v}_{Mike,river}\right| = 5\tfrac{m}{s}$$

$$\left|\vec{v}_{Mike,shore}\right| = \sqrt{29}\tfrac{m}{s}$$

$$\tan\theta = \frac{2}{5}$$

$$\theta = 21.8°\ \text{(East of North)}$$

Figure 3.3
The deterination of Mike's velocity with respect to the shore using vector addition.

D) Accelerating Reference Frames

In the last two sections we have related observations made in one reference frame with those in another. In both cases, one frame was moving relative to the other with a constant velocity. What would happen if this relative velocity was not constant? In other words, what happens in a reference frame which is accelerating?

We'll start with an everyday example. Suppose you have some fuzzy dice hanging from a string tied to the rear-view mirror of your car. When you step on the gas and accelerate forward you see the dice swing backward as shown in Figure 3.4, just as if something were pushing them toward the back of the car. Indeed, you will feel yourself being pushed back into your seat in exactly the same way. Of course, no one is actually pushing the dice backward, nor is there anything actually pushing you back into the seat either – it just seems that way because you are accelerating.
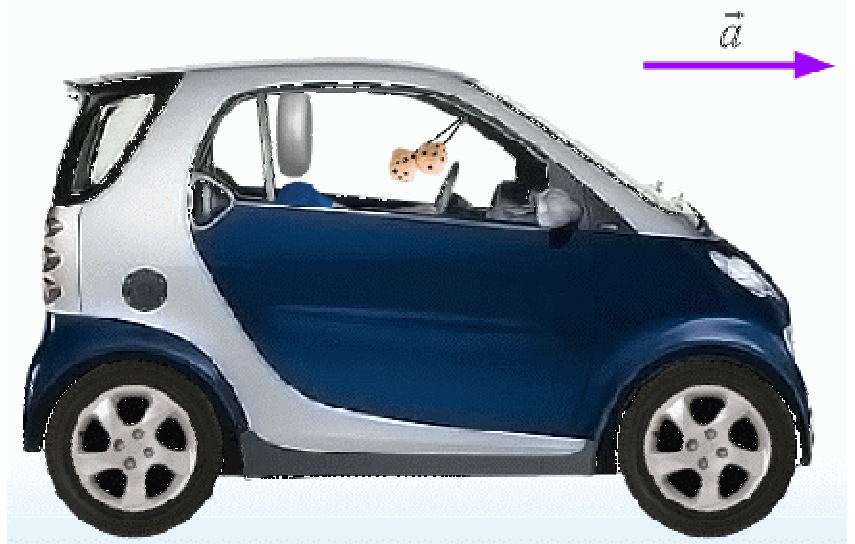

$\vec{a}$

Figure 3.4
As the car accelerates, the fuzzy dice swing backward.

Trying to understand physics in a reference frame that is accelerating can be confusing since the acceleration itself can easily be mistaken as a push or a pull. For this reason we usually only consider non-accelerating reference frames in this class, called "inertial" reference frames. We will develop this framework in the next unit. For now, though, we will take one more look at this example in order to discuss centripetal acceleration.

E) Rotating Reference Frames

Suppose you are in the same car discussed on the last section. Imagine that, instead of accelerating straight ahead, you are now driving with a constant speed but you are going around a circular track turning to the left. What happens now to the dice suspended from your rear-view mirror? Well, they will now swing over to the right! Since there is nothing actually pushing the dice, we conclude that the car, even though it is moving at a constant speed, must be accelerating.

We can visualize what is happening here if we consider a bird's eye view and draw in the velocity vector of the moving car as it moves as shown in Figure 3.5. The direction of this vector is clearly changing as a function of time, even though its magnitude, its speed, remains constant as a function of time. Since the acceleration vector is defined to be the time derivative of the velocity vector, we see that the

acceleration of the car is non-zero! The velocity vector changes in time because its direction changes in time.

$$\vec{a} \equiv \frac{d\vec{v}}{dt} \neq 0$$
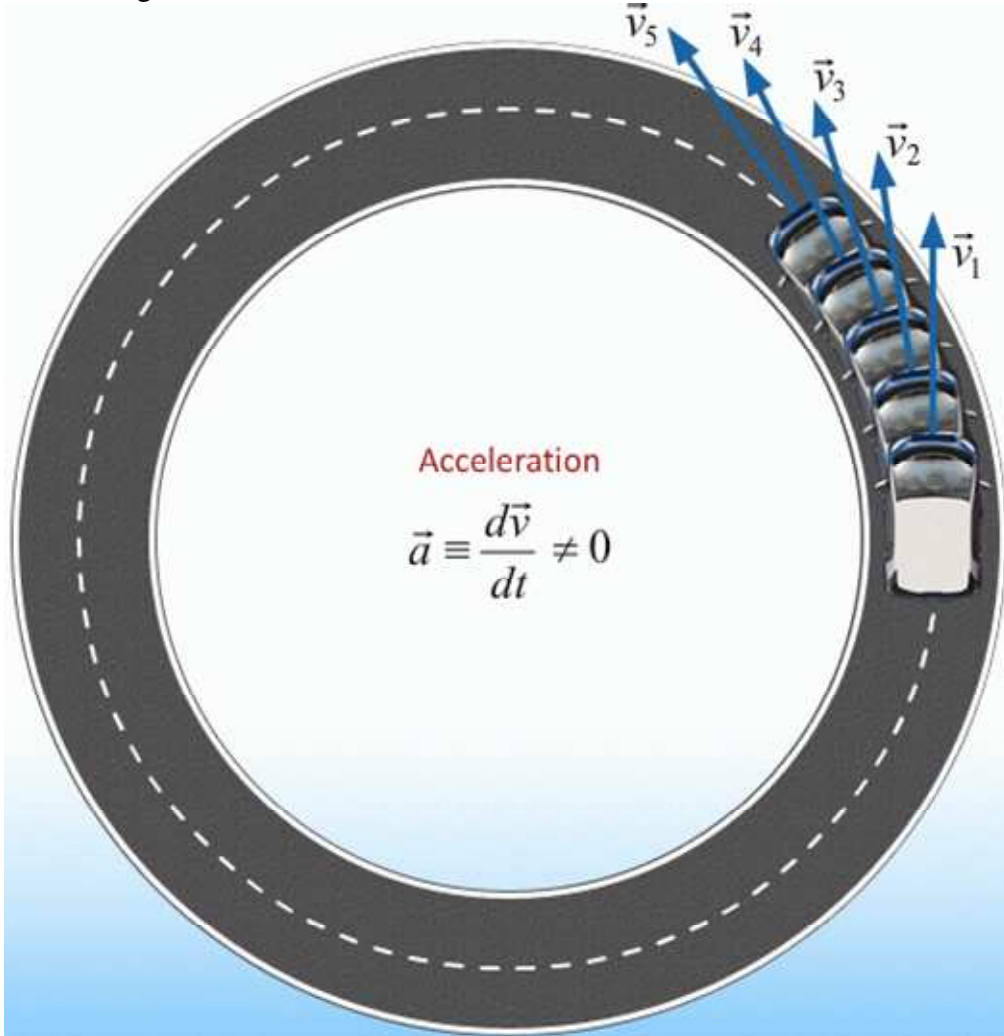
Acceleration

Figure 3.5
A car moves with constant speed around a circular track. Since the direction of the velocity vector is constantly changing, the acceleration of car is not zero. In fact its magnitude is constant and its direction is always towards the center of the circle.

What is the direction and magnitude of the car's acceleration? By subtracting successive velocity vectors in the figure, we can that the direction of this change in velocity is a vector that points toward the center of the circle. Since the direction of the acceleration is, by definition, the direction in which the velocity is changing, we see that the car's acceleration is towards the center of the circle. This result is consistent with the behavior of the dice hanging from the rear-view mirror. Namely, just as before they are tilted in the opposite direction of the acceleration.

What about the magnitude of this acceleration? To determine the magnitude, we have to do some geometry. We'll start by considering two velocity vectors separated in

time by a small amount equal to *dt* as shown in Figure 3.6. We define the angle between these velocity vectors during this time *dt* to be equal to *dθ*. We can use the small angle approximation for *dθ* to obtain the magnitude of *dv*, the change in velocity vector during the time *dt*.

$$dv = v\,d\theta$$

Now the magnitude of the acceleration is just equal to the magnitude of the time rate of change of the velocity, namely, *dv/dt*. Therefore, we see that the magnitude of the acceleration is given by:

$$|\vec{a}| \equiv \left|\frac{d\vec{v}}{dt}\right| = v\frac{d\theta}{dt}$$

We will discuss this acceleration and the meaning of *dθ/dt* in more detail in the next section.

F) Centripetal Acceleration

We just determined that an object following a circular path with a constant speed has an acceleration that points toward the center of the circle with a magnitude equal to the speed of the object times the time rate of change of its direction, *dθ/dt*. Since the acceleration vector always points toward the center, we say this motion has centripetal acceleration.

$$dv = v\,d\theta$$

$$\vec{v}(t+dt) \qquad d\theta \qquad \vec{v}(t)$$

Acceleration

$$a = v\,\frac{d\theta}{dt}$$

Figure 3.6
We use the small angle approximation to determine the magnitude of the centripetal acceleration.

It certainly makes sense that the acceleration the car experiences as it turns gets bigger as the speed increases, but what about the *dθ/dt* term? d*θ*/d*t*, by definition, is equal to the time rate of change of *θ*, the angle that locates the car with respect to the center. Therefore, *dθ/dt* is a measure of how fast the direction of the car is changing in time, how fast it is turning. Put this way, it does make sense that the acceleration depends on *dθ/dt*. Imagine driving at the same speed but turning the steering wheel harder and making your turn tighter; you are turning quicker so you expect the acceleration to feel bigger.

The time rate of change of *θ*, *dθ/dt*, is called the *angular velocity* and is usually denoted by the symbol *ω*. This angular velocity can be determined from the speed *v* and the radius *R* that defines a particular uniform circular motion. Namely, the speed is just equal to the distance the car goes in one revolution (the circumference of the circle) divided by the time it takes to make one revolution (the period).

$$v = \frac{2\pi R}{T}$$

Now the angular velocity is equal to the total angular distance the car travels (2π radians) divided by the time it takes to make one revolution (the period).
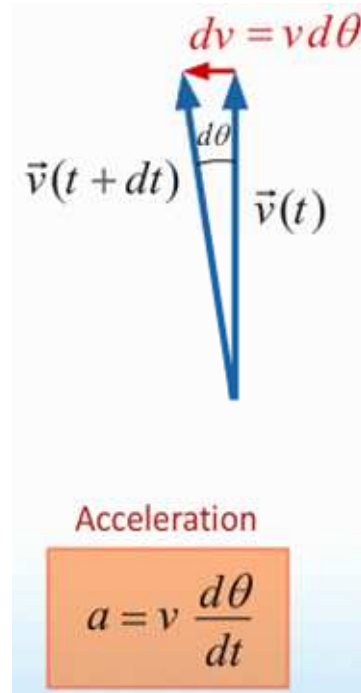
$$\omega = \frac{2\pi}{T}$$

Comparing these two equations, we see that the angular velocity is just equal to the velocity divided by the radius of the motion.

$$\omega = \frac{v}{R}$$

We can now write our expression for the centripetal acceleration in a more familiar way:

$$a = v\omega = \frac{v^2}{R}$$

G) Examples

We'll close this unit by doing a couple of examples to get a feeling for the magnitudes of accelerations that can be encountered.

We'll start by driving around a circular track whose radius is 100 m. How fast do we have go in order for the magnitude of our centripetal acceleration to be one g, the acceleration due to gravity?

Setting our expression for the centripetal acceleration equal to one g (9.81 m/s^2), we find that the required velocity is equal to about 31 m/s or about 70 mph.

$$v = \sqrt{aR} = \sqrt{gR} = 31 \quad m/s$$

You might be interested to know that racecar drivers regularly experience even higher accelerations. For example, at the Indianapolis 500, typical accelerations in the turns are about 4 g's.

For our second example, we will consider the rotation of a person at rest on the surface of the earth. Now this person, while at rest with respect to the surface of the earth is actually moving quite fast with respect to the axis of the earth. Indeed, this speed is easy to calculate: at the equator this speed is just equal to the circumference of the earth divided by the time in one day, which works out to be around 1000 miles per hour!

$$v_{equator} = \frac{2\pi R_{Earth}}{T} = \frac{2\pi\left(6.4\times10^6\,m\right)}{8.64\times10^4\,s} = 465 \quad m/s$$
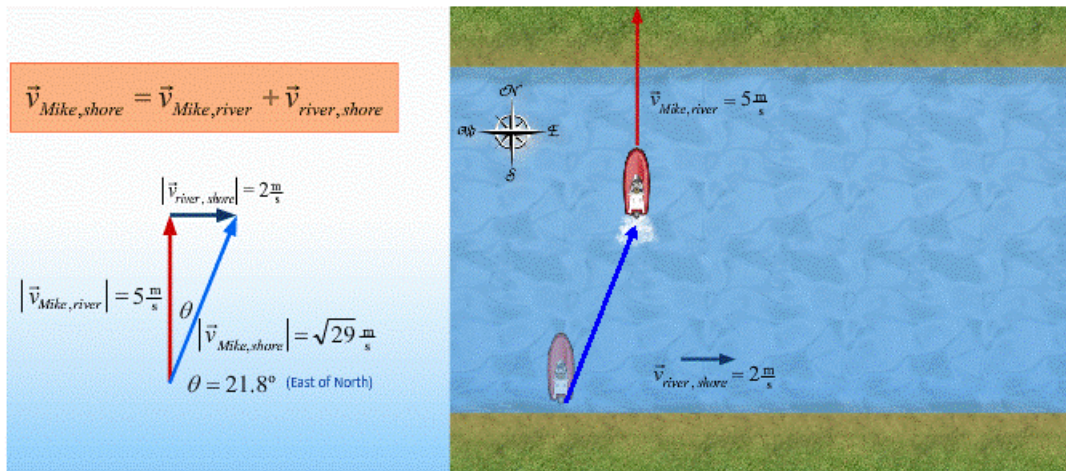
This velocity is certainly significant, but the quantity of interest here is the centripetal acceleration. When we divide the square of this speed by the radius of the earth, we find that the acceleration is, in fact, quite small: about one third of one percent of g. This value is small enough that we don't really notice it and we are quite justified in ignoring its effect for any measurements we will make. We can and will assume that the surface of the earth is a perfectly fine inertial reference frame.

By the way, we are also rotating about the Sun once per year. We leave it as an exercise to the student to calculate the centripetal acceleration of this motion. You should find that it is even smaller than that of the rotation of the earth about its axis.

Main Points

- ## **Relative motion**

  The velocity of an object in frame A can be found from its velocity in frame B by adding (as vectors) the relatie velocity of the two frames

  $$\vec{v}_{Mike,shore} = \vec{v}_{Mike,river} + \vec{v}_{river,shore}$$

  $$\left| \vec{v}_{river,shore} \right| = 2 \tfrac{m}{s}$$

  $$\left| \vec{v}_{Mike,river} \right| = 5 \tfrac{m}{s}$$

  $$\left| \vec{v}_{Mike,shore} \right| = \sqrt{29} \tfrac{m}{s}$$

  $$\theta = 21.8° \ \text{(East of North)}$$

  $$\vec{v}_{Mike,river} = 5 \tfrac{m}{s}$$

  $$\vec{v}_{river,shore} = 2 \tfrac{m}{s}$$

- ## **Uniform Circular Motion**

  An object moving with constant speed in a circular path has an acceleration whose direction is always towards the center of the circle (centripetal acceleration) and whose magnitude is proportional to the square of the speed and inversely proportional to the radius of the circle.

  $\vec{v}$

  R

  $\vec{a}_c$

  Centripetal Acceleration

  $$a_c = \frac{v^2}{R} = \omega^2 R$$