8. Conservative Forces and Potential Energy

A) Overview

This unit introduces an important new concept: potential energy. In particular, for any conservative force, we can define the change in potential energy of an object as minus the work done by this force. In this course, we deal with two conservative forces, gravity and springs. After defining the potential energy associated with each of these forces, we can rewrite the work-kinetic energy theorem so that it expresses a conservation law: the conservation of mechanical energy that applies whenever the only forces that do work in the situation are conservative forces.

B) Conservative Forces

In the last unit we introduced the concept of work as a force acting over some distance, and we showed that work done on an object will change its kinetic energy. We evaluated the work done by two of the forces we have discussed so far, gravity and springs, and we have found that for these forces, the work done on an object by these forces depends only on the starting and ending points of the motion, and not on the path taken between these points.

In general, forces that have the property that the work done by them is independent of the path used to integrate between the two points are called *conservative forces*. If the work done by a force between two points is independent of the path, then it must be true that the work done by such a force on a closed path, that is to say a path that ends up where it started, is zero. To prove this claim, consider any two points x_1 and x_2 on the closed path as shown in Figure 8.1.

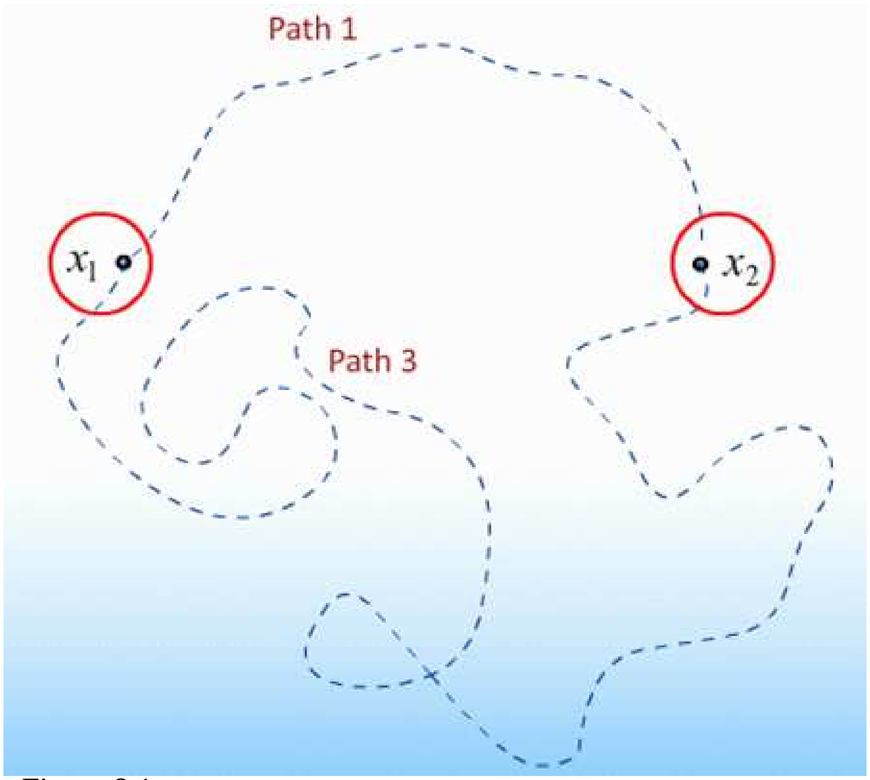

Figure 8.1
The work done by a conservative force around any closed path is zero. For example, the work done from x_1 to x_2 along Path 1 is equal to minus the work done from x_2 to x_1 along Path 3.

The work done from x_1 to x_2 along the top segment must be equal to the work done from x_1 to x_2 on the bottom segment, since the force is conservative. However, the work done from x_1 to x_2 on the bottom segment must be equal to minus the work done from x_2 to x_1 along the bottom segment, since we have just interchanged the endpoints. Now we can see that the work done around the closed loop is equal to the sum of the work done from x_1 to x_2 along the top segment and the work done from x_2 to x_1 along the bottom segment. This sum is zero!

C) Potential Energy

In general, the work done by a force on an object between two points does depend on the path taken by the object between the two points. For the special case of conservative forces, we have seen that the work does not depend on the path. Therefore, we can define, for conservative forces, an associated *potential energy* that, for a given object, depends only on its location. In particular, when a conservative force acts on an object as it moves between two points, we define the define the *change* in potential energy associated with that force as minus the work done by that force between those two points.

$$\Delta U = U_B - U_A \equiv -W_{A \to B}$$

At this point, this definition of potential energy must seem quite arbitrary to you. If we look at this definition in the context of the work-kinetic energy theorem, however, it will begin to make sense.

Recall the example from the last unit shown in Figure 8.2 in which we applied the work-kinetic energy theorem to determine that the speed that a ball, released from rest, attains while sliding down a frictionless surface only depends on the change in height of the ball and not on the details of the surface. In particular, we found that the change in

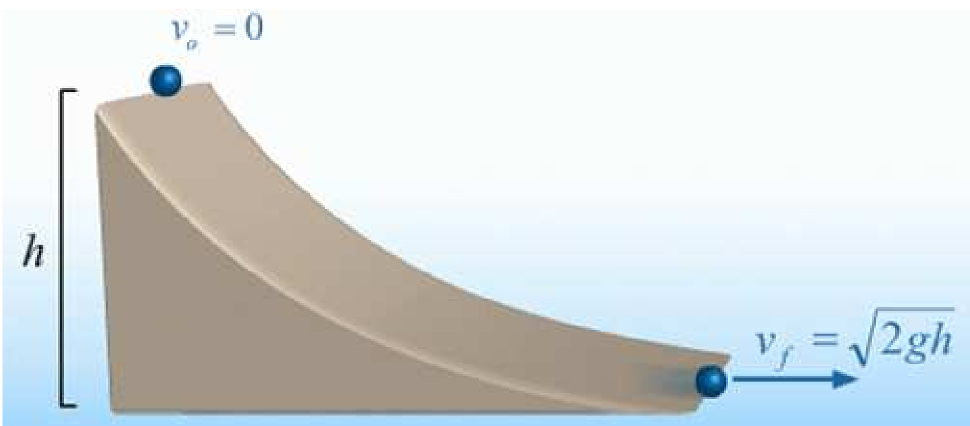

Figure 8.2
The speed of the ball at the botton of the frictionless surface depends only on *h*, the change in height of the ball.

kinetic energy of the ball was equal to the work done by the gravitational force. By our definition, the change in gravitational potential energy is equal to minus the work done by the gravitational force.

$$\Delta U_{gravity} = -W_{gravity} = -(-mg\Delta h) = mg\Delta h$$

Consequently, we see that the change in potential energy is just equal to the product of the weight of the ball and the change in height between the two points. Therefore, we see that the change in kinetic energy is just equal to minus the change in potential energy.

$$\Delta K = W_{gravity} = -\Delta U_{gravity}$$

Whatever kinetic energy the ball gains is exactly equal to the potential energy that the ball loses. If we define the total mechanical energy of an object as the sum of its kinetic and potential energy, then, in this case, we see that the total mechanical energy of the ball was conserved. That is, at any point during the motion of the ball, the sum of its kinetic energy plus its gravitational potential energy is a constant.

$$\Delta E_{mechanical} \equiv \Delta(K + U_{gravity}) = \Delta K + \Delta U_{gravity} = 0$$

D) Conservation of Mechanical Energy

In the lexample shown in Figure 8.2, we demonstrated that the total mechanical energy of the ball, the sum of its kinetic energy and potential energy, was constant throughout its motion. We will now examine the work-kinetic energy theorem to determine exactly when mechanical energy is conserved.

The work-kinetic energy theorem was derived from Newton's second law and states that the change in an object's kinetic energy is equal to the work done by the net force on that object. We can expand the work done by the net force as the sum of the work done by conservative forces and the work done by non-conservative forces.

$$\Delta K = W_{net} = W_C + W_{NC}$$

In this course, the only *conservative* forces we encounter are the *gravitational force* and the *spring force*. All other forces, for example, friction, tension, etc.., are non-conservative forces. If we now move the work done by conservative forces term to the left hand side of the equation and apply the definition of potential energy, we can see that the sum of the change in kinetic energy and the change in potential energy is equal to the work done by the non-conservative forces.

$$\Delta K - W_C = \Delta K + \Delta U = \Delta E_{mechanical} = W_{NC}$$

Now the sum of the change in kinetic energy and the change in potential energy is defined to be the change in the mechanical energy of the object. Therefore, we see that whenever the work done by non-conservative forces is zero, the change in mechanical energy is zero. That is, the *mechanical energy* is *conserved* whenever *the work done by all the non-conservative forces is zero*.

We will encounter many situations in this course in which the work done by the non-conservative forces is zero. In these cases, we can apply the conservation of mechanical energy to answer easily many questions that might be difficult to answer using Newton's laws directly. Indeed, the conservation of mechanical energy gives us the relationship between the location of the object and its speed.

We willl now do a couple of examples that illustrate the power of this conservation of mechanical energy law.

E) Gravitational Potential Energy

We've defined the change in potential energy as minus the work done by a conservative force between two points. We can convert this change in potential energy to a potential energy function defined at any single point by simply choosing some specific point as the zero of potential energy. For example, we can define the gravitational potential energy of a mass *m* near the surface of the Earth as simply *mgh*, where *h* is the height of the mass above a convenient, but arbitrary, point which we can choose to be the zero of potential energy as shown in Figure 8.3.

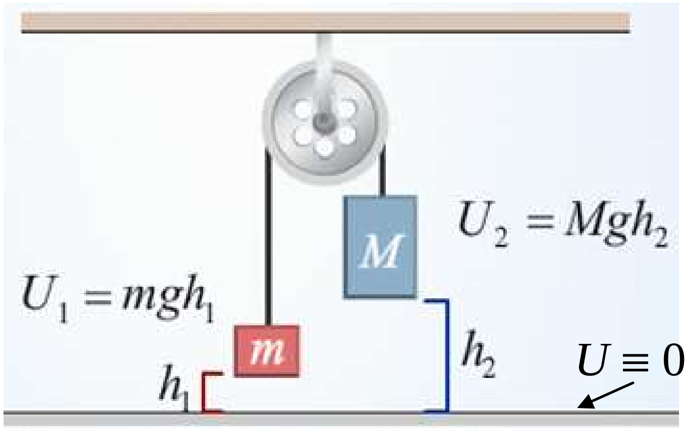

Figure 8.3
The gravitational potential energy U can be defined at any point by choosing a convenient height to have U = 0.

The table below shows the form of the potential energy function for all conservative forces we will deal with in this course. The arbitrary constant U_0 appears in the general form for the potential energy function since it will always cancel when we calculate the physically significant *change* in potential energy.

	Force $\vec{F}$	Work $W_{1\to 2}$	Change in P.E. $\Delta U = U_2 - U_1$	P.E. Function U
Gravity (Near Earth)	$m\vec{g}$	$-mg(h_2 - h_1)$	$mg(h_2 - h_1)$	$mgh + U_o$
Gravity (General Expression)	$-G\frac{m_1 m_2}{r^2}\hat{r}$	$Gm_1 m_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$	$-Gm_1 m_2\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$	$G\frac{m_1 m_2}{r} + U_o$
Spring	$-k\vec{x}$	$-\frac{1}{2}k(x_2^2 - x_1^2)$	$\frac{1}{2}k(x_2^2 - x_1^2)$	$\frac{1}{2}kx^2 + U_o$

We will now do an example using the potential energy associated with the universal gravitational force. Suppose we release a ball from a spot far away from Earth and want to know how fast will it be moving when it finally gets here. If the ball is released from rest its change in kinetic energy is proportional to the square of its final speed.

$$\Delta K = \tfrac{1}{2} m_{ball} v_f^2$$

Now the change in its potential energy can be found using the expression shown in the table. Since the initial distance is far from the earth, we can approximate one over the initial distance as zero. What is the final distance? To determine this distance we need to recall that when we discussed the application of Newton's universal gravitational force law between an object and the Earth, we said that we could consider all of the mass of the Earth to be located at its center. Therefore, the final distance here must be equal to the radius of the Earth!

$$\Delta U = -G\frac{M_E m_{ball}}{R_E}$$

Applying the conservation of mechanical energy, we obtain the result that the final speed is proportional to the square root of the ratio of the mass of the Earth to its radius. .

$$\Delta K + \Delta U = 0 \qquad \Rightarrow \qquad v_f = \sqrt{\frac{2GM_E}{R_E}}$$

It is not too hard to see that we would have arrived at exactly the same answer if we had started by asking another very interesting question: What is the initial speed we need to launch something with from the surface of Earth so that it never returns? In other words, after being launched from Earth, the object slows down and eventually stops when it is infinitely far away. This speed is called the Earth's escape velocity, and when you plug in the numbers you find that it is about 11,200 m/s!

F) Vertical Springs

We have already derived an expression for the change in potential energy of a spring.

$$\Delta U_{spring} = \tfrac{1}{2}k(x^2 - x_o^2)$$

If we choose x_0 to be the equilibrium length of the spring, we obtain a simple parabola centered on the origin as shown in Figure 8.4.

Suppose we now hang a spring vertically. As long as we assume the spring is massless, it will have the same equilibrium length as before, and the equation for its change in potential energy will have exactly the same form as long as we choose y_0 to be its equilibrium length.

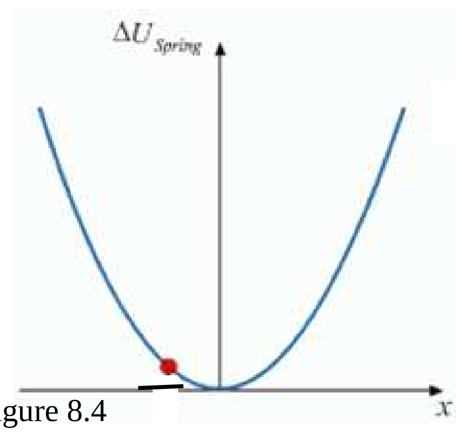

Figure 8.4
Defining the potential energy of a spring to be equal to zero at its equilibrium position results in the potential energy function being a parabola.

Now suppose we hang a box of mass *m* on the end of the spring. This will move the equilibrium position of the system downward to the point where the upward force of the spring balances the weight of the box as shown in Figure 8.5. The amazing thing is that the total change in potential energy of the system due to the spring *and* to gravity combined *still* has the same simple parabolic form as before as long as we make our measurements relative to the new equilibrium position y_e. We will now prove this claim.

Call the displacement from the new equilibrium position *y'*. We can now write down the expression for the change in potential energy of the spring as we move a distance *y'* from equilibrium position.

$$\Delta U_{spring} = \tfrac{1}{2}k((y' - y_e)^2 - y_e^2)$$

We can simplify this expression to obtain two terms.

$$\Delta U_{spring} = \tfrac{1}{2}ky'^2 + ky_e y'$$

The first term is what we want. We can rewrite the second term by replacing the ky_e factor (the magnitude of the force exerted by the spring) by the weight of the box. Once we make this replacement, we see that the second term is actually equal to minus the change in potential energy due to gravity.

$$ky_e y' = mgy' = -\Delta U_{gravity}$$

Therefore, if we move this second term to the left hand side of the equation, we obtain the expression that we want:

$$\Delta U_{spring} + \Delta U_{gravity} = \tfrac{1}{2}ky'^2$$

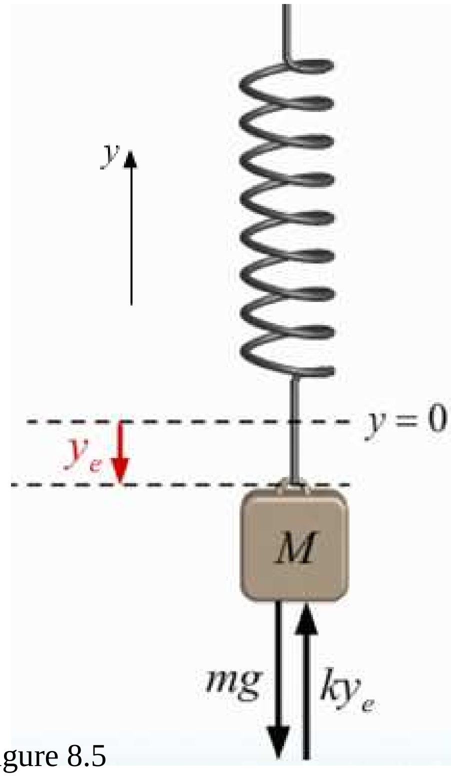

Figure 8.5
A mass M is hung from a spring stretching it a distance y_e from its unstretched length.

Namely, the sum of the change in potential energy due to the spring and the change in the potential energy due to gravity, *i.e.*, the change in the *total potential energy* of the system is just equal to the usual expression for the change in the potential energy of a spring if we choose the zero of potential energy to be the equilibrium position when the box is attached!

$$\Delta U_{total} = \tfrac{1}{2}ky'^2$$

The beautiful bottom line here is that the change in potential energy of masses hanging from vertical springs have the same simple formula as masses attached to horizontal springs, just as long as we measure the length of the spring *relative* to its *equilibrium length* in both cases.

G) Non-Conservative Forces

We will close this unit with a brief discussion of non-conservative forces in the context of the work-kinetic energy theorem and potential energy.

Figure 8.6 shows a box being pulled up a ramp through a displacement Δx. The forces acting are the weight, the tension, the normal force and the kinetic friction force. If we write down the work-kinetic energy theorem applied to the box as a single rigid object and expand the work done by the net force as the sum of the works done by the individual forces, we obtain:.

$$\Delta K = W_{gravity} + W_{tension} + W_{normal} + W_{friction}$$

The only conservative force acting is the weight. We can bring its term to the left hand side of the equation and call it the change in potential energy.

$$\Delta K + \Delta U_{gravity} = W_{tension} + W_{normal} + W_{friction}$$

The left hand side of the equation is now the change in the total mechanical energy of the box. The right hand side of the equation is the sum of the work done by the non-conservative forces. It's clear now that the work done by the non-conservative forces on an object is the thing that *changes* the total mechanical energy of an object. No potential energy can be associated with a non-conservative force because the work it does depends not only on the endpoints of the movement but also on the exact path taken.

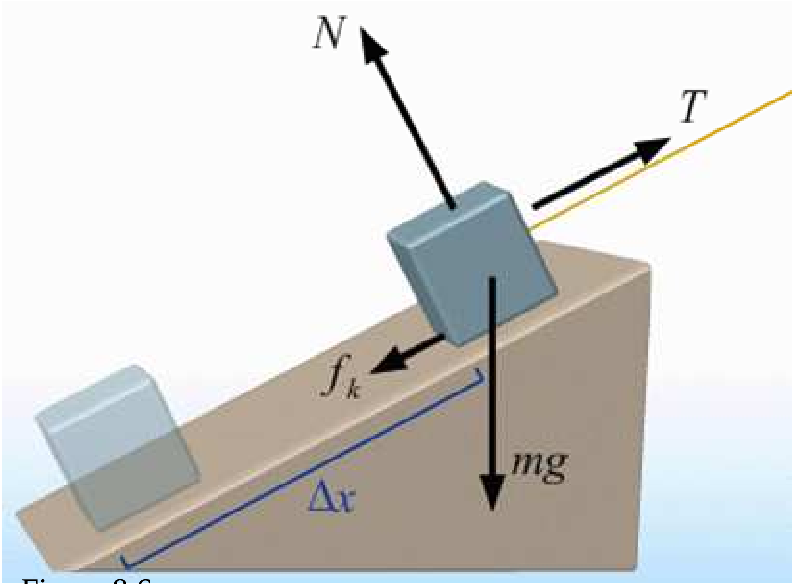

Figure 8.6
The forces acting on a box being pulled through a displacement Δx up a ramp are the tension T, the weight mg, the normal force N, and the kinetic friction force f_k.

In the next unit we will discuss in detail the calculation of the work done by non-conservative forces, especially friction, that is responsible for the change in the mechanical energy of an object.

Main Points

- **Potential Energy**

The change in potential energy that is associated with a specific conservative force as an object moves between two locations is defined as minus the work done by that force between those two locations.

$$\Delta U = U_B - U_A \equiv -W_{A\to B}$$

A potential energy function can be defined for the object and the particular force by choosing a specific location as the zero of the function.

$$U = -W_{r\to r_o} + U_o$$

- **Conservation of Mechanical Energy**

The mechanical energy of an object is defined to be the sum of its kinetic and potential energies

$$E_{Mechanical} \equiv K + U$$

The Work-Kinetic Energy Theorem can be reformulated as a conservation law. Whenever, the work done by non-conservative forces is ZERO, the mechanical energy of that object is conserved

$$\Delta E_{Mechanical} = 0$$

(When the work done by non-conservative forces is zero.)