16. Rotational Dynamics

A) Overview

In this unit we will address examples that combine both translational and rotational motion. We will find that we will need both Newton's second law and the rotational dynamics equation we developed in the last unit to completely determine the motions. We will also develop the equation that is the rotational analog of the center of mass equation. Namely, we will find that the change in the rotational kinetic energy is determined by the integral of the torque over the angular displacement. We will close by examining in detail the motion of a ball rolling without slipping down a ramp.

B)Example: Disk and String

In the last unit we developed the vector equation that determines rotational dynamics, that the net torque on a system of particles about a given axis is equal to the product of the moment of inertia of the system about that axis and the angular acceleration.

$$\vec{\tau}_{Net} = I\vec{\alpha}$$

We will now apply this equation to a number of examples. We will start with the solid cylinder, mounted on a small frictionless shaft through its symmetry axis, as shown in Figure 16.1. It has a massless string wrapped around its outer surface. The string is pulled with a force ***F*** causing the cylinder to turn. Our task is to determine the resulting angular acceleration of the disk.

We will start by defining the system to be the disk and calculating the torque exerted on this system about the rotation axis. The torque is produced by the applied force ***F*** which always acts at a distance *R* from the axis. Furthermore, the direction of the force is always perpendicular to ***R***, the vector from the axis to the point of application of the force. Therefore the torque vector (***R***×***F***) has magnitude equal to the product of *R* and *F* and a direction, obtained from the right hand rule, that points along the axis, to the right in the figure.

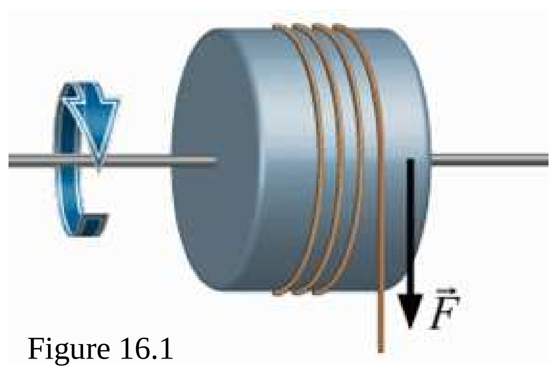


Figure 16.1
A force ***F*** is applied to a string wrapped around a solid cylinder mounted on a frictionless shaft producing an angular acceleration of the cylinder about its axis.

$$\vec{R}\times\vec{F} = I\vec{\alpha}$$

The direction of the angular acceleration must be the same as that of the torque. Consequently, since the disk was initially at rest, the disk rotates in the direction shown and its speed increases with time. Since we know the moment of inertia of a solid disk about its axis of symmetry, we can solve for the magnitude of the angular acceleration.

$$I = \frac{1}{2}MR^2 \quad \Rightarrow \quad \alpha = \frac{2F}{MR}$$

C) Combining Translational and Rotational Motion

Figure 16.2 shows the disk from the last section with a weight added to the end of the string. When we release the weight, the weight falls, pulling the string and causing the disk to rotate. In this example, we must deal with both the translational motion of the weight and the rotational motion of the disk. We want to calculate the resulting linear and angular accelerations.

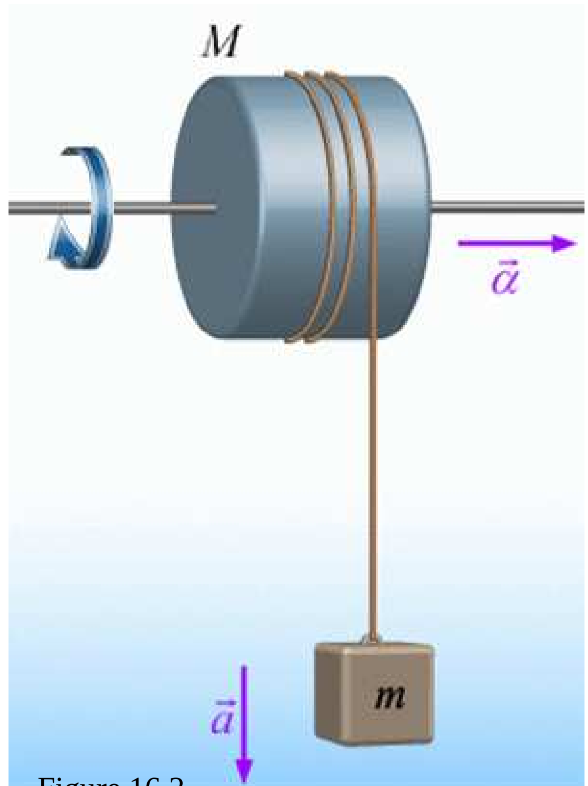


Figure 16.2
A a mass *m* is attached to a string which is wrapped around a solid cylinder. As the mass falls, the string unwinds, producing an angular acceleration of the cylinder about its axis.

How do we go about starting the calculation? To determine the motion of the weight, we will start by writing down Newton's second law. There are two forces acting on the weight: the tension force exerted by the string pointing up and the gravitational force exerted by the Earth pointing down. We will choose the positive y axis to point down here which will result in a positive linear acceleration.

$$mg - T = ma$$

For the rotation of the disk, we have the same equation as before, with the applied force ***F*** replaced by the tension force ***T***.

$$RT = I\alpha$$

We now have two equations and three unknowns: the tension and the linear and angular accelerations. We need another equation in order to solve the problem. The key here is to realize that since the string does not slip, the length of string that unwinds is equal to the arc length through which the disk turns! Therefore, we can use our result from the last unit that relates the linear acceleration of a point on the rim to the angular acceleration of the disk.

$$a = R\alpha$$

We now have three equations and three unknowns. All that is left to do is simply to solve these equations. For example, we can first replace the angular acceleration in the rotational equation by the ratio of the linear acceleration to the radius of the disk to obtain:

$$RT = I\frac{a}{R} \quad \Rightarrow \quad T = I\frac{a}{R^2}$$

We can now add this equation to the Newton's second law equation for the weight in order to eliminate the tension.

$$mg = a\left(m + \frac{I}{R^2}\right)$$

We can now eliminate the moment of inertia by substituting in its value in terms of the mass and radius of the disk to obtain our result for the acceleration of the weight:.

$$mg = a\left(m + \tfrac{1}{2}M\right) \;\Rightarrow\; a = g\left(\frac{m}{m + \tfrac{1}{2}M}\right)$$

We see that the acceleration of the weight is less than g by a factor determined by the masses of the weight and the disk. We can now use this value for the linear acceleration to determine the tension in the string:

$$T = m(g - a) = mg\left(\frac{M}{M + 2m}\right)$$

Here we see that the tension is less than the weight by another factor determined by the masses of the weight and the disk.

D) Work and Energy in Rotations

We now want to look at the rotational dynamics equation in the context of energy. Recall that by integrating Newton's second law for a system of particles, we obtained the center of mass equation, namely that the total macroscopic work done on the system is equal to the change in the center of mass kinetic energy, calculated as if the system were a point particle having the total mass of the system and moving with the velocity of the center of mass.

$$\int \vec{F}_{Net} \cdot d\ell_{CM} = \Delta\left(\tfrac{1}{2} m v_{CM}^2\right)$$

We can obtain an exactly analogous equation for rotational motion relative to the center of mass. The derivation follows closely the previous derivation of the center of mass equation. Namely, if we replace the angular acceleration ($d\omega/dt$) in the rotational equation by the product of ω and $d\omega/d\theta$,

$$\alpha \equiv \frac{d\omega}{dt} = \frac{d\theta}{dt}\frac{d\omega}{d\theta} = \omega\frac{d\omega}{d\theta}$$

we obtain an equation that relates the net torque about an axis passing through the center of mass to the rate of change of the angular velocity to the angular displacement.

$$\tau_{Net} = I_{CM}\,\omega\frac{d\omega}{d\theta}$$

If we now integrate this equation, we find the relationship we are looking for.

$$\int_{\theta_1}^{\theta_2} \tau_{Net}\, d\theta = \int_{\omega_1}^{\omega_2} I_{CM}\,\omega\, d\omega = \Delta\left(\frac{1}{2} I_{CM}\,\omega^2\right)$$

Namely, that the integral of the torque over the angular displacement is equal to the change in the rotational kinetic energy. This relationship is completely general and it will prove to be a powerful tool in solving rotational problems.

This result is actually more familiar than it might seem. For example, if we evaluate the integral of the torque over the angular displacement for the rotating disk in the last section, we find that it is just equal to the work done by the tension force!

$$\int_{\theta_1}^{\theta_2} \tau_{Net} d\theta = TR\int_{\theta_1}^{\theta_2} d\theta = TR\Delta\theta = TD$$

Namely, the torque is constant and equal to the product of the tension and the radius of the cylinder, while the change in angular displacement as the weight falls through a distance *D* is just equal to *D/R*. Consequently we see that the integral of the net torque over the angular displacement is indeed equal to the product of the tension and the displacement of the weight which is just equal to the work done by the tension force!

E) Total Kinetic Energy of a Rolling Ball

We have previously shown that the total kinetic energy of a solid object is just equal to the kinetic energy of the center of mass of the object plus the kinetic energy due to the rotation of the object around an axis through the center of mass.

$$K_{Total} = \frac{1}{2}M_{Total}v_{CM}^2 + \frac{1}{2}I_{CM}\omega^2$$

The first term is called the translational kinetic energy of the object; the second term is called the rotational kinetic energy. For cases in which the object is rolling without slipping, we can simplify this expression since the angular velocity and the center of mass velocity are related in a very simple way.

Figure 16.3 shows the ball rolling through one revolution. As the ball rotates through an angular displacement θ, the center of mass moves through a distance equal to the arc

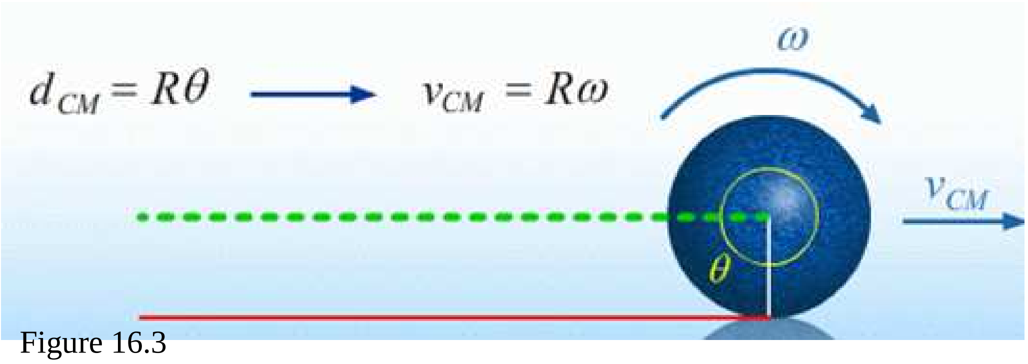


Figure 16.3
A ball rolls without slipping through a distance that corresponds to one complete revolution of the ball about its center. The center of the mass has traveled a distance = $v_{CM}t = v_{CM}(2\pi/\omega)$ which is also equal to $2\pi R$. Consequently, $v_{CM} = R\omega$.

length which is equal to the product of *R* and θ. Therefore, we see that velocity of the center of mass is just equal the product of the angular velocity of the ball and its radius!

We can now combine the kinetic energy of the center of mass with the kinetic energy of the rolling ball relative to the center of mass to obtain the total kinetic energy of the ball. Since the angular velocity of a ball that is rolling without slipping is simply related to its translational velocity, we can rewrite the total kinetic energy totally in terms of the ball's translational velocity.

$$K_{Total} = \frac{1}{2} M_{Total} v_{CM}^2 + \frac{1}{2} I_{CM} \left(\frac{v_{CM}}{R} \right)^2$$

Substituting in the moment of inertia for a solid sphere about its axis (2/5 $M_{Total}R^2$) into this equation we obtain our final result:

$$K_{Total} = \frac{7}{10} M_{Total} v_{CM}^2$$

Note that the total kinetic energy is now bigger than what it would be if the ball were sliding with the same speed, since we need to account for the additional kinetic energy due to rotation.

F) Ball Rolling Down a Ramp

We have just determined the total kinetic energy of a ball that rolls without slipping. We will now apply this result to the situation shown in Figure 16.4 where we see a solid sphere, released from rest at the top of a ramp, that then rolls without slipping to the bottom. Our task is to determine its speed when it reaches the bottom.

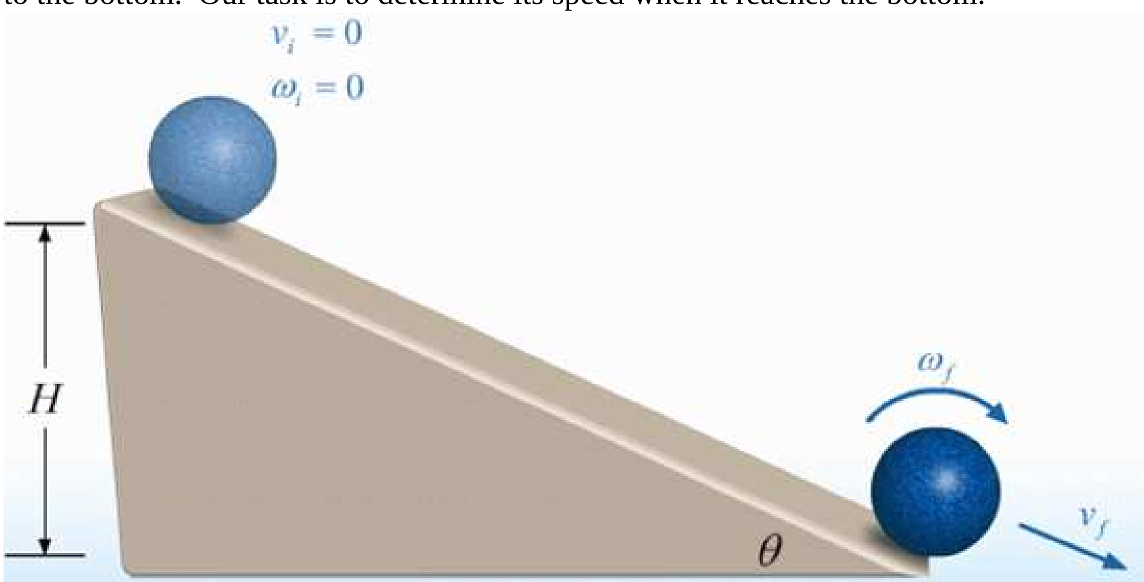


Figure 16.4
A ball is released from rest at the top of a ramp and rolls without slipping to the bottom. The speed of the ball at the bottom can be determined from energy considerations.

How do we go about solving this problem? We have certainly solved similar problems when the object was sliding, rather than rolling, down the ramp. In those cases, we applied the center of mass equation that says that the change in kinetic energy is equal to the macroscopic work done by all forces acting on the object.

$$\int \vec{F}_{Net} \cdot d\ell_{CM} = \Delta\left(\tfrac{1}{2} m v_{CM}^2\right)$$

The center of mass equation still applies here: the macroscopic work done on the ball is equal to the product of the displacement of the center of mass and the difference between the component of the weight down the ramp and the frictional force.

$$\int \vec{F}_{Net} \cdot d\ell_{CM} = (mg\sin\theta - f_K)\Delta x_{CM} \;\;\Rightarrow\;\; (mg\sin\theta - f_K)\Delta x_{CM} = \Delta\left(\tfrac{1}{2} m v_{CM}^2\right)$$

We cannot determine the final velocity of the center of mass from this equation, though, because we do not know the magnitude of the frictional force.

We can determine the magnitude of the frictional force, however from a consideration of the rotational energy equation we have recently derived, namely that the product of the net torque and the angular displacement is equal to the change in the rotational kinetic energy.

$$\int_{\theta_1}^{\theta_2} \tau_{Net}\, d\theta = \tau_{Net}\Delta\theta = \Delta\left(\tfrac{1}{2} I_{CM}\,\omega^2\right)$$

The net torque is just equal to the product of the frictional force and the radius of the ball.

$$\tau_{Net} = f_K R$$

We demonstrated in the last section that the product of the radius of the ball and the angular displacement is just equal to the displacement of the center of mass. Therefore, we can relate the change in the rotational kinetic energy of the ball to the product of the frictional force and the displacement of the center of mass.

$$f_K \Delta x_{CM} = \Delta\left(\tfrac{1}{2} I_{CM}\,\omega^2\right)$$

Combining this information with the center of mass equation, we obtain our final result: the change in the kinetic energy of the center of mass plus the change in the rotational kinetic energy relative to the center of mass is equal to the work done by the gravitational force.

$$\Delta\left(\tfrac{1}{2} m v_{CM}^2\right) + \Delta\left(\tfrac{1}{2} I_{CM}\,\omega^2\right) = mg\sin\theta\Delta x_{CM}$$

G) Acceleration of a Rolling Ball

In the last section we determined that the change in the kinetic energy of a ball rolling down a ramp was just equal to the work done by gravity. We would now like to determine the speed of the ball at any arbitrary time. In other words, we would like to calculate the acceleration of the ball.

We will start by drawing the free-body diagram as shown in Figure 16.6 and then writing down Newton's second law that determines the motion of the center of mass.

$$Mg\sin\theta - f_K = Ma_x$$

We would like to solve this equation for the acceleration of the center of mass, but we don't know the magnitude of the frictional force! The only thing we know about the frictional force is that it is big enough to keep the ball from slipping.

The key to finding the magnitude of this force is to realize that it is the frictional force that supplies the torque that produces the angular acceleration of the ball. Therefore, the other equation we need is the rotational equation about the center of mass of the ball.

$$f_K R = I_{CM}\alpha_{CM}$$

The magnitude of the torque is just equal to the product of the radius of the ball and the frictional force. The direction of the torque is into the page. Therefore, the angular

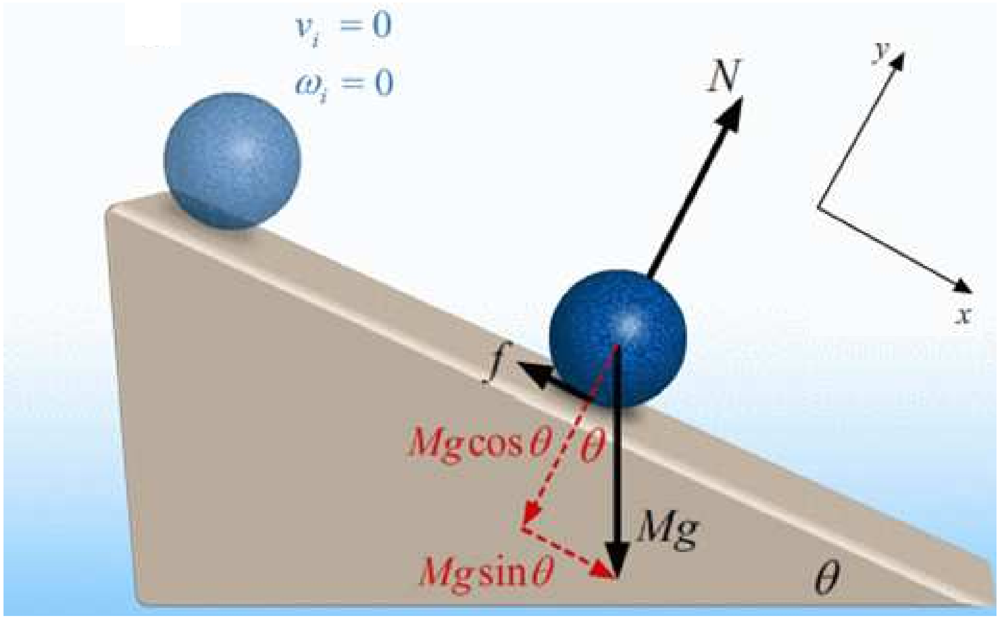


Figure 16.5
The free-body diagram for a ball rolling without slipping down a ramp.

acceleration of the ball is just equal to the magnitude of the torque divided by the moment of inertia about an axis passing through the center of mass of the ball.

$$\alpha_{CM} = \frac{f_K R}{I_{CM}}$$

Looking at the two equations we now have (Newton's second law for translations and rotations), we see we have three unknowns, the frictional force and the linear and angular accelerations. We need to eliminate one more unknown and that we can do because we know that in rolling without slipping, the angular acceleration about the center of mass is just equal to the acceleration of the center of mass divided by the radius of the ball.

$$\alpha_{CM} = \frac{a_x}{R}$$

Making this substitution for the angular acceleration, we can now solve for both the magnitude of the frictional force and the acceleration of the center of mass.

$$\frac{a_x}{R} = \frac{f_K R}{I_{CM}} = \frac{f_K}{\frac{2}{5}MR} \quad \Rightarrow \quad f_K = \frac{2}{5}Ma_x$$

$$Mg\sin\theta - f_K = Ma_x$$

We find that the acceleration of the center of mass is smaller than that of an object sliding down a frictionless ramp inclined at the same angle (which was equal to $g\sin\theta$).

$$a_x = \frac{5}{7}g\sin\theta$$

H) Why Did That Last Derivation Work?

We just derived the acceleration of a ball rolling without slipping down a ramp by applying the rotational equation ($\boldsymbol{\tau}_{Net} = I\boldsymbol{\alpha}$) in which we evaluated the torque about an axis passing through the center of mass of the ball. Recall that we obtained the rotational equation from Newton's second law and that Newton's second law is valid in inertial reference frames. The reference frame of the ball is clearly *not* an inertial reference frame. What is going on here?

The surprising answer to this question is that we can *always* apply this rotational equation, even for an object that is accelerating, as long as we are considering only rotations about the center of mass of the object! This result is certainly *not* obvious and its proof requires the introduction of a concept, *angular momentum*, that we will discuss in a future unit. For the benefit of those of you who are curious, we will present a proof of this claim now.

We will start with the definition of the angular momentum $\boldsymbol{L}$ of a particle about some axis as the as the cross product of $\boldsymbol{r}$, the vector from the axis to the particle, with the momentum vector.

$$\vec{L} \equiv \vec{r} \times \vec{p}$$

Taking the derivative of $\boldsymbol{L}$ with respect to time, we get two terms. The first term is zero since the velocity vector ($d\boldsymbol{r}/dt$) and the momentum vector are parallel. We can use Newton's second law ($\boldsymbol{F} = d\boldsymbol{p}/dt$) to write the second term as the net torque on the particle.

$$\frac{d\vec{L}}{dt} = \left(\frac{d\vec{r}}{dt} \times \vec{p}\right) + \left(\vec{r} \times \frac{d\vec{p}}{dt}\right) = 0 + \vec{r} \times \vec{F} = \vec{\tau}$$

We will use this equation which determines the time dependence of the angular momentum vector of the particle, to show that the rotational equation ($\boldsymbol{\tau} = I\boldsymbol{\alpha}$) holds in the frame of the ball that is rolling without slipping down the ramp.

The first step is to obtain an important result concerning the angular momentum of a system of particles, namely:

$$\vec{L} = \vec{L}_{CM} + \vec{L}^*$$

Where $\boldsymbol{L}$ is the total angular momentum of a system of particles in a particular reference frame, $\boldsymbol{L}_{CM}$ is the angular momentum of the center of mass of the system in that frame, and $\boldsymbol{L}^*$ is the angular momentum of the system in the center of mass frame.. The proof is straightforward, but a little lengthy. We start by expressing the displacement vector is the specified frame in terms of the displacement vector of the center of mass of the system in that frame and the displacement vector in the center of mass frame.

$$\vec{r}_i = \vec{r}_{CM} + \vec{r}_i^{\,*}$$

We now write down the expression for the angular momentum of a system of particles:

$$\vec{L} = \sum\left(\vec{r}_{CM} + \vec{r}_i^{\,*}\right) \times p_i = \left(\vec{r}_{CM} \times \sum \vec{p}_i\right) + \sum\left(\vec{r}_i^{\,*} \times \vec{p}_i\right)$$

The first term on the right hand side of the last equation is equal to the angular momentum of the center of mass since the sum of all the individual momenta is just the total momentum of the system!

$$\vec{L}_{CM} = \left(\vec{r}_{CM} \times \sum \vec{p}_i\right)$$

The second term on the right hand side must be the angular momentum of the system in the center of mass frame. To demonstrate this claim, we need to first expand the individual momenta in the specified frame in terms of the individual momenta in the center of mass frame.

$$\vec{p}_i = m_i \frac{d\vec{r}_i}{dt} = m_i\left(\frac{d}{dt}\vec{r}_{CM} + \frac{d}{dt}\vec{r}_i\,*\right) = m_i\vec{v}_{CM} + m_i\frac{d\vec{r}_i^{\,*}}{dt}$$

$$\sum\left(\vec{r}_i * \times \vec{p}_i\right) = \sum\left(\vec{r}_i^{\,*} \times m_i\vec{v}_{CM}\right) + \sum\left(\vec{r}_i^{\,*} \times m_i\vec{v}_i^{\,*}\right)$$

The first term on the right hand side of the last equation is zero since we can take the velocity of the center of mass vector outside the sum, leaving the mass-weighted sum of the displacements in the center of mass which must just be zero!

$$\sum\left(\vec{r}_i^{\,*} \times m_i\vec{v}_{CM}\right) = \left(\sum m_i\vec{r}_i^{\,*}\right) \times \vec{v}_{CM} = 0$$

We are left then with our result:

$$\sum\left(\vec{r}_i * \times \vec{p}_i\right) = \sum\left(\vec{r}_i^{\,*} \times m_i\vec{v}_i^{\,*}\right) = \vec{L}^*$$

$$\Rightarrow \quad \vec{L} = \vec{L}_{CM} + \vec{L}\,*$$

We are now finally ready to prove what we set out to prove in the beginning of this section, that we can *always* apply the rotational equation ($\tau = I\alpha$), even for an object that is accelerating, as long as we are considering only rotations about the center of mass of the object!

Suppose we fix our *x*-*y* coordinate system to the ramp as shown in Figure 16.6. This system is clearly an inertial reference frame; therefore our angular momentum

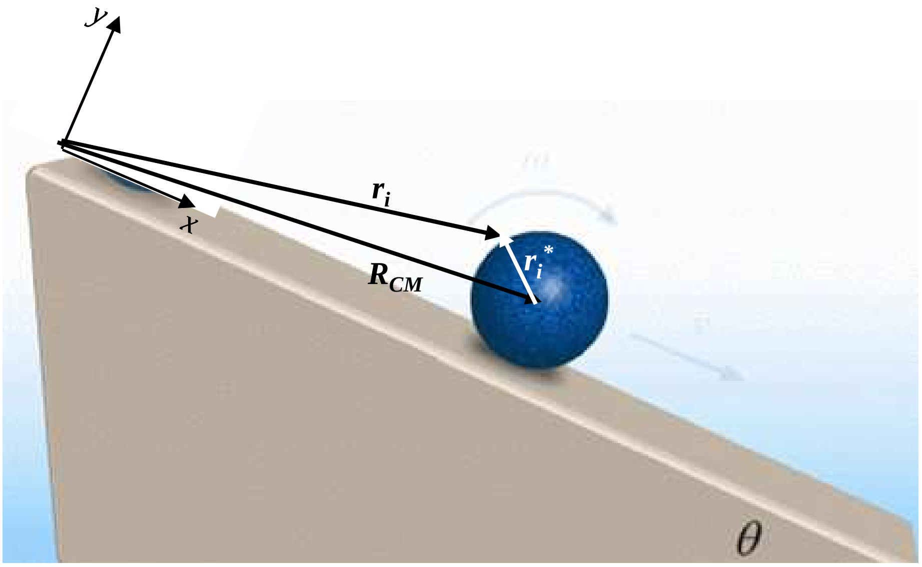


Figure 16.6
The displacement vector $\boldsymbol{r_i}$ in the inertial reference frame fixed to the ramp (*x*-*y*) is equal to the vector sum of the the displacement vector of the center of mass $\boldsymbol{R_{CM}}$ plus the displacement vector relative to the center of mass, $\boldsymbol{r_i^*}$.

equation holds in this frame. Considering the ball to be a system of particles, we can write down the equation of motion in terms of the time rate of change of the angular momentum of the system:

$$\sum \vec{\tau}_i = \sum \frac{d\vec{L}_i}{dt}$$

We will now expand the torques in terms of the individual displacements:

$$\sum \vec{\tau}_i = \sum \left(\vec{r}_i \times \vec{F}_i\right) = \sum \left(\left(\vec{r}_{CM} + \vec{r}_i^{\,*}\right) \times \vec{F}_i\right)$$

Similarly, we can expand the angular momentum of the system:

$$\sum \frac{d\vec{L}_i}{dt} = \vec{L}_{CM} + \vec{L}^*$$

Putting this altogether, we obtain the equation:

$$\sum \left(\left(\vec{r}_{CM} + \vec{r}_i^{\,*}\right) \times \vec{F}_i\right) = \frac{d}{dt}\left(\vec{L}_{CM} + \vec{L}^*\right)$$

We now have the equation we need. We just have to expand the terms and simplify. Unfortunately, once again this process is a little lengthy. We start by expanding the time rate of change of the angular momentum of the center of mass term:

$$\frac{d}{dt}\vec{L}_{CM} = \frac{d}{dt}\left(\vec{r}_{CM} \times M_{Tot}\vec{v}_{CM}\right) = \left(\vec{v}_{CM} \times M_{Tot}\vec{v}_{CM}\right) + \left(\vec{r}_{CM} \times M_{Tot}\vec{a}_{CM}\right)$$

The first term on the right hand side of the last equatin is zero since the cross product of any vector with itself is zero. The second term is equal to the net force on the system from Newton's second law. Therefore, we have obtained the expression for the time rate of change of the angular momentum of the center of mass:

$$\frac{d}{dt}\vec{L}_{CM} = \left(\vec{r}_{CM} \times \sum \vec{F}_i\right)$$

If we now substitute this form bak into our master equation (three above), we get cancellations that leave us with:

$$\sum \left(\vec{r}_i^{\,*} \times \vec{F}_i\right) = \frac{d}{dt}\vec{L}^*$$

We have now obtained our result: that the sum of the torques about an axis through the center of mass is equal to the time rate of change of angular momentum of the system in the center of mass frame. For a rigid, symmetric, solid object (such as our rolling ball),

$$\vec{L}^* = I_{CM}\,\vec{\omega}_{CM}$$

Differentiating this angular momentum with respect to time, we obtain our result:

$$\frac{d}{dt}\vec{L}^* = I_{CM}\frac{d}{dt}\vec{\omega}_{CM} = I_{CM}\,\vec{\alpha}$$

$$\Rightarrow \quad \vec{\tau}_{CM} = I_{CM}\,\vec{\alpha}$$

It's been a long haul, but we have finally proved the important result that the sum of the torques about the center of mass of a system of particles is always equal to the product of the moment of inertia about the center of mass and the angular acceleration about the center of mass, even if the system itself is accelerating!!

Main Points

- **Center of Mass Equation for Rotational Motion**

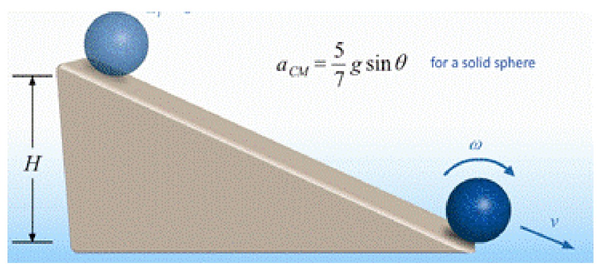
Integrating the rotational dynamics equation, we determined that the change in the rotational kinetic energy is equal to the integral of the torque over the angular displacement.

$$\int_{\theta_i}^{\theta_f} \tau_{Net}\, d\theta = \Delta\left(\frac{1}{2} I_{CM}\, \omega^2\right)$$

- **Translation + Rotation: Rolling Without Slipping**

The total kinetic energy of an object is equal to the sum of the translational kinteic energy of the center of mass and the rotational motion about thecenter of mass.

$$K_{Total} = \frac{1}{2} M v_{CM}^2 + \frac{1}{2} I_{CM}\, \omega^2$$



Applying the center of mass equation and its rotational analog, we determined that the change in the total kinetic energy of the ball rolling without slipping down the ramp is equal to the work done by gravity.

$$\Delta\left(\frac{1}{2} M v_{CM}^2\right) + \Delta\left(\frac{1}{2} I_{CM}\, \omega^2\right)$$

Applying Newton's second law and the rotational dynamics equation, we determined the acceleration of the ball down the ramp.

$$a_{CM} = \frac{5}{7} g \sin\theta$$