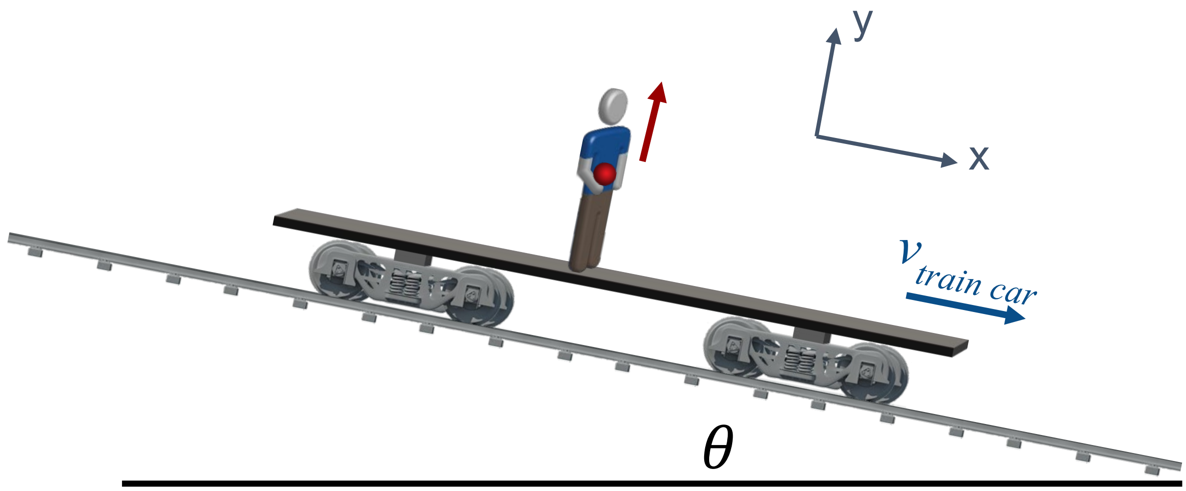
Solving the “tilted track” problem



In class on Thursday (1/23), we discussed a problem related to the figure above, where a ball is thrown “up” (as seen by the person, with a velocity relative to the train car in the direction perpendicular to the train car’s motion) and then comes back down at the same location as the person who launched it. This is a pretty non-intuitive result, and led to a lot of questions, and so we figured we’d send out some solutions for it in case there was still any confusion.

Rotated Coordinate System

In class, we discussed the more elegant (and far simpler) way to demonstrate that the ball should land back at the position of the person who launched it. In this argument, you first make your life a lot easier, by choosing a new set of rotated coordinate axes, as shown in the figure above. Here, the x axis points along the rail track, and y points perpendicular to x. From here, one can say that along the x direction, the ball and person have the **same initial position along x** (because the ball starts in the person’s hand), they have the **same initial velocity along x** (because the person gives the ball and extra initial velocity along a direction perpendicular to the car’s motion), and they each have the **same acceleration along x**. This last part may seem a bit strange. For the ball, which will just have gravity acting on it, acceleration points straight down. In these “rotated” coordinates, however, this has a projection along the x axis, with a magnitude of $g \sin \theta$, where g is the usual rate of gravitational acceleration near Earth’s surface, and the angle θ is labeled above. Similarly, the train car will also experience an acceleration along the track (along x) with a magnitude of $g \sin \theta$, ignoring any friction or any other forces (as you can see, there’s no engine or drive train on the rail car). **Given the same initial position, velocity, and acceleration along x, as well as the independence of x and y motion (which is true whenever the acceleration is fixed in both magnitude and direction – true even in this rotated coordinate system), one finds that the ball and person will always have the same position along the rail track.** In the perpendicular direction, the ball then just goes up and comes down at some later time.

“Usual” Coordinate System (gravitational acceleration pointing in negative y direction)

Now, just like in all of the problems you’ll work out in this course, **you should get the same answer independent of what coordinate system you work in**. While it’s easier in the rotated coordinate system, it also works fine if we work with our usual x and y coordinate system, where y points straight up and x points to the right. In this canonical coordinate system, gravity points only in the negative y direction. The math shown below works out this case, and shows that there’s a time t^* at which the ball and the person have the same x and y coordinates (i.e., the ball lands back at the position of the person).

Initial conditions:

$$\boldsymbol{x_0^{ball} = x_0^{person} = x_0}$$

$$\boldsymbol{y_0^{ball} = y_0^{person} = y_0}$$

$$\boldsymbol{\vec{v}_0^{ball} - \vec{v}_0^{person} = \overrightarrow{\Delta v} = \Delta v(\sin\theta\,\hat{x} + \cos\theta\,\hat{y})} \quad (i.e., \text{perpendicular to the track})$$

$$\boldsymbol{v_{x,0}^{ball} = v_{x,0}^{person} + \Delta v \sin\theta}$$

$$\boldsymbol{v_{y,0}^{ball} = v_{y,0}^{person} + \Delta v \cos\theta}$$

Acceleration of ball & person on car:

$$\boldsymbol{a_x^{ball} = 0}$$

$$\boldsymbol{a_y^{ball} = -g}$$

$$\boldsymbol{\vec{a}^{person} = g\sin\theta(\cos\theta\,\hat{x} - \sin\theta\,\hat{y})} \quad (i.e., \text{down the track})$$

$$\boldsymbol{a_x^{person} = g\sin\theta\cos\theta}$$

$$\boldsymbol{a_y^{person} = -g\sin^2\theta}$$

Difference in position:

$$\boldsymbol{\left[x^{ball}(t) - x^{person}(t)\right] = \Delta v \sin\theta\, t - g\sin\theta\cos\theta\, t^2/2}$$

$$\boldsymbol{\left[y^{ball}(t) - y^{person}(t)\right] = \Delta v \cos\theta\, t - \frac{gt^2}{2} + \frac{g\sin^2\theta\, t^2}{2} = \Delta v\cos\theta\, t - \frac{gt^2}{2}\cos^2\theta}$$

The ball and person are in contact (i.e., both of the above expressions equal 0), at two times. The first is trivial, at time t = 0 (by definition, just as the ball is released). The second happens at time $t^* = 2\Delta v/(g\cos\theta) = 2\Delta v/a_\perp$, an expression for the hang time in the air that should look vaguely familiar from Thursday's lecture, where $a_\perp$ is the acceleration acting on the ball in the direction perpendicular to the rail track due to gravity.