2. Two Dimensional Kinematics

A) Overview

We will begin by introducing the concept of vectors that will allow us to generalize what we learned last time in one dimension to two and three dimensions. In particular, we will define vector addition and subtraction and relate the component representation of a vector to the magnitude and direction representation.

We will then introduce one physics example, namely the description of free fall near the surface of the earth as motion of constant acceleration in the vertical direction and motion at constant velocity in the horizontal direction. We will use these descriptions to calculate some properties of projectile motion. Finally, we will use the principle of superposition to relate the descriptions of projectile motion in two different reference frames.

B) Kinematic Definitions in Three Dimensions

To this point, we have restricted ourselves to discussions of motions in one-dimension; we have defined velocity as d*x*/d*t* and acceleration as d*v*/d*t*. How do we generalize these definitions to more than one dimension? The generalization we make is most easily understood in terms of Cartesian co-ordinates. Figure 2.1a shows a Cartesian coordinate system, with the mutually orthogonal directions labeled *x*, *y*, and *z*. To

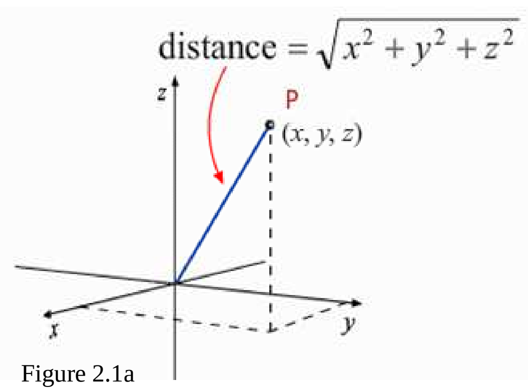

Figure 2.1a
A point *P* is specified in a Cartesian co-ordinate system by its components (*x, y, z*).

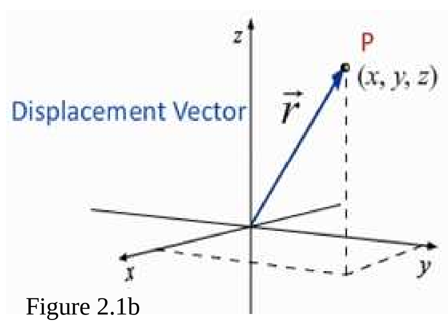

Figure 2.1b
A point *P* is specified by its displacement vector ***r*** whose Cartesian coordinates are (*x, y, z*)

identify a point *P* in this space, we can specify its three co-ordinates (*x*,*y*,*z*). These co-ordinates represent how far the point is from the origin in the *x*, *y*, and *z* directions. If we draw an arrow from the origin to the point, as shown in Figure 2.1b, we can define this arrow as the *displacement vector* that locates the point; the coordinates (*x*,*y*,*z*) are called the components of the displacement vector in this system. With this definition of the displacement vector, it is natural to define the components of the *velocity vector* and the *acceleration vector* similarly.

$$v_x \equiv \frac{dx}{dt} \qquad v_y \equiv \frac{dy}{dt} \qquad v_z \equiv \frac{dz}{dt}$$

$$a_x \equiv \frac{dv_x}{dt} \qquad a_y \equiv \frac{dv_y}{dt} \qquad a_z \equiv \frac{dv_z}{dt}$$

With these definitions, we can see that everything we did last time for one dimension (x) is just repeated for the other two dimensions (y and z). For example, we can immediately write down the equations for all components for motion at constant acceleration.

$$v_x = v_{ox} + a_x t \qquad v_y = v_{oy} + a_y t \qquad v_z = v_{oz} + a_z t$$

$$x = x_o + v_{ox} t + \tfrac{1}{2} a_x t^2 \qquad y = y_o + v_{oy} t + \tfrac{1}{2} a_y t^2 \qquad z = z_o + v_{oz} t + \tfrac{1}{2} a_z t^2$$

We can formalize this generalization from one dimension to three dimensions by defining these kinematic quantities, displacement, velocity and acceleration as *vector quantities*. For example, we can write down a single equation for the velocity vector as a function of time for the special case of constant acceleration.

$$\vec{v} = \vec{v}_o + \vec{a} t$$

The "arrow" notation is apt here since it indicates that, like an arrow, a vector has both a *length* and a *direction*. The length of a vector is also called its magnitude and is often represented as the absolute value of the vector. This single *vector equation* is equivalent to the three scalar equations we wrote down earlier..

$$v_x = v_{ox} + a_x t \qquad v_y = v_{oy} + a_y t \qquad v_z = v_{oz} + a_z t$$

We have introduced these vectors in terms of one representation, their Cartesian components. In fact, you should think of these *vectors* as the primary object. They can have several different scalar component representations. In the next section, we will support this claim by introducing some important properties of vectors that we will use often in this course.

C) Vectors

You know how to perform many operations on scalar quantities. For example, you know how to add, subtract, multiply and divide numbers. You also know how to differentiate and integrate scalar functions. We can define similar operations for vectors.

For example, Figure 2.2 shows the procedure for defining the sum of two vectors ***A*** and ***B***. Namely, ths sum is defined to be another vector ***C*** which is obtained from ***A*** and ***B*** using the following prescription: place the tail of vector ***B*** at the head of vector ***A*** and then draw the arrow from the tail of vector ***A*** to the head of vector ***B***. Note that the vector sum depends on the directions of the vectors as well as their magnitudes. For example, if you were to rotate vector ***B*** through some angle, its magnitude would not change, but both the direction and the magnitude of the vector sum ***C*** will change! Clearly the magnitude of

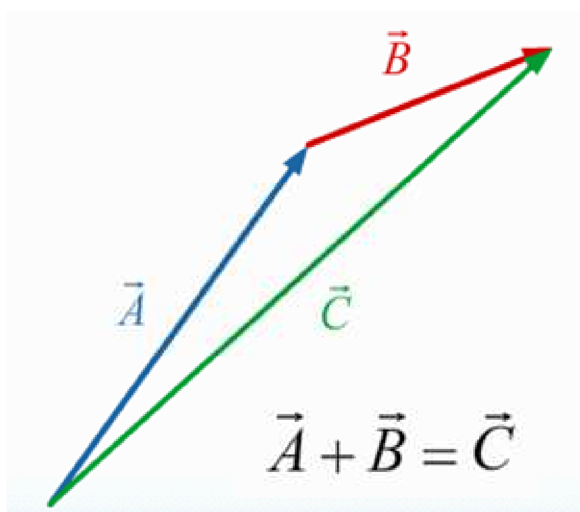

Figure 2.2
The sum of two vectors ***A*** and ***B*** is defined to be another vector ***C*** formed by placing the tail of ***B*** at the head of ***A*** and drawing a vector from the tail of ***A*** to the head of ***B***.

the vector sum ***C*** is ***not*** equal to the sum of the magnitudes of vectors ***A*** and ***B***.

You must be thinking that this is a pretty strange prescription to be given the name of something simple like addition. This prescription becomes more clear if we look at the Cartesian components of the vectors as shown in Figure 2.3.

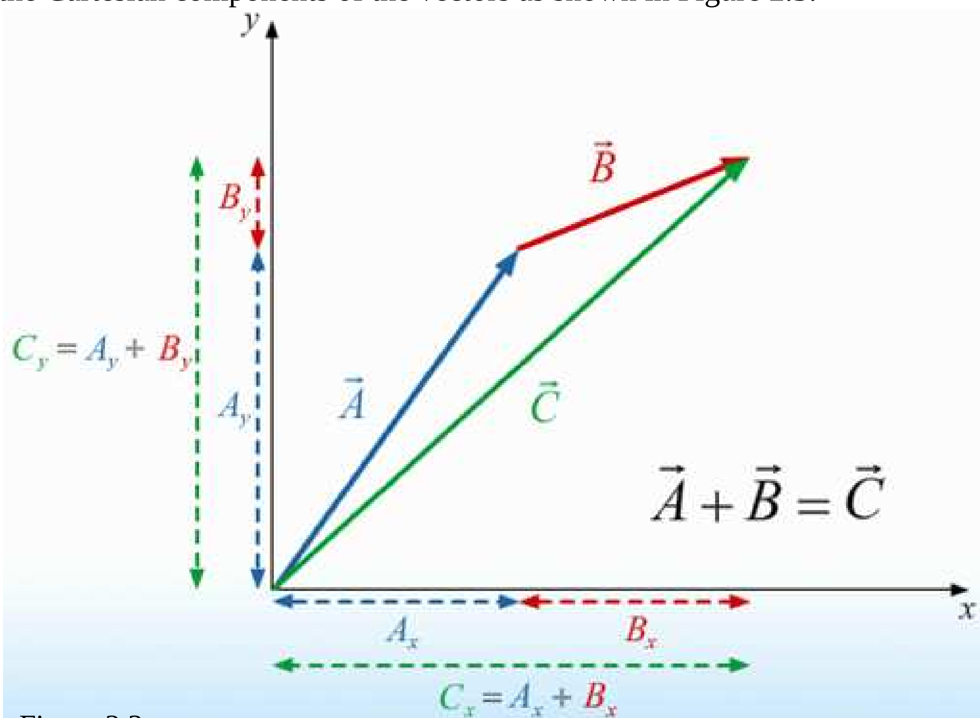

Figure 2.3
The components of the vector sum ***C*** are equal to the sum of the components of the vectors ***A*** and ***B***.

Aha, there is a method to this madness! It's clear that this definition of vector addition gives the result that the Cartesian components simply add!

$$C_x = A_x + B_x \qquad\qquad C_y = A_y + B_y$$

With this definition of vector addition, we see we can also write a general expression of any vector ***A*** in terms of its Cartesian components and the unit vectors $(\hat{i}, \hat{j}, \hat{k})$ in the (x, y, z) directions, as shown in Figure 2.4. Here we have used the fact that multiplying a vector by a scalar is the same as multiplying each of its components by the same scalar, which simply changes its length. Multiplying a vector by a negative scalar reverses its direction.

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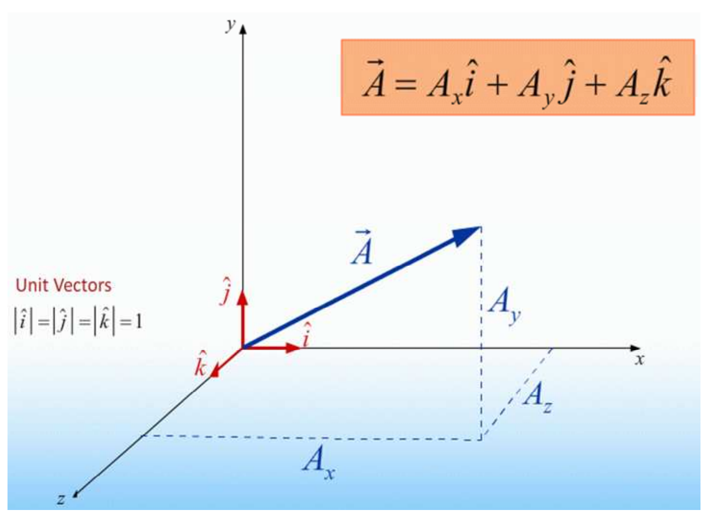


Figure 2.4
The vector ***A*** represented as the vector sum of the product of its components and the corresponding unit vector.

The Cartesian component representation of a vector is a common representation, but certainly not the only one. For example, a vector can also be specified in spherical components, in which the length of the vector and the angles describing its orientation are used to specify the vector as in Figure 2.5a. In two dimensions, the orientation of a

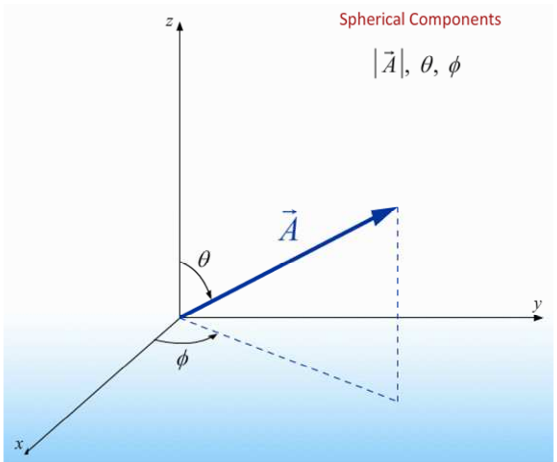


Figure 2.5a
The representation of a 3-dimensional vector ***A*** in spherical coordinates

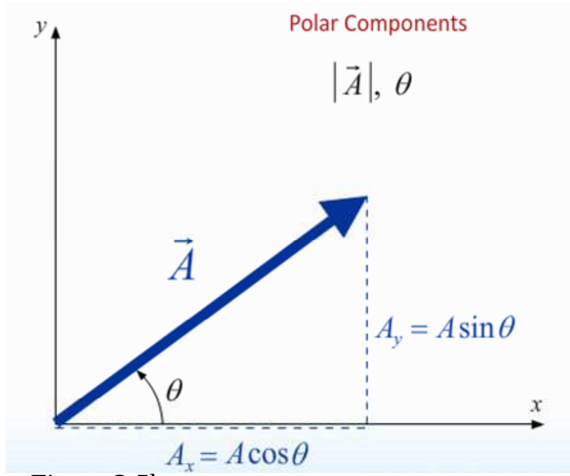


Figure 2.5b
The representation of a 2-dimensional vector ***A*** in polar coordinates

vector can be specified by the angle θ it makes with the x-axis as shown in Figure 2.5b.. Using trigonometry, we can determine completely the relation between the Cartesian components (A_x, A_y) and the polar components (A, θ).

In all cases, though, you should think of the vector itself as an object – the arrow. The different coordinate systems we invent are just different ways of describing this object in terms of scalar quantities.

D) Example: Free Fall (gravity)

We will now return to do some physics by considering an example of motion in three dimensions with constant acceleration, namely the throwing of a ball across a room. Once the ball leaves our hand the only force acting on it is gravity. We will learn more about gravity in a few units – for now the only thing we need to know is that, near the surface of the earth, any object under the influence of just gravity (*i.e.*, in free fall) will experience the same downward acceleration: of 9.8 m/s^2. It is customary to refer to the magnitude of the acceleration of gravity as g.

Figure 2.6 shows the familiar parabolic trajectory followed by the ball once it is in the air.

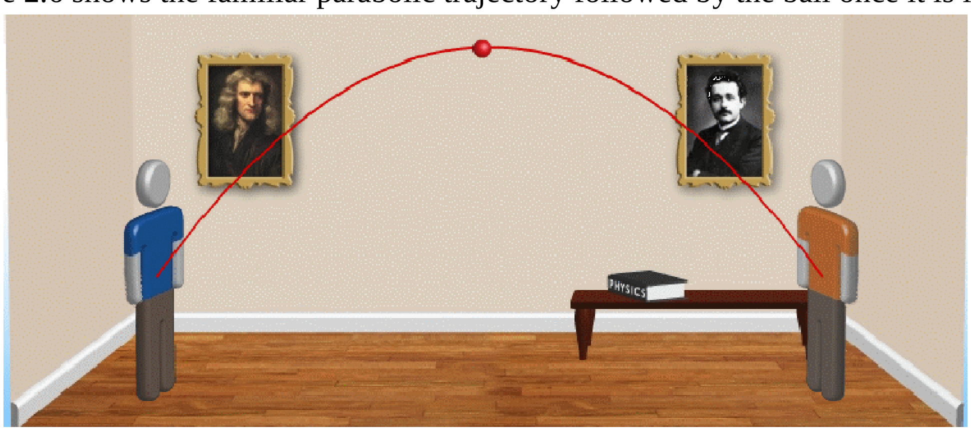


Figure 2.6
The parabolic trajectory followed by a thrown ball

Before attempting to describe this motion using our new 3D kinematics equations we need to define our coordinate system. It is customary to pick the *y* axis to point vertically upward and the *x* axis to point horizontally in the direction of the throw. With this choice, our kinematics equations simplify considerably as shown in Figure 2.7. Since the

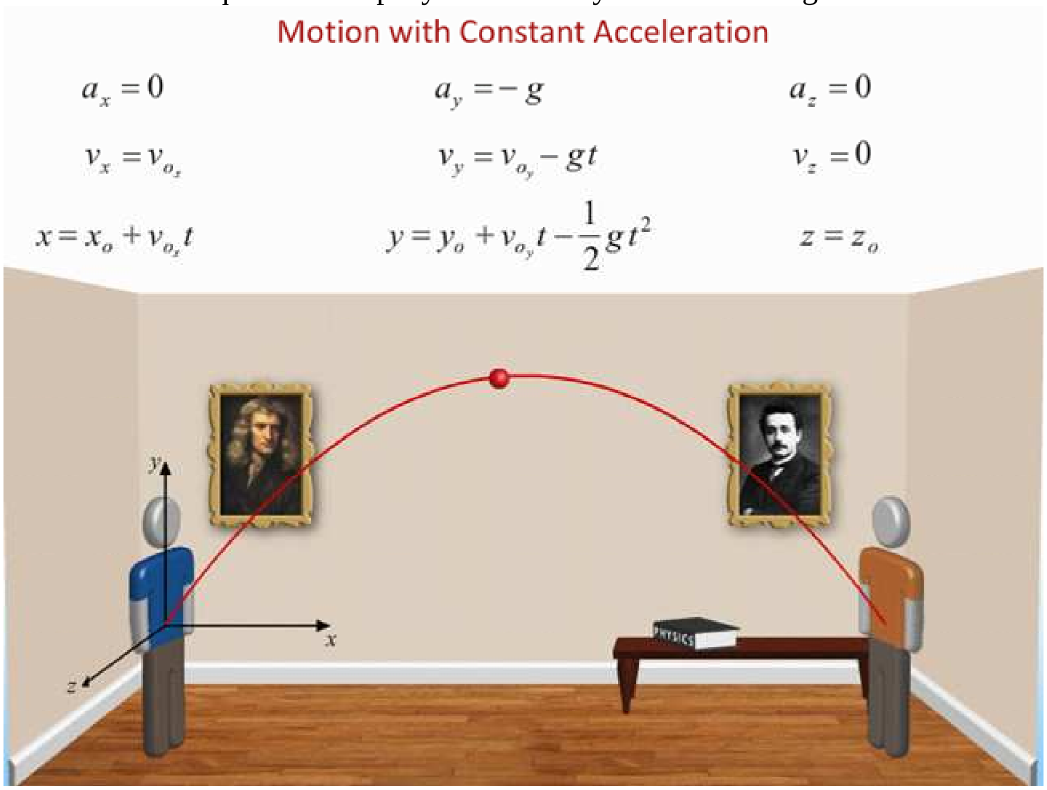


Figure 2.7
The kinematic equations for a ball thrown across the room.

acceleration is only in the –*y* direction, a_x and a_z are zero. Therefore, the velocities in the *x* and *z* directions cannot change; the motion in these directions is just motion at constant velocity. Since we chose v_{0z} to be zero, we have no motion along the z direction at all. The motion of the ball will be restricted to the *x*-*y* plane; we have reduced the problem to a two-dimensional problem. Indeed, we might as well choose the initial *z* position (z_0) to be zero which results in these simplified equations.

$$a_x = 0 \qquad v_x = v_{ox} \qquad x = x_o + v_{ox} t$$

$$a_y = -g \qquad v_y = v_{oy} - gt \qquad y = y_o + v_{oy} t - \tfrac{1}{2} g t^2$$

Before we use these equations in a calculation, let's first make a few observations. First, note that there are minus signs in all equations where *g* appears. The explanation is simply that the acceleration due to gravity is downward, in the –*y* direction, and we always take the value of *g* itself to be positive since *g* represents the magnitude of the acceleration, which we know to be 9.8 m/s^2. Second, note that the *x* component of the velocity is constant – it never changes from is initial value. Last, but not least, note again that the equations for the *x* and *y* components of the motion are totally independent – neither one cares about the other. We will see soon that this independence has important implications.

E) Example: Soccer Ball Kick

We will now use our knowledge of the motion of an object in free fall near the surface of the earth to make a calculation. Suppose you kick a ball off the ground at an angle θ with an initial speed v_0. How far away will it land?

The equations we developed in the last section already reflect the fact that we have chosen the *y* axis to be up and the *x* axis to be in the direction of the kick. To make things even simpler, lets kick the ball at t=0 and choose the origin of our coordinate system to be at the initial position of the ball so that $x_0 = y_0 = 0$. We now have equations for all three quantities that change as a function of time (x, y, *and* v_y).

$$x = v_{ox}t \qquad y = v_{oy}t - \tfrac{1}{2}gt^2 \qquad v_y = v_{oy} - gt$$

We want to determine the horizontal distance the ball travels before hitting the ground, call it D. Suppose the ball hits the ground at time $t = t_f$. The distance D then is really just the x position of the ball at time $t = t_f$. Therefore, we use the x-equation to tell us that D is just equal to the product of the time t_f and the v_{0x}, the x-component of the initial velocity.

$$x = v_{ox}t_f$$

Therefore to determine D, we must first determine t_f and v_{0x}. How do we do that?

We can certainly determine v_{0x} from trigonometry. Namely, all we have to do is to decompose the initial velocity vector into its Cartesian components as shown in Figure 2.8. Since these components form the sides of a right triangle whose hypotenuse is equal to v_0, the magnitude of the initial velocity, we see that:

$$v_{ox} = v_o \cos\theta \qquad v_{oy} = v_o \sin\theta$$

Since both v_0 and θ are given in the problem statement, we now know both v_{0x} and v_{0y} as well. The only remaining task is to determine t_f, the time the ball stays in the air. Since we have an equation for y as a function of t we can just solve this equation for the times at which $y = 0$.

$$0 = v_{oy}t_f - \tfrac{1}{2}gt_f^2$$

Since this is a quadratic equation, we will find two solutions:

$$t_f = 0 \qquad t_f = \frac{2v_{oy}}{g}$$

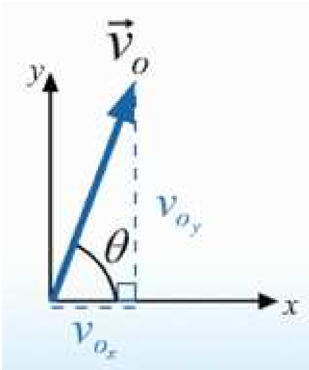


Figure 2.8
The decomposition of the initial velocity vector $\boldsymbol{v_o}$ into its components, v_{ox} and v_{oy}

These are the times at which the height of the ball was zero: one is when the ball was kicked ($t_f = 0$), and the other is when the ball landed ($t_f = 2v_{0y}$/g).

Finally, we just plug this last value for t_f in our equation to determine that the distance the ball travels in the air is proportional to the product of the x and y components of the initial velocity and inversely proportional to the acceleration due to gravity, g.

$$D = \frac{2v_{ox}v_{oy}}{g}$$

F) The Range Equation

When we solved the problem in the last section we found that the distance the ball travels was proportional to both v_{0x} and v_{0y}. It's always a good idea to check your results to see if they make sense. .

If we increase v_{0x} the ball moves further along the x direction in a given amount of time. If we increase v_{0y} the ball will be in the air longer and will travel further for any given velocity in the x direction. This all makes sense, the catch is that for a given initial speed v_0, both v_{0x} and v_{0y} depend on the angle at which the ball is kicked. Therefore, an interesting question is: for what angle is this distance D a maximum? How would we go about making this calculation? The first step is to determine a general expression for the distance D in terms of the angle θ. We can then examine this expression to determine the angle that maximizes D.

We'll start with the expression for D that we obtained in the last section and write both v_{0x} and v_{0y} in terms of v_0 and θ.

$$D = \frac{2v_o^2 \cos\theta \sin\theta}{g}$$

We see that D is proportional to the product of $\sin\theta$ and $\cos\theta$. We can simplify this expression a bit further by realizing the product of $\sin\theta$ and $\cos\theta$ is proportional the $\sin(2\theta)$ to obtain the usual form of the range equation.

$$D = \frac{v_o^2 \sin 2\theta}{g}$$

Figure 2.9a shows a plot of the range D as a function of θ. We see that D reaches its maximum value when $\theta = 45^{o}$. Figure 2.9b shows trajectories for different values of θ.

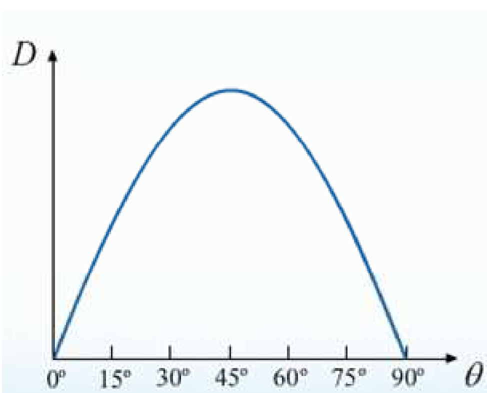


Figure 2.9a
A plot of the range D of a kicked soccer ball as a function of θ, thae angle the ball's initial velocity vector makes with the horizontal. .

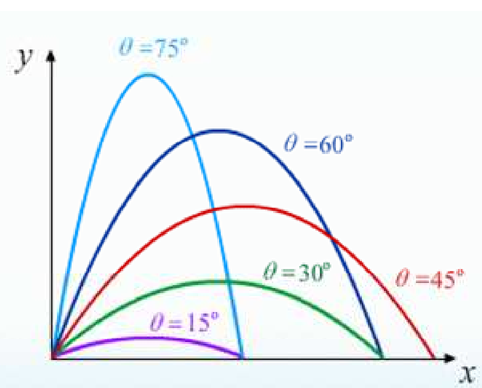


Figure 2.9b
Trajectories for different angles made by the ball's initial velocity vector and the horizontal.. .

We see that the range for complementary angles is the same and that the maximum range is indeed obtained when $\theta = 45^{o}$.

G) Superposition

We have already seen that the *x* and *y* equations of projectile motion are independent. In practical terms this means that the behavior of a projectile in the vertical direction is the same no-matter how fast it is moving in the *x* direction, and vice versa. A bullet shot horizontally out of a gun will take the same amount of time to hit the level ground as a bullet dropped from the same height.

Therefore we can consider the motion of the kicked ball to be the superposition of two simpler motions, the first being that of a ball moving vertically with constant acceleration, and the second being that of a ball moving horizontally with constant velocity. We can actually see this superposition if we simply consider a single motion as viewed from two different reference frames. For example,

If a man throws a ball vertically upward, we know the ball will go straight up and then straight down, all the time moving with constant acceleration of 9.8 m/s^2 pointed downward. Now suppose this man is sitting in a train while the train is moving with a constant speed past an observer at a station. What will he see? Well, if the train is really moving with constant speed, then the man on the train must see exactly what he saw before; the ball goes straight up and returns to his hand. The speed of the train makes no difference, as long as it's constant. What does the observer at the stations see? He can't really see the same thing. Figure 2.10 shows the trajectory of the ball as seen by the observer on the ground. He does not see the ball go simply straight up and down!

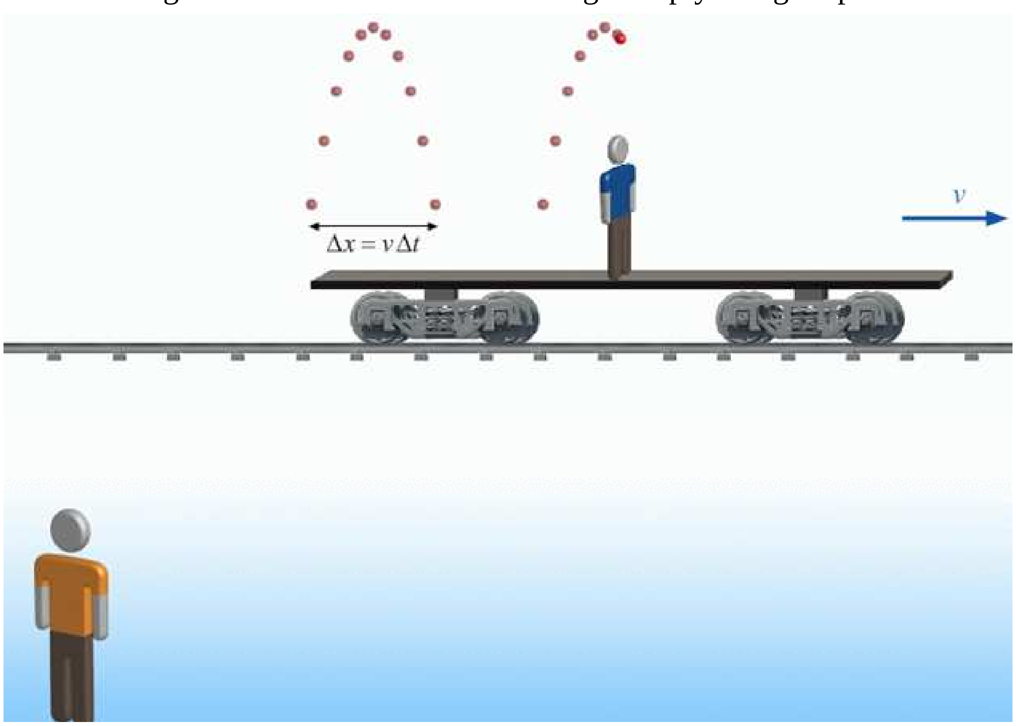


Figure 2.10
The trajectory of a ball thrown straight up by a man at rest on a train .moving at constant velocity *v* with respect to an observer on the ground. In the time Δt it takes the ball to go stright up and down with respect to the man on the train, the train has travelled a distance $\Delta x = v\Delta t$. The ball is always diretly above the man on the train and therefore appears to have the trajectory shown to the observer n the ground.

He sees the train moving so that the horizontal positions of the ball when it leaves the man's hand and when it returns are separated by the distance the train travels during the flight time of the ball. In fact, what he sees is the combined motions of constant acceleration in the vertical direction and constant velocity in the horizontal direction. What he sees is exactly the trajectory a soccer ball would have if it were kicked with an initial velocity such that its vertical component were the initial velocity of the ball with respect to the man sitting on the train and its horizontal component were the velocity of the train with respect to the observer at the station.

The amazing conclusion we take away from this analysis is that projectile motion can be explained as simply free fall as viewed from a moving reference frame!

For example, in this case, we can predict what the man at the station will see by combining the information of what the man on the train sees with the known motion of the train.   In particular, we can write a vector equation that relates the velocity of the ball as measured by the observer at the station to the velocity of the ball as measured by the man on the train.

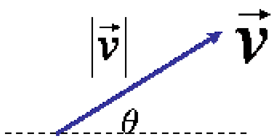
$$\vec{v}_{ball-wrt-ground} = \vec{v}_{ball-wrt-train} + \vec{v}_{train-wrt-ground}$$

This vector equation relates observations in two different reference frames that are moving relative to each other, and will be the topic of our next unit.

Main Points

- **Kinematic Quantities as Vector Quantities**

Vectors have magnitude and direction

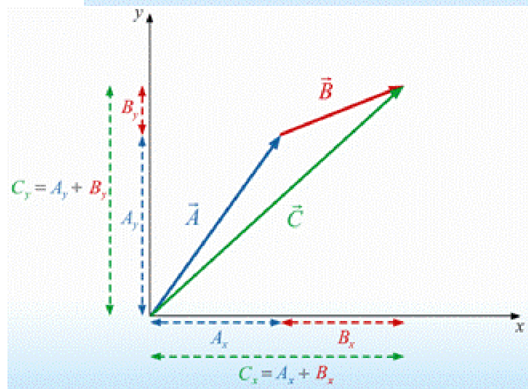


Displacement	$\vec{x}$
Velocity	$\vec{v} \equiv \dfrac{d\vec{x}}{dt}$
Acceleration	$\vec{a} \equiv \dfrac{d\vec{v}}{dt}$

- **Vector Addition**

The vector sum of two vectors A and B is another vector C

$$C_x = A_x + B_x$$
$$C_y = A_y + B_y$$



- **Special Case: Projectile Motion**

Projectile motion is the superposition of two independent motions:

a) Horizontal: constant velocity

b) Vertical: constant acceleration

Projectile motion can be understood simply as freefall viewed from a moving reference frame