10. Center of Mass

A) Overview

This unit expands our study of mechanics from single particles to systems of particles. We will introduce the very important concept of the center of mass of a system of particles and determine the center of mass for both discrete and continuous mass distributions. We will use Newton's second law to obtain the equations of motion for the center of mass of a system of particles. We will also obtain a version of the work-kinetic energy theorem, called the *center of mass equation*, that can be applied to a system of particles,

B) Systems of Particles and the Center of Mass

So far we have only considered the motion of simple objects. We have intentionally not considered the motion of, for example, an object composed of two different sized balls connected to the ends of a rod   In the next few units we will develop the tools to understand the motion of more complicated systems of objects such as these. We will discover that their behavior can be understood by applying what we already know, and we will see that the equations describing their motion are remarkably similar to those we have already developed.

We will start by introducing a new concept which will play a key role in what follows, namely that of the *center of mass*. Quite simply put, the *center of mass* of an object is just the average location of the mass that makes up the object. For a simple symmetric object like a ball or box of uniform density we will see that the center of mass is just at the center of the object. For less simple shapes we will have to perform a calculation to determine the location of the center of mass.

The procedure we will adopt for finding the average position of the all of the mass contained in some system of objects will be to simply take a mass-weighted average of the positions of the individual parts.   Namely, we will define the location of the center of mass of a system of particles to be equal to the sum of the positions of the individual particles with each one weighted by its own fraction of the total mass of the system as shown for a system of discrete masses in Figure 10.1.

In the next section we will determine the center of mass for a two particle system. This concrete example will illustrate the general procedure and hopefully make clear why this definition makes sense.

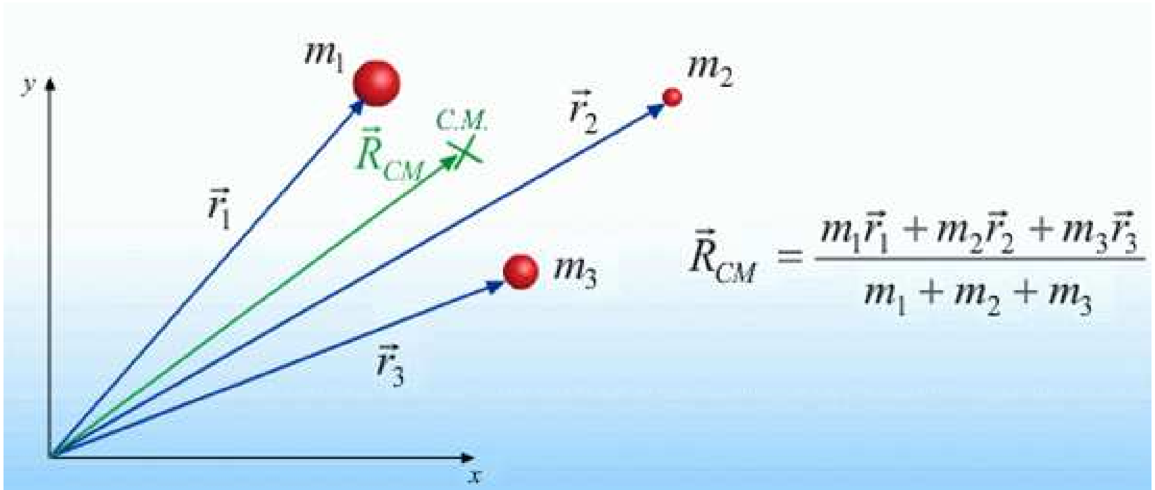


Figure 10.1
The definition of the center of mass for a system of three discrete masses.

C) Center of Mass for a Two-Body System

We'll start by considering an object made of only two point particles, labeled 1 and 2. We will assume that we know the masses of the two particles as well as their locations along the *x* axis as shown in Figure 10.2.

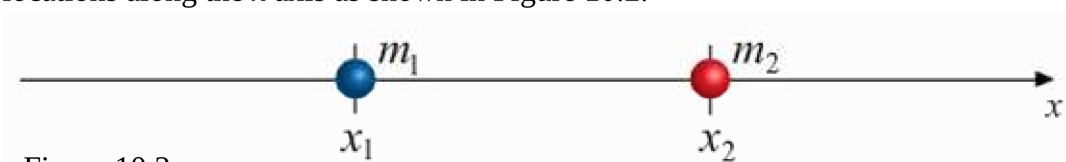


Figure 10.2
Two masses located along the x-axis.

Since the particles lie along the *x*-axis, the calculation of the position of their center of mass is straightforward.

$$\vec{R}_{CM} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \quad \Rightarrow \quad x_{CM} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

If the masses are equal, we see that the center of mass is located at the average value of x_1 and x_2. This position is halfway between them, which certainly makes sense. If, on the other hand, one mass is twice as big as the other, it will count for twice as much in the average, which means the center of mass will be closer to the heavier particle than the lighter one, which also seems reasonable.

So far we have considered just the one-dimensional case in which both particles lie along the *x* axis. This procedure can be easily extended to more than one dimension, though, using vector addition. We start by writing the expression for the location of the center of mass in terms of the masses of the two particles and the vectors that locate each of the two particles.

$$\vec{R}_{CM} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

We can rewrite this formula so that the vector locating the center of mass is equal to the sum of two vectors, the displacement vector of one particle and another vector that is proportional to the difference in the displacement vectors of the two particles.

$$\vec{R}_{CM} \equiv \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} = \vec{r}_1 + \frac{m_2}{m_1 + m_2}(\vec{r}_2 - \vec{r}_1)$$

We can think of this equation as a map that tells us how to get to the center of mass: We first go to one of the objects and then we go a fraction of the way to the other object, where this fraction is determined by the masses, as shown in Figure 10.3. If the second

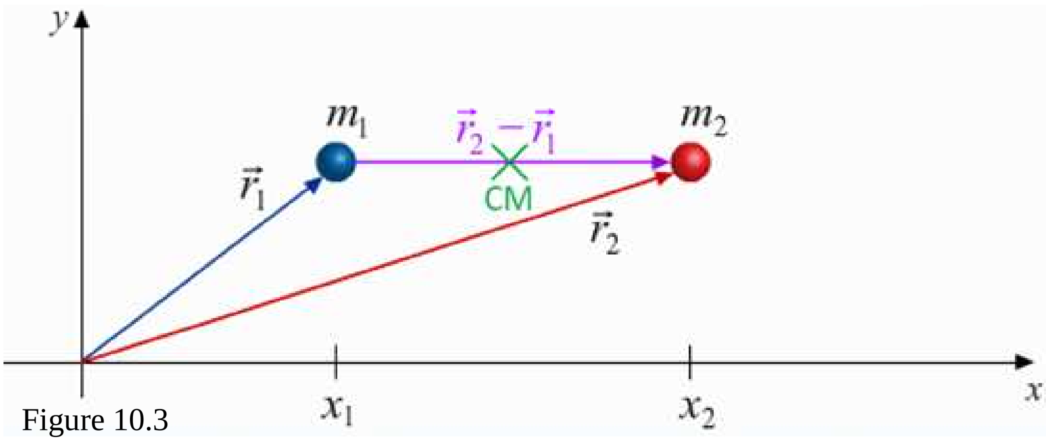


Figure 10.3
The location of the center of mass of a two body system is located along the vector difference ($\boldsymbol{r_2} - \boldsymbol{r_1}$) of the two displacements..

object has the same mass as the first, we go half way. If the second object is heavier than the first, we go more than half way, and if the second is lighter than the first we go less than halfway.

The beauty of this approach is that we can see that the location of the center of mass *does not* depend on our choice of the origin or the orientation of our coordinate system; the center of mass always lies at the same fixed point along the line connecting the two objects. We have just demonstrated an important result, namely, that the center of mass is a property of the system itself; it does not depend on the way we choose to look at the system. Indeed, we will show in a later unit that the center of mass of a rigid system is the same as its balance point!

D) Center of Mass for Systems of More than Two Particles

We can extend our definition of the center of mass for systems containing more than two particles by simply summing up the mass-weighted displacement vectors for each particle.

$$\vec{R}_{CM} \equiv \frac{1}{M_{total}}\sum_i m_i\vec{r}_i$$

It is usually easier to break this vector equation into components and evaluate each component separately. Just to make sure we know how this works, let's do an example involving eight equal mass particles located on the corners of a cube, as shown in Figure 10.4.

To find the x-coordinate of the center of mass we need to sum the x-coordinates of each of the eight particles weighted by the ratio of the mass of each particle to the total mass of the system. In this example the particles all have the same mass and their x-coordinates are either zero or L, so that the sum is easy to evaluate, and we find that the x-coordinate of the center of mass is just equal to ½ L.

$$X_{CM} \equiv \frac{1}{M_{ttoal}} \sum_i m_i x_i = \frac{1}{8m} 4mL = \frac{L}{2}$$

We will get the same results for both the y and z coordinates, so that we see that the center of mass is at the center of the box, as expected.

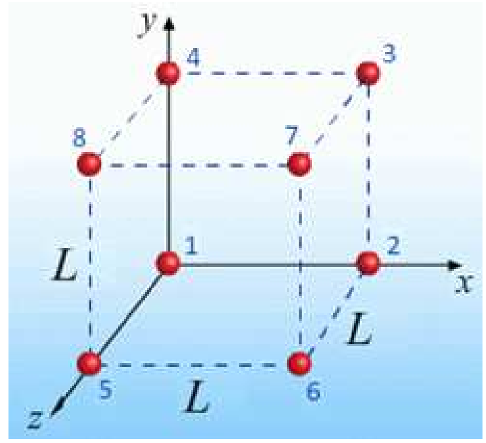


Figure 10.4
A system of eight equal mass particles located at the corners of a cube of side L.

Now suppose the cube was a solid, made up of millions of atoms rather than just eight particles on the corners. We suspect the answer would be the same, that the center of mass is still in the middle, but how can we prove this conjecture when actually performing the sum over millions of atoms seems difficult, if not impossible? Once again calculus comes to the rescue!

Rather than calculating the product of position and mass for individual particles, we just integrate the position vector over all infinitesimal mass elements dm contained in the cube. We will do this calculation in the next section.

E) Center of Mass for Continuous Mass Distribution

Before we actually evaluate any integral, let's make sure we understand exactly how we adjust our definition of the center of mass when we are dealing with a continuous mass distribution. The big idea is that we have to replace a discrete sum by a continuous sum, an integral. In the discrete sum we evaluated the product of the mass of each part of the system and its position and then added them up. In the continuous sum we are doing the same thing, the only difference now is that are dividing a continuous object up into an infinite number of tiny volume elements each having a mass dm.

$$\vec{R}_{CM} \equiv \frac{1}{M_{total}} \int \vec{r}\, dm$$

Since here we are integrating over the volume of a cube, a 3 dimensional object, the integral itself must be evaluated in all three dimensions, x, y and z. To evaluate the mass element dm, we take the product of the volume of the element and the mass density (the mass per unit volume ρ) of the cube.

$$dm = \rho dV = \rho dxdydz$$

Our job now is to evaluate this triple integral. The first step is to break our equation for the center of mass vector $\boldsymbol{R}_{cm}$ into x, y and z components. We will start with the x-equation and first determine the limits of integration. In each direction, the cube is located between the origin and a distance L from the origin. If we assume the mass density ρ is a constant, then it can be taken outside of the integral.

$$X_{CM} = \frac{1}{M_{total}} \rho \int_0^L x dx \int_0^L dy \int_0^L dz$$

The resulting three-dimensional integral is equal to the product of three one-dimensional integrals, each of which is evaluated separately. Requiring the product of the mass density and the volume of the cube to be equal to the total mass of the cube, we obtain the expected result, that the c-coordinate of the center of mass of the cube is just equal to ½ L.

$$X_{CM} = \frac{1}{M_{total}} \rho \left(\tfrac{1}{2} L^2\right)(L)(L) = \frac{\rho L^3}{M_{total}} \left(\tfrac{1}{2} L\right) = \tfrac{1}{2} L$$

The y and z component calculations are absolutely identical to the x-component calculation, giving us the expected result, that the center of mass of the box is at its center!

F) Center of Mass of a System of Objects

We now know how to find the center of mass of a collection of point particles as well as that of a continuous solid object. What happens when we want to find the center of mass of a collection of solid objects? Figure 10.5 shows two objects, labeled *a* and *b*. By definition, the center of mass of the system is found by integrating the position vector over all of the mass in the system. Since the system is made of two objects, the total integral is just the sum of two separate integrals, one for each object.

$$\vec{R}_{CM} = \frac{1}{M_{total}} \left(\int_a \vec{r} dm + \int_b \vec{r} dm \right)$$

If we multiply and divide each of these integrals by the mass of the object, we certainly haven't changed anything, but we can now see that the numerator just becomes the mass of each object times the position of its center of mass.

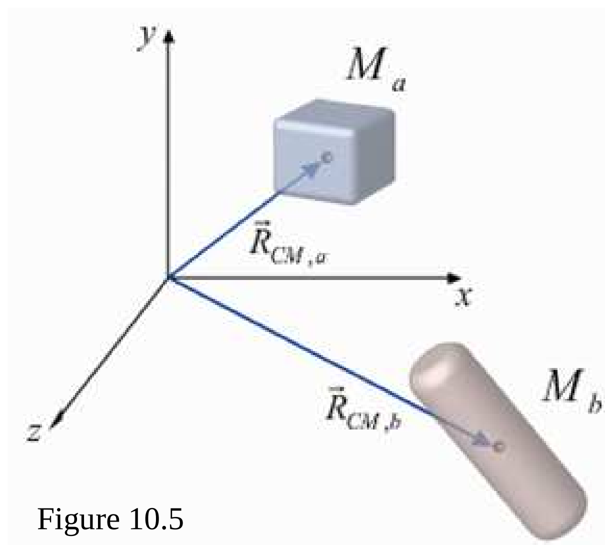


Figure 10.5
The center of mass of the two objects can be calculated simply by treating each object as a point particle having a mass equal to the total mass of the object

$$\vec{R}_{CM} = \frac{1}{M_{total}} \left(M_a \left(\int_a \frac{\vec{r} dm}{M_a} \right) + M_b \left(\int_b \frac{\vec{r} dm}{M_b} \right) \right) = \frac{1}{M_{total}} \left(M_a \vec{R}_{CM,a} + M_b \vec{R}_{CM,b} \right)$$

Therefore, we have just arrived at a simple procedure for finding the center of mass of a system of solid objects. Namely, we just treat each object as a point particle with all of its mass located at its center of mass! That’s all there is to it!

G) Dynamics of the Center of Mass

To this point we have defined the concept of the center of mass and have shown how to find it for any system of objects. With this knowledge in hand, we can finally do some physics.

We will start with our definition for the center of mass of a system of objects and take the derivative of this expression with respect to time.

$$\frac{d\vec{R}_{CM}}{dt} = \frac{1}{M_{total}} \sum_i m_i \frac{d\vec{r}_i}{dt}$$

The left hand side of the equation becomes the velocity of the center of mass, and the numerator on the right hand side becomes the sum of the mass times velocity for each object in the system. We have already defined the product of the mass and velocity of an object as its momentum. Therefore, the numerator on the right hand side is just equal to the total momentum of the system.

$$\vec{V}_{CM} = \frac{1}{M_{total}} \sum_i m_i \vec{v}_i$$

We will now take another derivative with respect to time. The left hand side of the equation becomes the acceleration of the center of mass, and the numerator on the right hand side becomes the sum of the mass times acceleration for each object in the system.

$$\frac{d\vec{V}_{CM}}{dt} = \frac{1}{M_{total}} \sum_i m_i \frac{d\vec{v}_i}{dt}$$

The product of the mass and acceleration of an object is just equal to the total force on that object.

$$\vec{A}_{CM} = \frac{1}{M_{total}} \sum_i m_i \vec{a}_i = \frac{\sum_i \vec{F}_{net,i}}{M_{total}}$$

Now this sum of the total forces acting on all the objects in the system could get unwieldy if the number of objects gets large. The really good news, though, is that we can simplify this sum significantly by realizing that any forces that act between two objects that are both in the system will *cancel* in this sum Newton's third law requires that all such forces always come in pairs of equal magnitude and opposite direction!

$$\sum_i \vec{F}_{net,i} = \sum_i \vec{F}_{external,i} + \sum_{i \neq j} F_{ij} = \sum_i \vec{F}_{external,i}$$

Hence the sum of all forces acting on all objects in the system just reduces to the sum of all forces acting on the system from the outside, or the total external force.

$$\vec{A}_{CM} = \frac{\sum_i \vec{F}_{external,i}}{M_{total}} = \frac{\vec{F}_{Net,External}}{M_{total}}$$

We have finally arrived at something that looks exactly like Newton's second law, but rather than applying just to point particles as before, this expression relates the total external force on the whole system to the acceleration of the center of mass of the system! Consequently, we can say that no matter how complicated a system of objects may be, the center of mass of the system behaves in the same simple way that a point particle does.

Indeed, in the next section, we will use this equation to obtain a generalization of the work-kinetic energy theorem for systems of particles.

H) Center of Mass Equation

In section *E* of unit 7, we integrated Newton's second law to obtain the work-kinetic energy theorem for point particles: namely, that the change in kinetic energy of a particle is equal to the work done on that particle by the net force.

$$\Delta K = W_{net}$$

We will now extend this result to systems of particles. We start from our result from the last section: that the acceleration of the center of mass of a system of particles is equal to the total external force acting on a system of particles divided by the total mass of the system.

$$\vec{A}_{CM} = \frac{\sum_i \vec{F}_{external,i}}{M_{total}} = \frac{\vec{F}_{Net,External}}{M_{total}}$$

This equation looks exactly like Newton's second law for a point particle that has the total mass of the system (M_{total}) and is located at the center of mass of the system. Consequently, we can perform exactly the same derivation we made in unit 7 to obtain the work-kinetic energy theorem for a point particle. The result here is an equation, often called the center of mass equation, that looks exactly like the work-kinetic energy theorem we derived for point particles.

$$\Delta\left(\frac{1}{2}M_{total}V_{CM}^2\right)\vec{A}_{CM} = \int \vec{F}_{Net,External} \bullet d\vec{\ell}_{CM}$$

The only differences lie in the subscripts. These subscripts *are important*, however. The change in kinetic energy term is calculated as if the system were a particle of mass M_{total} and were moving with the velocity of the center of mass. We define the right hand side of the equation as the ***"macroscopic work done by the net force"***. This macroscopic work is calculated as if all forces were acting on this particle located at the center of mass.

We can now see why we did not need to worry about the microscopic work done by kinetic friction when we were calculating the motion of the box skidding to a stop in the last unit. If we consider the box to be a system of particles, we see that the change in kinetic energy of the box is exactly equal to the macroscopic work done by the net force, the kinetic friction force. The microscopic work done by the kinetic friction force at the interface of the surfaces of the box and the floor determines the additional thermal energy in the box and the floor, but does *not* determine the motion of the *center of mass* of the box.

I) Example: the Astronaut and the Wrench

We will end with a simple example to illustrate some of the concepts we have developed in this unit. Imagine you are an astronaut far out in space. You have just finished fixing a space telescope using a big wrench whose mass is one tenth as big as yours. You realize you have no way to get back to your spaceship which is 20 meters away from you, so you throw the wrench as hard as you can in a direction away from the spaceship which causes you to move in the opposite direction, toward the spaceship. When you finally reach the space ship, how far away are you from the wrench?

The key concept needed to answer this question is that the acceleration of the center of mass of a system will be ***zero*** if the external force on the system is zero. In this case, we define the system to be you and the wrench, and the center of mass of the system is initially at rest a distance of 20 meters from your spaceship. Since there are no external forces acting on the system and the center of mass is initially at rest, the location of the center of mass of the system can never change! If we choose the initial location of the center of mass to be at $x = 0$, the center of mass will always be at $x = 0$.

The location of the center of mass of the system is determined from its definition.

$$X_{CM} = \frac{M_{astronaut}x_{astronaut} + M_{wrench}x_{wrench}}{M_{astronaut} + M_{wrench}} = 0$$

Multiplying both sides by the total mass, we obtain our result that the product of the position and mass of the wrench is always equal to minus the product of your position and mass.

$$M_{wrench}x_{wrench} = -M_{astronaut}x_{astronaut}$$

Since the mass of the wrench is 1/10 of your mass, the wrench will always be ten times as far away from the center of mass as you are, and it will always be on the opposite side of the center of mass from you.

$$x_{wrench} = -\frac{M_{astronaut}}{M_{wrench}}x_{astronaut}$$

When you are at the spaceship, 20 meters to the left of the center of mass, the wrench will be 200 meters to the right of the center of mass, which is 220 meters from the spaceship

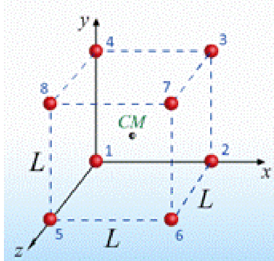
Main Points

- **Definition of Center of Mass**

The center of mass of a system of objects is defined to be the mass-weighted average of its components.

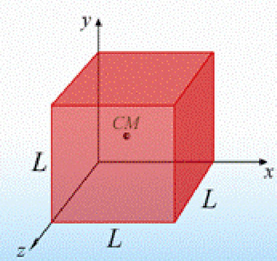
(Discrete Particles)

$$\vec{R}_{CM} = \frac{1}{M_{Total}} \sum_{i=1}^{N} m_i \vec{r}_i$$



(Continuous Mass Distribution)

$$\vec{R}_{CM} = \frac{1}{M_{Total}} \int \vec{r}\, dm$$



- **Equation of Motion for Center of Mass**

Applying Newton's second law to a system of particles, we obtain the equation of motion for the center of mass.

$$\vec{F}_{Net,External} = M_{total}\vec{A}_{CM}$$

- **The Center of Mass Equation**

Integrating the equationof motion for the center of mass, we obtainthe "center of mass equation" that relates the change in the kinetic energy of the center of mass (calculated as if the system were a particle having the total mass of the system and moving with the velocity of the center of mass) to the "macroscopic work" done by the total external force (calculated as if all forces were acting at the center of mass).

$$\Delta\left(\frac{1}{2} M V_{CM}^2\right) = \int \vec{F}_{Net,External} \cdot d\vec{l}_{CM}$$