11. Conservation of Momentum

A) Overview

This unit introduces the important concept of the conservation of momentum. Namely, the total momentum of a system of particles will be conserved whenever the sum of the external forces acting on the system is zero. We will apply this conservation law to collisions of particles and investigate sources of energy loss in these collisions. We will also introduce a special reference frame, associated with a system of particles, called the center of mass frame, in which the total momentum of all the particles in the system is zero. The description of collisions is often simple in this frame.

B) Momentum Conservation

In the last unit we introduced the concept of the center of mass of a system of particles as the mass-weighted average of their positions. Taking the derivative of this expression with respect to time, the left hand side of the equation becomes the velocity of the center of mass, and the numerator on the right hand side becomes the sum of the mass times velocity for each object in the system.

$$\frac{d\vec{R}_{CM}}{dt} = \frac{1}{M_{total}} \sum_i m_i \frac{d\vec{r}_i}{dt}$$

We have already defined the product of the mass and velocity of an object as its momentum. Therefore, the numerator on the right hand side is just equal to the total momentum of the system. Multiplying both sides of the equation by the total mass of the system, we obtain the result that the total momentum of the system is equal to the product of the total mass and the velocity of the center of mass.

$$\vec{P}_{total} = M_{total}\vec{V}_{CM}$$

Differentiating once more with respect to time, we see that the time rate of change of the total momentum of the system is equal to the product of the total mass of the system and the acceleration of the center of mass.

$$\frac{d\vec{P}_{total}}{dt} = M_{total}\vec{A}_{CM}$$

Now we showed in the last unit that the product of the total mass and the acceleration of the center of mass is just equal to the total external force applied to the system. Therefore, we see that the time rate of change of the total momentum of the system is just equal to the total external force applied to the system.

$$\frac{d\vec{P}_{total}}{dt} = \vec{F}_{Net,External}$$

This deceptively simple looking equation is extremely important and we will spend several units exploring its meaning. $\boldsymbol{P_{total}}$ is the total momentum vector of the system and is equal to the *vector sum* of the momenta of all of the parts of the system. Likewise, the total external force is the *vector sum* of all external forces acting on all parts of the system.

Note that when there are no external forces acting on the system, the time rate of change of the total momentum is zero. In other words, if the total external force is zero, then the total momentum of the system does not change in time. In this case, we say that the *momentum of the system is conserved*. We will now work out a couple of simple examples that illustrate momentum conservation.

C) Momentum Example: Astronaut and Wrench

We'll start by revisiting the problem we ended with in the last unit – that of an astronaut throwing a wrench. Since the astronaut and the wrench are both initially at rest, the initial momentum of the system is zero, and since there are no external forces acting on the system of the astronaut and the wrench, the total momentum of the system is conserved and will therefore *always* be *zero*. The total momentum of the system has two contributions – one from the astronaut and one from the wrench – and the vector sum of these is zero.

We see that in order for the total momentum to be zero, the astronaut must move in the opposite direction of the wrench with a speed fixed by the ratio of the masses.

$$\vec{P}_{astronaut} = -\vec{P}_{wrench} \quad \Rightarrow \quad \vec{v}_{astronaut} = -\frac{m_{wrench}}{m_{astronaut}}\vec{v}_{wrench}$$

For example, if the mass of the astronaut is ten times as big as the mass of the wrench, the speed of the wrench will be ten times the speed of the astronaut. This requirement ensures both that the magnitudes of the momentum of the wrench and the astronaut are the same, and also that the center of mass of the system does not move since the distance the wrench moves in any given time interval will be ten times that moved by the astronaut.

Let's now examine our momentum equation a bit more carefully. Suppose the total external force is zero in some direction but not in others – what can we say about the momentum of the system?

$$\frac{d\vec{P}_{total}}{dt} = \vec{F}_{Net,External}$$

Since our equation is a vector equation, we know that the only component of momentum which will be conserved will be the one that lies along the direction in which the total external force is zero.

D) Example: Inelastic Collision

We will now move on to consider an interesting class of problems that can be addressed using this conservation of momentum principle. Namely, we will look at collisions between particles. We'll start with the example shown in Figure 11.1. A box of mass m_1 slides with velocity v_1 along a horizontal frictionless floor and collides with a second box of mass m_2 which is initially at rest. After the collision the boxes stick together and move with a final velocity v_f. Our job is to determine this final velocity.

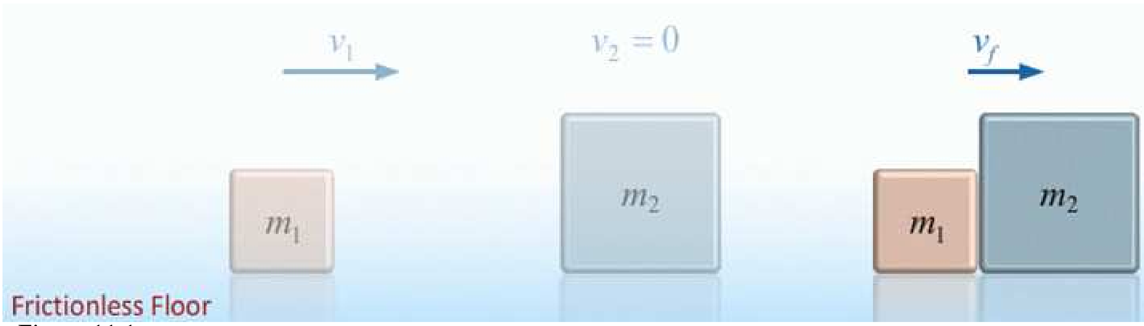


Figure 11.1
An inelastic collision: Box 1 moves with speed v_1 and collides with Box 2 that is initially at rest. The two boxes stick together and move off with speed v_f. The momentum of the system of two boxes is conserved in this collision which allows us to determine the final speed v_f.

In this problem the system we are interested in is made up of the two boxes. Since the floor is horizontal and frictionless, the total external force on the system in the horizontal direction is *zero*. Therefore the total momentum of the two box system is conserved. That is, this total momentum will be the same before and after the collision.

$$\vec{P}_{initial} = \vec{P}_{final}$$

At this point you might be wondering about the forces that will act between the boxes during the actual collision itself – won't these forces, which will definitely have components in the horizontal direction, change the momentum of the system? The answer is no – they will not – and the reason is simple: The forces between the boxes are not external forces; these forces are internal forces, being exerted by the boxes, the objects that make up the system. In other words, the force by box 1 on box 2 will definitely change the momentum of box 2 and the force by box 2 on box 1 will definitely change the momentum of box 1, but the total momentum of the two boxes will not change since these forces are equal and opposite by Newton's third law! For this reason we never actually have to worry about what happens during the instant when the boxes collide – we can just focus on the total momentum before and after

In this example the initial momentum of the system in the horizontal direction is due entirely to box 1. The final momentum of the system is due to *both* boxes. Since the initial and final momentum of the system has to be the same, we can solve for the final velocity of boxes 1 and 2 in terms of the initial velocity of box 1.

$$m_1 v_1 = (m_1 + m_2) v_f \quad \Rightarrow \quad v_f = \frac{m_1}{(m_1 + m_2)} v_1$$

E) Energy in Collisions

We see that momentum is conserved in this collision, but what happens to the total kinetic energy of the system? Is it conserved also? The answer to this question depends on what we call the kinetic energy of the system. Certainly, if there are no external forces acting on the system, then there is no macroscopic work done on the system and the kinetic energy of the system, defined as ½ the total mass times the square of the velocity of the center of mass cannot change either. However, this kinetic energy of the center of mass is *not* equal to the sum of the kinetic energies of the objects making up the system. We will demonstrate this claim now as we explicitly calculate the sum of the kinetic energies of boxes 1 and 2 before and after the collision.

Before the collision the sum of the kinetic energy of the boxes is just equal to the initial kinetic energy of box 1.

$$K_{initial} = \frac{1}{2} m_1 v_1^2$$

After the collision the sum of the kinetic energies of the boxes is equal to the final kinetic energy of the object composed of the two boxes stuck together.

$$K_{final} = \frac{1}{2}(m_1 + m_2) v_f^2$$

In the last section, we used the conservation of momentum to determine the final velocity of the boxes in terms of the initial velocity of box 1. Therefore, we can determine the final kinetic energy in terms of the initial kinetic energy.

$$K_{final} = \frac{1}{2}(m_1 + m_2)\left(\frac{m_1}{m_1 + m_2} v_1\right)^2 = K_{initial}\left(\frac{m_1}{m_1 + m_2}\right)$$

We see here that the final energy is smaller than the initial kinetic energy by exactly the same factor that related the final and initial velocities. In other words, the kinetic energy of the system, defined as the sum of the kinetic energies of the boxes, was *not* conserved in the collision. We call this kind of a collision "inelastic". In the next section, we will take a look at how this energy is lost.

F) Energy Loss in Collisions

We saw in the last section that the kinetic energy of the boxes after the collision was less than the kinetic energy of the boxes before the collision. How can we understand this loss of energy? Where did the energy go?

To understand this loss of energy, we need to look at the collision in more detail. Let's first focus our attention on box 1. We can define box 1 to be our system and apply the center of mass equation to determine that the change in the kinetic energy of the box is equal to the macroscopic work done on the box during the collision.

$$\Delta K_1 = \int \vec{F}_{21} \cdot d\ell_{1_{CM}}$$

What force is responsible for this work? Clearly, the force that box 2 exerts on box 1 during the collision must be responsible for this work. This work is done during the time of the collision and it may be hard to visualize since the idealized diagram we have drawn seems to suggest that the boxes themselves are not crushed or deformed during the collision.

To get a better feeling for what is going on, just consider what happens to two cars after a collision. The obvious deformation of the cars as a result of the collision shows where the energy was lost during the collision. This energy loss can be understood in terms of the work done during this collision by the force that one car exerts on the other times the distance that the front of the other car was deformed.

Returning to our example of the sliding boxes, we can see that if we actually wanted the boxes to stick together we would have to provide some mechanism for non-conservative work to be done during the collision. Perhaps we could put a bit of putty on the surface of one of the boxes that could be compressed during the collision. The details of the nature of the internal forces acting during the collision can influence the amount of energy lost in a collision, but as long as there are no external forces acting, then we can be sure that the total momentum of the system will be conserved!

G) Center of Mass Reference Frame

We will now return to the concept of the center of mass since we will find that it can play a useful role in collisions as well. We have already derived the important relationship between the total momentum of a system and the velocity of its center of mass.

$$\vec{P}_{total} = M_{total}\vec{V}_{CM}$$

If we know that the total momentum does not change in time, for example, then it must be true that the velocity of the center of mass also does not change in time!

Recall the example of the astronaut throwing the wrench. We determined that the velocity of the center of mass of the system (astronaut + wrench) was constant and, in fact, equal to zero. The total momentum of the system was zero implying that the momentum of the wrench was exactly equal and opposite to the momentum of the astronaut. The reference frame in which we presented this example is called the *center of mass reference frame*, since the velocity of the center of mass is zero in this frame.

What about the more general case when the center of mass is moving with some constant velocity? We already know how to compare measurements in different reference frames. We learned in unit 3 that if the velocity of an object is known in reference frame A, and reference frame A is moving relative to reference frame B with a constant velocity, then the velocity of the object in reference frame B, is just equal to the vector sum of these velocities.

$$\vec{v}_{O,B} = \vec{v}_{O,A} + \vec{v}_{A,B}$$

Therefore, once we determine the velocity of the center of mass in the given frame, we can always transform the problem to the center of mass frame, if doing so makes the problem easier to solve. We will do such an example in the next section.

H) Example: Center of Mass Reference Frame

Suppose an asteroid is moving with a constant velocity of 4 km/s in the $+x$ direction as observed by a spaceship. An explosive device inside the asteroid suddenly blows it into two chunks, one having twice the mass of the other as shown in Figure 11.2. In the reference frame of the asteroid the lighter chunk moves in the $+y$ direction with a speed of 6 km/s. What is the speed of the heavier chunk of the asteroid as measured by someone on the spaceship?

The total momentum is *always zero* in the center of mass reference frame. Now the center of mass frame for the two chunks is clearly the frame in which the asteroid was at rest before it exploded. Since the total momentum is zero in this frame, the momentum of the two chunks after the explosion must be equal and opposite. If the lighter chunk has a velocity of 6 km/s upward then the bigger chunk must be moving downward and must have half the speed of the lighter chunk since it has twice the mass.

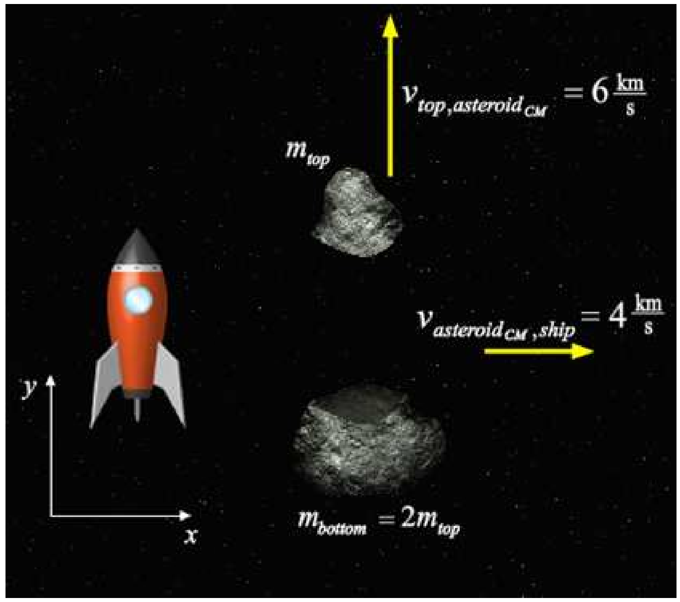


Figure 11.2
An asteroid moving in the x-direection suddenly explodes into two pieces. Conservation of momentum is most conveniently applied in the asteroid center of mass to determine the speed of the heavier chunk in the spacship frame.

The velocity of any object in the reference frame of the spaceship is equal to the velocity of that object in the center of mass reference frame plus the velocity of the center of mass in the reference frame of the ship.

$$\vec{v}_{O,ship} = \vec{v}_{O,CM} + \vec{v}_{CM,ship}$$

For the big chuck of asteroid, the velocity relative to the center of mass is 3 km/s in the –*y* direction and the velocity of the center of mass relative to the ship is 4 km/s in the +*x* direction. We can add these vectors using the Pythagorean theorem to find that the speed of the big chink is 5 km/s in the spaceship frame as shown in Figure 11.3.

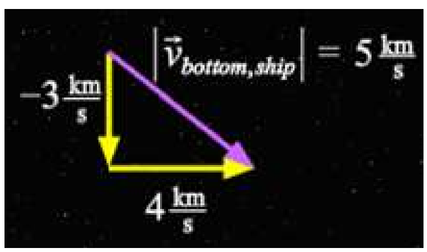


Figure 11.3
To find the velocity of the chunk with respect to the spaceship, we take the vector sum of the velocity of the chunk with respect to the asteroid CM and the veloicty of the asteroid CM with respect to the spaceship.

Main Points

- **Conservation of Momentum**

$$\frac{d\vec{P}_{Total}}{dt} = \vec{F}_{Net,External}$$

If the sum of the external forces acting on any system of particles is zero, then the total momentum of the system, defined as the vector sum of the momenta of the individual particles, is conserved.

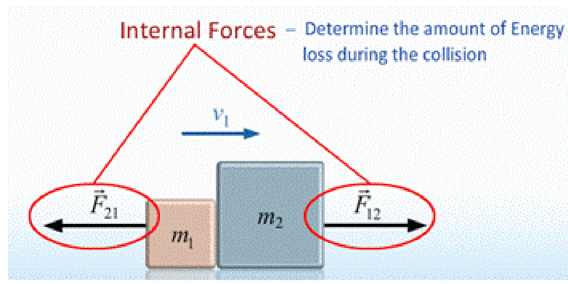
When $\vec{F}_{Net,External} = 0$

$$\vec{P}_{Total} \equiv \sum_i \vec{p}_i = \text{Constant}$$

- **Forces in a Collision**

Internal forces determine the amount of energy lost in a collision.

If only internal forces act during a collision, the total momentum of the system will be conserved



- **Center of Mass Reference Frame**

The Center of Mass Reference Frame is defined to be that frame in which the total momentum of all particles in the system is zero. In other words, it is the frame in which the center of mass of the system is at rest.

Conservation of momentum calculations are often simplified in the center of mass frame.

Center of Mass Reference Frame

$$\vec{P}_{Total} = M_{Total}\vec{V}_{CM} = 0$$