12. Elastic Collisions

A) Overview

In this unit, our focus will be on elastic collisions, namely those collisions in which the only forces that act during the collision are conservative forces. In these collisions, the sum of the kinetic energies of the objects is conserved. We will find that the description of these collisions is significantly simplified in the center of mass frame of the colliding objects. In particular, we will discover that, in this frame, the speed of each object after the collision is the same as its speed before the collision.

B) Elastic Collisions

In the last unit, we discussed the important topic of momentum conservation. In particular, we found that when the sum of the external forces acting on a system of particles is zero, then the total momentum of the system, defined as the vector sum of the individual momenta, will be conserved. We also determined that the kinetic energy of the system, defined to be the sum of the individual kinetic energies, is not necessarily conserved in collisions. Whether or not this energy is conserved is determined by the details of the forces that the components of the system exert on each other. In the last unit, our focus was on inelastic collisions, those collisions in which the kinetic energy of the system was not conserved. In particular non-conservative work was done by the forces that the individual objects exerted on each other during the collision.

In this unit, we will look at examples in which the only forces that act during the collision are conservative forces. In this case, the total kinetic energy of the system is conserved. We call these collisions, *elastic* collisions. As an example, consider the collision we discussed in the last unit with one modification – instead of having the boxes stick together, we'll put a spring on one of the boxes as shown in Figure 12.1. The spring will compress during the collision, storing potential energy, and when it relaxes back to its original length it will turn this stored potential energy back into kinetic energy. In this way, no mechanical energy is lost during the collision so that the final kinetic energy of the system will be the same as its initial kinetic energy.

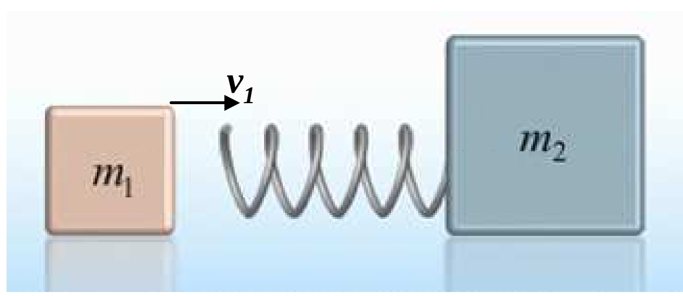


Figure 12.1
An elastic collision: Box 1 moves with speed v_1 and collides with Box 2 that is initially at rest. A spring is connected to Box 2 which is compressed during the collision and then extends to send the two boxes in opposite directions. The mechanical energy of the two box plus spring system is conserved.

C) One Dimensional Elastic Collisions

We will start with the example from the last section. Knowing that neither the momentum nor the kinetic energy of the system will change during this collision allows us to write down two independent equations that relate the initial and final velocities of the boxes. .

$$m_1v_{1,i} + m_2v_{2,i} = m_1v_{1,f} + m_2v_{2,f}$$

$$\tfrac{1}{2}m_1v_{1,i}{}^2 + \tfrac{1}{2}m_2v_{2,i}{}^2 = \tfrac{1}{2}m_1v_{1,f}{}^2 + \tfrac{1}{2}m_2v_{2,f}{}^2$$

These two equations contain six variables (the initial and final velocities of box 1, the initial and final velocities of box 2, and the masses of the two boxes). Therefore, if we know any four of these quantities, these two equations will allow us to solve for the other two. For example, if we know the masses and the initial velocities of both boxes then we can solve these two equations for the final velocities of both boxes.

There is a complication, however, that will make the actual solution of these equations tedious, at best. For example, if we solve the momentum equation for the velocity of box 2 after the collision in terms of the velocity of box 1 after the collision, and plug the result back into the energy equation, we get a pretty messy quadratic equation that, with some effort, can certainly be solved.

$$\tfrac{1}{2}m_1v_{1,i}{}^2 + \tfrac{1}{2}m_2v_{2,i}{}^2 = \tfrac{1}{2}m_1v_{1,f}{}^2 + \tfrac{1}{2}m_2\left(v_{2,i} + \frac{m_1}{m_2}(v_{1,i} + v_{1,f})\right)^2$$

There is, however, a better way. Physics can rescue us from this tedious mathematical chore! Namely, if we solve this problem in the center of mass frame, we can avoid solving any quadratic equations. We can then determine the final velocities in the initial frame by simply transforming the velocities in the center of mass frame back into the initial frame. We will perform this calculation in the next section.

D) The Center of Mass View

Figure 12.2 shows the collision as viewed in the center of mass frame. We have labeled the velocities in this frame with an asterisk.

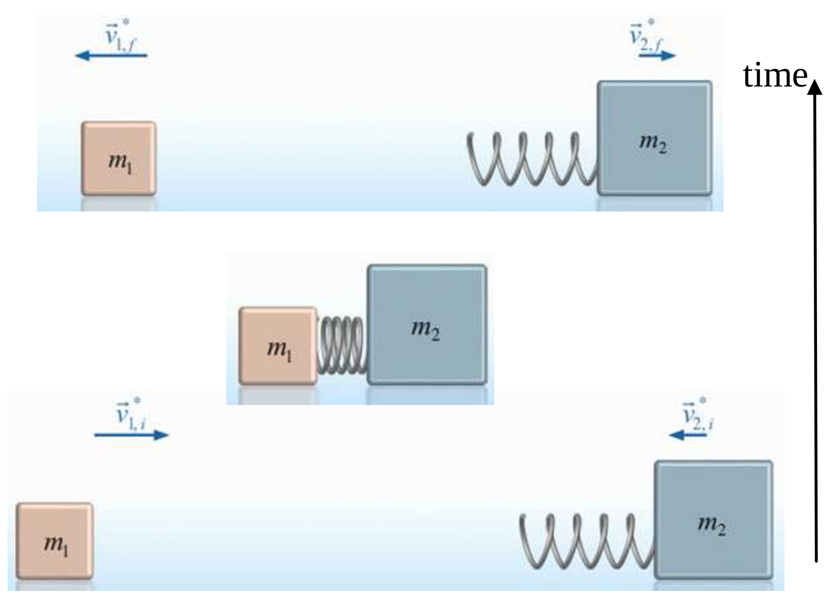


Figure 12.2
The elastic collision as viewed in the center of mass frame.

We know the total momentum is conserved in *any* inertial reference frame. What distinguishes the center of mass frame, though, and what simplifies the calculation, is the additional constraint that the *total momentum is always zero in the center of mass frame*. Consequently, the single momentum conservation equation has become two equations, one for the initial state and one for the final state.

$$m_1 v^*_{1,i} - m_2 v^*_{2,i} = 0$$

$$-m_1 v^*_{1,f} + m_2 v^*_{2,f} = 0$$

Note that we have chosen to use the speed variables (the magnitude of the velocities) in these equations. Therefore, we have explicitly inserted the minus signs to indicate the direction of the velocities. To see how this constraint simplifies the problem, we will multiply and divide each term in the energy equation by the appropriate mass. The result of this operation is that each term now is proportional to the square of an individual momentum.

$$\frac{1}{2m_1}(m_1 v^*_{1,i})^2 + \frac{1}{2m_2}(m_2 v^*_{2,i})^2 = \frac{1}{2m_1}(m_1 v^*_{1,f})^2 + \frac{1}{2m_2}(m_2 v^*_{2,f})^2$$

We can now use the momentum equations to write each side of the energy equation in terms of the square of the momentum of just one of the particles.

$$\left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)(m_1 v^*_{1,i})^2 = \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)(m_1 v^*_{1,f})^2$$

In fact we see that the magnitude of the momentum of each of the objects individually is now also the same before and after the collision, although the direction of each one has changed. In other words, when the collision is viewed in this reference frame the *speed* of each object is the *same* before and after the collision.

$$v^*_{1,i} = v^*_{1,f}$$

$$v^*_{2,i} = v^*_{2,f}$$

This result is very important; it provides us with a very simple strategy to solve any elastic collision problem. We will work through such an example in the next section.

E) Center of Mass Example

We will now work out an example that demonstrates the use of the center of mass frame in elastic collisions. In the collision shown in Figure 12.1, we will assume m_1 = 2kg, v_1 = 5 m/s, and m_2 = 3kg. The boxes collide elastically and both move along the axis defined by the initial velocity vector (call it the *x*-axis). Our job is to determine the final velocities of both boxes in this reference frame, which we will call the lab frame.

Our first step is to transform this problem to the center of mass system. In order to make this transformation, we need to know the velocity of the center of mass in the lab frame. In unit 10 we determined that this velocity was just equal to the vector sum of the individual velocities, weighted by the fraction of the total mass each particle carries.

$$\vec{V}_{CM,lab} = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} = \frac{m_1}{m_1 + m_2}\vec{v}_1$$

Plugging in the values for the masses and the initial velocities, we find that the center of mass is moving at 2 m/s in the +*x* direction.

We can now use this value for the velocity of the center of mass to determine the initial velocities of the boxes as viewed in the center of mass frame. We know that the velocity of an object in the center of mass frame is equal to the velocity of the object in the lab frame plus the velocity of the lab frame in the center of mass frame.

$$\vec{v}^*_{object} = \vec{v}_{object.lab} + \vec{V}_{lab,CM}$$

We know the velocity of the lab in the center of mass frame must just be equal to minus the velocity of the center of mass in the lab frame. We can now find the initial velocities of both boxes in the center of mass frame by simply adding numbers that we now know. Namely, we find the initial velocity of box 1 in the center of mass frame is equal to 5 m/s -2m/s = 3m/s in the positive x-direction, and the initial velocity of box 2 in the center of mass frame is equal to 0 m/s -2m/s = 2m/s in the negative x-direction. From the last section we know that the final speeds in the center of mass frame are equal to initial speeds in that frame. Therefore, we know the final velocity of Box 1 is equal to 3 m/s in the *negative x*-direction and the final velocity of Box 2 is equal to 2 m/s in the *positive x*-direction.

$$\vec{v}^*_{1,f} = -3 \quad m/s \ \ \hat{i}$$

$$\vec{v}^*_{2,f} = +2 \quad m/s \ \ \hat{i}$$

Our final step is to transform these results to the lab frame. We can make this transformation by simply adding the velocity of each object in the center of mass frame to the velocity of the center of mass in the lab frame. When we make these additions, we see that after the collision, box 1 moves with speed 1 m/s in the negative x direction and box 2 moves with speed 4 m/s in the + x direction. You can verify, using these values, that both momentum and energy are indeed conserved in this collision!

If we now replace the masses and initial velocities in this problem by variables and follow the identical procedure, we arrive at the general expressions for the final velocities of any two objects undergoing an elastic collision under the assumption that the second object is at rest to begin with and that all motion is in one dimension.

$$\vec{v}_{1,f} = \vec{v}_{1,i} \frac{m_1 - m_2}{m_1 + m_2}$$

$$\vec{v}_{2,f} = \vec{v}_{1,i} \frac{2m_1}{m_1 + m_2}$$

We can learn a couple of interesting things from these equations. First, if the masses are the same we find that final velocity of the first object is zero and the final velocity of the second object is just equal to the initial velocity of the first. In other words, the objects trade roles!

Second, we see that the final velocity of the first object changes sign if m_2 is greater than m_1. In other words, if the first object is lighter than the second it will bounce back. If, on the other hand, the first object is heavier than the second, it will continue in its initial direction with a reduced speed.

F) Elastic Collisions in Two Dimensions

The last example assumed that the motion of the colliding objects was constrained to one dimension. We found that knowing the masses and the initial velocities of both objects was enough to completely determine the final velocities. We will now extend this analysis to two dimensions by considering the collision of objects on a frictionless horizontal surface without any constraint that the motion is along a single axis. An actual example of such as situation might be billiard balls colliding on a pool table or pucks colliding on an air-hockey table.

We will start by making the simplification that the mass of both objects is the same and that one of the objects is initially at rest. Even in this restricted case, we can see that the final velocity vectors cannot be determined from a knowledge of the initial velocity vectors. Indeed, the final directions depend on the orientation of the objects when they collide. If the collision is head-on the final velocities will be along the x-axis just like in the previous example, but if the collision is not head-on, as illustrated in Figure 12.3, the final velocities can have y-components as well

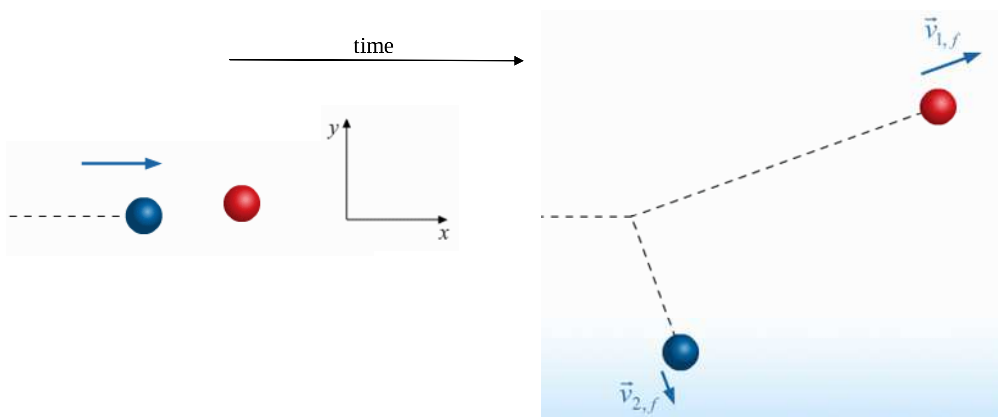


Figure 12.3
An elastic collision between two equal mass balls. If the centers of he balls are not aligned, the collision becomes two-dimensional; the final velocities can develop *y*-components.

What can we say about the final velocities in these cases? Just as in the previous example, we get the simplest view of the collision from the center of mass reference frame as shown in Figure 12.4. Prior to the collision we see the objects approaching each

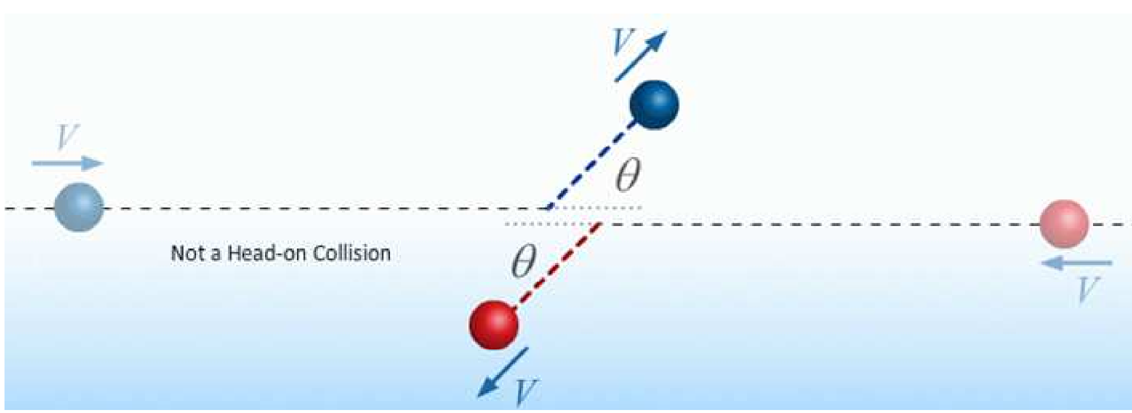
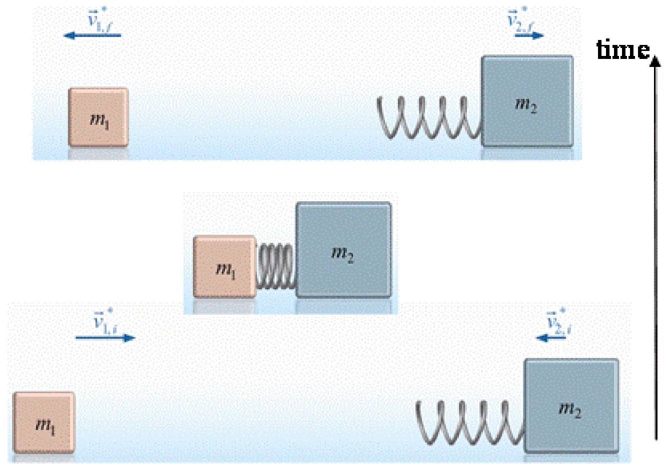


Figure 12.4
An elastic collision viewed from the center of mass frame. The angle θ is *not* determined from conservation of momentum and energy.

other head on, and afterward we see them leaving the collision point back to back with exactly the same speed they had before the collision. The one parameter that is not determined from the conservation of momentum and energy is the angle between the initial and final velocities of one of the objects. Indeed, this angle can vary all the way from 180 degrees in the case of a head-on collision, to 0 degrees in the case where the objects simply miss each other.

Main Points

- **Elastic Collisions**



If the only forces acting during a collision are conservative forces, then the kinetic energy of the system, defined to be the sum of the kinetic energies of the colliding objects, is conserved. Such collisions are called elastic collisions.

$$\sum_i K_i = \sum_f K_f$$

- **Center of Mass Frame**

Elastic collisions are most simply described in the center of mass frame of the colliding objects.

The colision may cause objects to be deflected through some angle in this frame, but their speeds will always remain the same.

Center of Mass Reference Frame

$$\vec{P}_{Total} = M_{Total}\vec{V}_{CM} = 0$$