13. Collisions, Impulse and Reference Frames

A) Overview

In this unit we will conclude our discussion of collisions and look at the energy of a system of particles in more detail. In particular, we will start by developing a useful relation between relative velocities that must hold in an elastic collision. We will then look at the details of the collision process and introduce the concept of the impulse that describes the change in momentum of one of the objects in a collision. Finally, we will investigate the kinetic energy of a system of particles and will find that the total kinetic energy can be expressed as the sum of the kinetic energy of the center of mass and the kinetic energy of the particles relative to the center of mass.

B) Relative Speed in Elastic Collisions

In the last unit we discovered that the description of collisions is often simplified when viewed in the center of mass reference frame. In particular, we showed that the speed of an object before and after an elastic collision is the same when viewed in this frame even though its direction will be changed as is shown in Figure 13.1.. We will now use this result to obtain a relation between relative speeds in a collision that will hold in *all* reference frames.

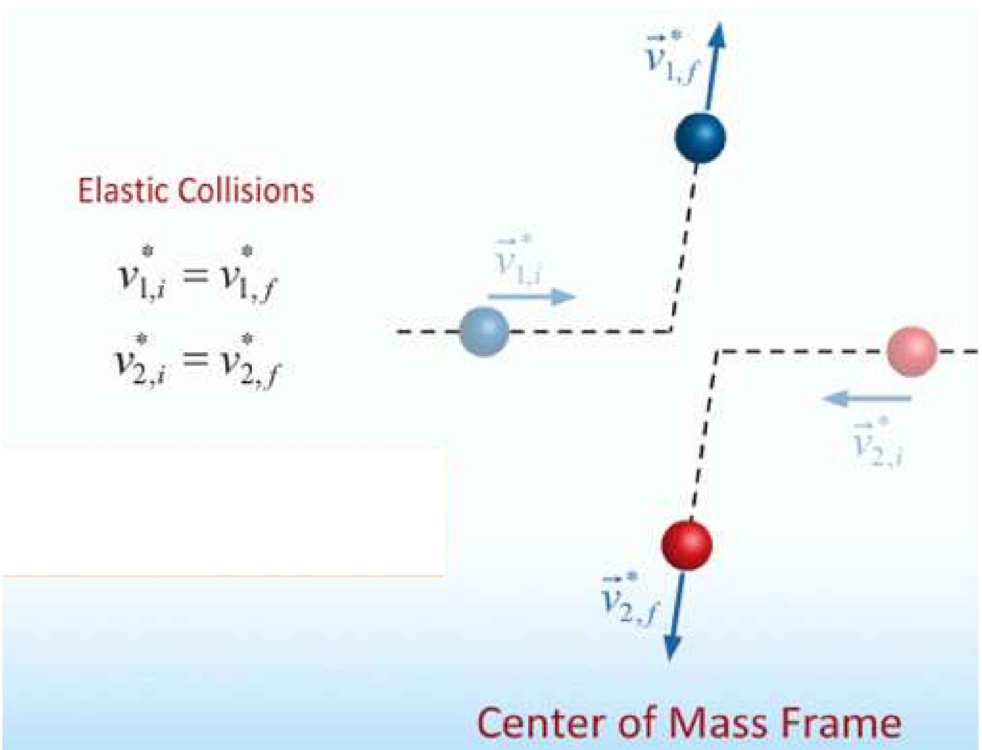


Figure 13.1
An elastic collision as viewed in the center of mass frame. In this frame, the speeds of each particle do not change.

In particular, since the speed of an object before and after an elastic collision is the same if viewed in the center of mass frame, then it is also true that the relative speed of the two objects is the same before and after the collision in this frame.

$$\left|\vec{v}_{2,i} - \vec{v}_{1,i}\right| = \left|\vec{v}_{2,f} - \vec{v}_{1,f}\right|$$

That is, the rate at which two objects approach each other before an elastic collision is the same as the rate at which they separate afterward.

We can now use this result to identify elastic collisions in any inertial reference frame. Namely, the relative velocity of two objects at a given time, that is, the difference in the velocity vectors of the objects, must be the same in all inertial reference frames. This claim follows from the fact that to transform both velocity vectors to a different inertial frame, we simply add the same vector (the relative velocity vector for the two frames) to each initial velocity vector. This relative velocity vector then cancels when we take the difference of the velocities of the objects.

$$\vec{v}_{2,B} - \vec{v}_{1,B} = \left(\vec{v}_{2,A} + \vec{v}_{A,B}\right) - \left(\vec{v}_{1,A} + \vec{v}_{A,B}\right) = \vec{v}_{2,A} - \vec{v}_{1,A}$$

If the relative *velocity* of two objects at a given time is the same in all inertial reference frames, then the relative *speed* of the two objects must also be the same in all inertial reference frames. Since we have just shown that the relative speed of the two objects in an elastic collision is the same before and after the collision in the center of mass frame, then it follows that the relative speed of the two objects in an elastic collision is the same before and after in any inertial reference frame!

Indeed, if we look back to the one dimensional example in section *E* of the last unit, we see that the relative speeds of the two objects, that is the difference in the magnitudes of their velocities, is equal to 5 m/s both before *and* after the collision, in both the center of mass *and* the lab reference frames!

C) Elastic Collision Examples

We just showed that, in an elastic collision between two objects, the rate at which the objects approach each other before the collision is the same as the rate at which they separate after the collision and that this statement is true in all inertial reference frames!

$$\left|\vec{v}_{2,i} - \vec{v}_{1,i}\right| = \left|\vec{v}_{2,f} - \vec{v}_{1,f}\right|$$

For example, suppose we throw a ball against the wall of a building. If the wall is hard and solid and the ball is made of good hard rubber then the collision will be almost elastic and we expect the speed of the ball to be about the same before and after it bounces off the wall.

Suppose we now consider a bowling ball, moving with speed *V*. colliding head-on with a ping pong ball that is initially at rest. If we assume the collision to be elastic and the motion to be constrained to one dimension, what will be the final velocities of the balls?

How do we go about solving this problem? The one thing we do know is that if the collision is elastic, the speed of the ping pong ball relative to the bowling ball must be the same after the collision as it was before the collision. Before the collision, the speed of the ping pong ball relative to the bowling ball was just equal to *V*. Therefore, the speed of the ping pong ball relative to the bowling ball after the collision must also be equal to *V*.

We can obtain an approximate solution by assuming the velocity of the bowling ball will not change much during the collision since it is much heavier than the ping pong ball. In this approximation, we expect the final speed of the ping pong ball to be about twice the initial speed of the bowling ball.

As a check, we can look at the exact solution which we obtained in section *E* of the last unit.

$$\vec{v}_{1,f} = \vec{v}_{1,i} \frac{m_1 - m_2}{m_1 + m_2}$$

$$\vec{v}_{2,f} = \vec{v}_{1,i} \frac{2m_1}{m_1 + m_2}$$

Here we see that in the limit that $m_1 >> m_2$, we recover our approximate solution, that the speed of the ping pong ball after the collision is about twice the speed of the bowling ball.

D) Forces During Collisions

In applying conservation of momentum to collisions between two objects, we have been concerned only with the velocities of the objects before and after the collision. We now want to investigate exactly what Newton’s laws can tell us about the details of the collision process itself.

We start with the differential form of Newton's second law which relates the total force on an object to the time rate of change of its momentum.

$$\vec{F}_{Net} = \frac{d\vec{p}}{dt}$$

We can rewrite this expression to determine that the change in the momentum of an object during a small time *dt* is just equal to the total force acting on the object multiplied by this time interval. If we now integrate this expression over the time of the collision itself, we see that the total change in the momentum of the object during the collision is equal to the integral of the total force acting on that object during this time.

$$\int_{t_1}^{t_2} \vec{F}_{Net} dt = \int d\vec{p} = \vec{p}(t_2) - \vec{p}(t_1) \equiv \Delta\vec{p}$$

This integral is usually called the *impulse* delivered by the force.

We can use this result to define the average force acting on the object during the collision to be equal to the change in the momentum of the object divided by the duration of the collision.

$$\Delta\vec{p} = \int_{t_1}^{t_2} \vec{F}_{Net} dt \equiv \vec{F}_{avg} \Delta t$$

This result simply reflects the differential form of Newton's second law that we used to get started.

E) Impulse Examples

We have just determined that the change in momentum of an object during a collision is equal to the product of the average force acting on that object and the time over which it acts. Therefore, we can achieve the same change in momentum by having a large force acting for a short time as we can having a small force acting for a long time.

Figure 13.2 depicts an example to illustrate this observation. A ball of mass 1 is released from rest from an initial height of 1 meter above the floor. It bounces back to half its original height. If we assume the ball is in contact with the floor for a time of 10 ms, what is the average force on the ball during the collision?

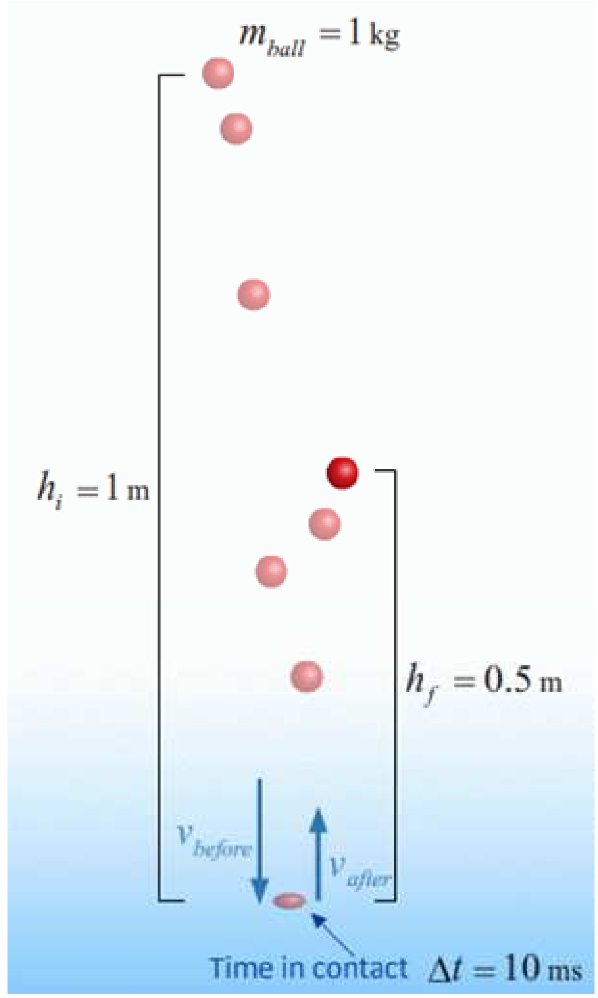


Figure 13.2
A ball is released from rest from a height of 1m and rebounds to a height of 0.5 m. Assuming the ball was in contact with the floor for 10ms, what was the average force exerted by the floor on the ball?

To determine the average force acting during the collision, we need to first determine the change in the momentum of the ball. We can use the conservation of energy during the ball's initial free fall to determine its speed just before it hits the floor and we find that it is proportional to the square root of the height from which it was released.

$$m_{ball} g h_i = \tfrac{1}{2} m_{ball} v_{before}^2$$

$$v_{before} = \sqrt{2gh_i}$$

Putting in the numbers, we obtain a speed of 4.43 m/s. We can also use energy conservation to determine that for the ball to rebound to a height of 0.5 m, it must have had a speed of 3.13 m/s immediately after it left the floor.

$$v_{after} = \sqrt{2gh_f}$$

The change in the momentum of the ball during the collision is therefore equal to 7.56 kg-m/s, since the initial direction is downward and the final direction is upward. We can now determine the average force acting during the collision by dividing this change in momentum by the duration of the collision to obtain the value of 756 N.

$$\Delta p = m_{ball} \Delta v_{ball}$$

Suppose we were to repeat the exact same experiment with a harder ball that flexes less and consequently spends less time in contact with the floor. If, for example, the time of the collision is reduced by a factor of ten, the average force on the ball must be increased by the same factor of ten to keep the change in momentum the same. In other words, the average force on the ball during such a collision would be 7560 N.

F) Energy of a System of Particles

We have seen that often the simplest description of collisions occurs in a reference frame in which the center of mass of the colliding objects is at rest. We will now extend this approach to the discussion of the kinetic energy of a system of particles

Consider a simple system made up of two point particles of mass m_1 and m_2 connected by a massless rod. If we throw this object in our laboratory reference frame we know it will tumble in some complicated way, but we also know that the center of mass will move in a very simple way, namely that the center of mass will behave as though it were a point particle having the total mass of the object as shown in Figure 13.3.

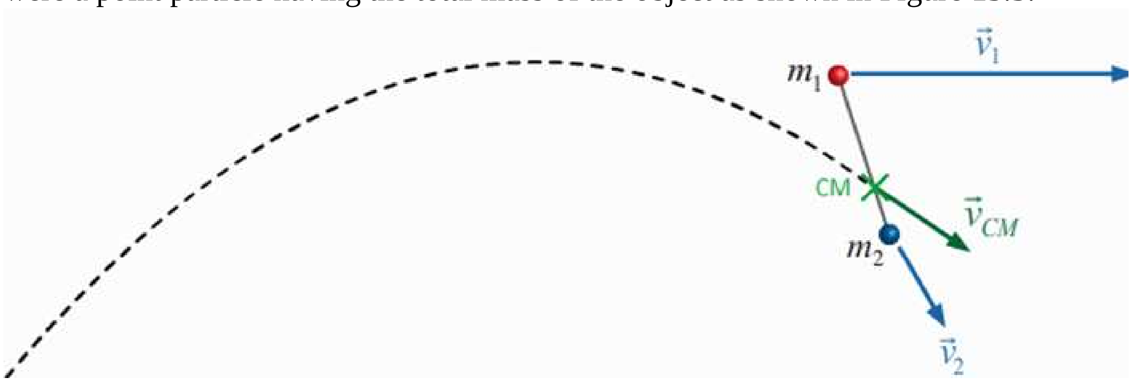


Figure 13.3
An object consisting of balls connected by a massless rod is in free fall. Although the motion of the individual balls is complicated, the center of mass of the system must follow the parabolic trajectory of any object in free fall.

At any instant the kinetic energy of the system is equal to the sum of the kinetic energies of the two particles. We can express the velocity of an object in the lab frame as the vector sum of the velocity of the object in the center of mass reference frame plus the velocity of the center of mass in the lab reference frame.

$$K_{system,lab} = \sum_i \tfrac{1}{2} m_i (\vec{v}_i \cdot \vec{v}_i) = \sum_i \tfrac{1}{2} m_i \left(\vec{v}_i^{\,*} + \vec{V}_{CM}\right) \cdot \left(\vec{v}_i^{\,*} + \vec{V}_{CM}\right)$$

$$K_{system,lab} = \sum_i \tfrac{1}{2} m_i v_i^{*2} + \left(\sum_i \tfrac{1}{2} m_i\right) V_{CM}^2 + \left(\sum_i \tfrac{1}{2} m_i \vec{v}_i^{\,*}\right) \cdot \vec{V}_{CM}$$

$$K_{system,lab} = K_{REL} + K_{CM} + \vec{P}_{Total,CM} \cdot \vec{V}_{CM}$$

$$K_{system,lab} = K_{REL} + K_{CM}$$

When we make this sum, we see that the total energy of the system as viewed in the lab frame can be written as the sum of just two terms: The first term is sum of the kinetic energies of the objects as viewed in the center of mass reference frame, and the second term is the kinetic energy of the center of mass as viewed in the lab reference frame. The remaining terms involve the total momentum in the center of mass reference frame, which by definition is always zero.

The result we have just derived is completely general: The total kinetic energy of any system of objects as viewed by any observer is simply equal to the total kinetic energy of the objects as viewed in the center of mass reference frame, often called the relative kinetic energy, plus the total kinetic energy of the center of mass in the observer's reference frame, often called the center of mass kinetic energy.

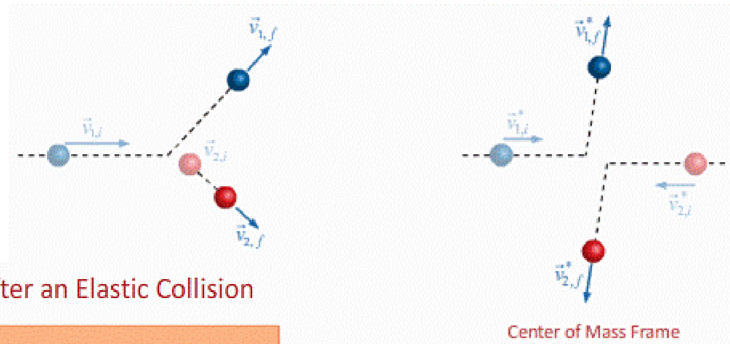
$$K_{system,lab} = K_{REL} + K_{CM}$$

This result has two profound implications. First of all, the total kinetic energy of a system of particles will, in general, have two distinct components. This result will become central to our discussion of rotations in the next unit. Second, we see that the kinetic energy of a system of particles *does* depend on the reference frame of the observer. In other words, the relative kinetic energy will be the same for all observers, but the center of mass kinetic energy will be different for different observers since it will depend on the speed of the center of mass in the frame of the observer.

Main Points

- ## Elastic Collisions: Relative Speeds

The rate at which two objects approach each other before an elastic collision is equal to the rate at which they separate afterward.



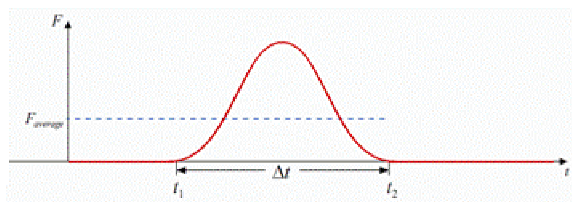
Relative Speeds before and after an Elastic Collision

$$\left|\vec{v}_{2,f}-\vec{v}_{1,f}\right|=\left|\vec{v}_{2,i}-\vec{v}_{1,i}\right|=\left|\vec{v}^{*}_{2,i}-\vec{v}^{*}_{1,i}\right|=\left|\vec{v}^{*}_{2,f}-\vec{v}^{*}_{1,f}\right|$$

- ## Impulse

The Impulse is defined as the integral of the force over the time of the collision.

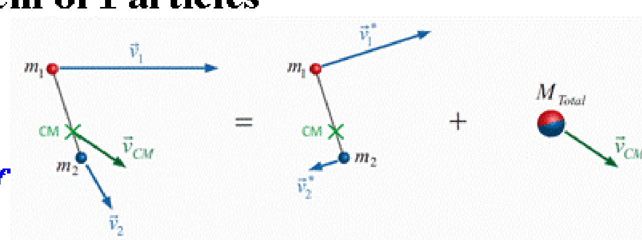
Integrating Newton's second law over time, we find that the impulse is equal to the change in momentum during the collision



Impulse $$\int_{t_1}^{t_2}\vec{F}_{Net}\,dt=\Delta\vec{p}$$

- ## Kinetic Energy of a System of Particles

The kinetic energy of a system of particles, defined as the sum of the kinetic energies of the particles in the system, is equal to the kinetic energy of the particles relative to the center of mass, a term that is the same in all reference frames, plus the energy of the center of mass, a term that does depend on the reference frame of the observer.



$$K_{system,lab}\equiv\sum_i\frac{1}{2}m_i v_i^2=\sum_i\frac{1}{2}m_i v_i^{*2}+\sum_i\frac{1}{2}m_i v_{CM}^2$$