14. Rotational Kinematics and Moment of Inertia

A) Overview

In this unit we will introduce rotational motion. In particular, we will introduce the angular kinematic variables that are used to describe the motion and will relate them to our usual one dimensional kinematic variables. We will also define the moment of inertia, the parameter in rotational motion that is analogous to the mass in translational motion. We will evaluate the moment of inertia for a collection of discrete particles as well as for symmetric solid objects.

B) Rotational Kinematics

Until now our studies of dynamics have been restricted to linear motion of objects described in a Cartesian coordinate system (*x, y, and z*). In our recent discussions of systems of particles, though, we have discovered that the motion can be described as having two components: (1) the motion *of* the center of mass and (2) the motion *relative to* the center of mass. As an illustration of the motion relative to the center of mass, we will look at the rotation of an object about an axis through the center of mass. Our first step is to develop a coordinate system in which these rotations can be described naturally.

Figure 14.1 shows a disk rotating about an axis though its center. The orientation of the disk at any time can be described by a single parameter: the angle θ through which the disk has rotated relative to its initial orientation. We call this angle the *angular displacement*. The time rate of change of the angular displacement is called the *angular velocity* ω, and the time rate of change of the angular velocity is called the angular acceleration α.

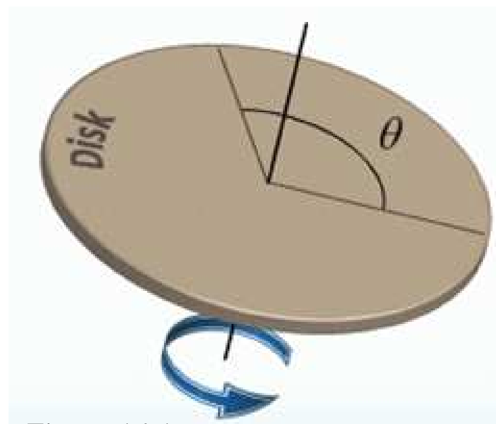


Figure 14.1
The rotation of a disk about an axis through its center is described by an angular displacement θ.

$$\omega \equiv \frac{d\theta}{dt}$$

$$\alpha \equiv \frac{d\omega}{dt}$$

These equations look strikingly similar to those we have used to describe one dimensional kinematics.

$$v \equiv \frac{dx}{dt} \qquad a \equiv \frac{dv}{dt}$$

The reason for this similarity is simply that this rotational motion can be described by a single angular displacement, θ, just as linear motion can be described by a single spatial displacement, *x*.

If we now consider the special case of a constant angular acceleration α, we can derive the equations for ω and θ for this motion by integrating the defining equations. The resulting equations for ω and θ are absolutely identical in form to those describing one dimensional motion at constant acceleration with the substitutions α for *a*, ω for *v*, and θ for *x*.

$$\theta = \theta_o + \omega_o t + \tfrac{1}{2}\alpha t^2$$

$$\omega = \omega_o + \alpha t$$

$$\omega^2 - \omega_o^2 = 2\alpha(\theta - \theta_o)$$

C) Relating Linear and Rotational Parameters

We can now make another useful connection between rotation and one dimensional kinematics by obtaining the relationships between the angular and linear kinematic parameters used to describe the motion of a point that is a fixed distance *R* from the rotational axis.

In the case of one dimensional motion along the *x* axis we needed to specify which direction we choose to be positive so that the signs of displacement and velocity and acceleration have meaning. In exactly the same way, we need to specify which direction of rotation we choose to be positive so that the signs of angular displacement and angular velocity and angular acceleration have meaning.

Although we are always free to choose either direction of rotation to be positive, it is customary to pick the counterclockwise as the positive direction. With this choice, the angular displacement θ agrees with that usually used in trigonometry. Since all points on the disk are rotating together, we can determine the linear displacement, speed and acceleration of any point on the disk in terms of the corresponding angular parameters. Figure 14.2 shows a disk turned through an angular displacement θ. We can see that a point located a distance *R* from the rotation axis moves through a distance *s* along a circular path of radius *R*. This distance *s* is determined from geometry to be just equal to the product of the radius and the angular displacement, measured in radians.

$$s = R\theta$$

Taking the derivative of this linear distance with respect to time, we find a simple relationship between the speed of the point and the angular velocity of the disk.

$$\frac{ds}{dt} = R\frac{d\theta}{dt} \qquad\qquad v = R\omega$$

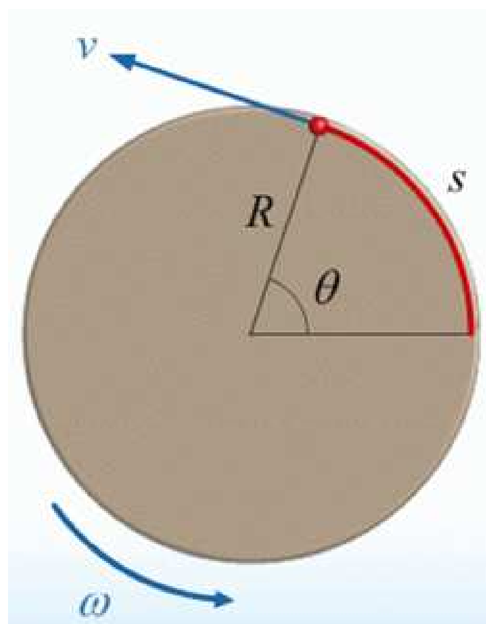


Figure 14.2
A disk rotates with angular velocity ω through an angular displacement θ.

Taking the derivative of this linear speed with respect to time, we find a simple relationship between the tangential acceleration of the point along its circular path and the angular acceleration of the disk.

$$\frac{dv}{dt} = R\frac{d\omega}{dt} \qquad\qquad a = R\alpha$$

D) Kinetic Energy in Rotations

We will now expand our discussion of rotations by considering the motion of a rigid object made up of a set of point particles connected by massless rods as shown in Figure 14.3. This object rotates about the fixed axis with a constant angular velocity ω. We will assume we know the masses of each particle (m_i) and also the distances of each particle from the axis of rotation (r_i).

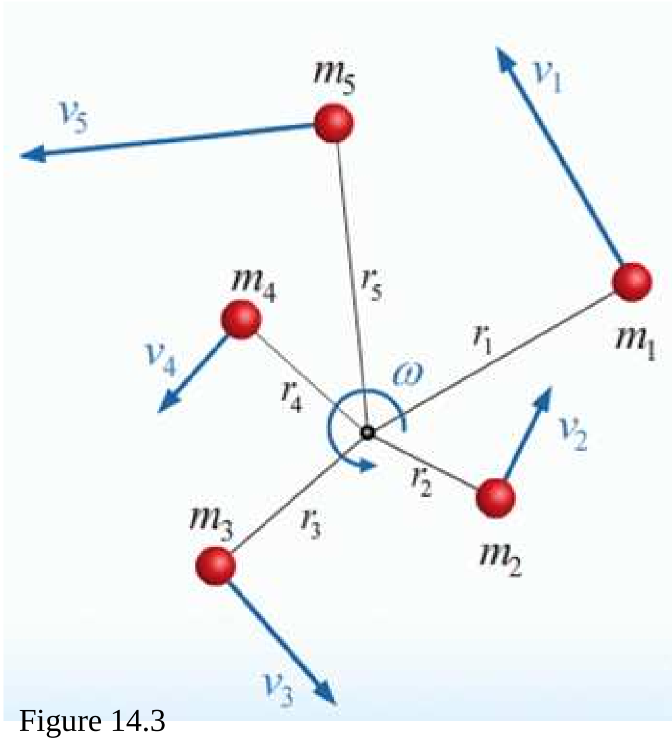


Figure 14.3
A rigid object consisting of five point particles connected by massless rods rotates with angular velocity ω.

The total kinetic energy of this object is defined to be the sum of the kinetic energies of each of its parts. We found in the last section that the speed of a rotating object relative to the axis of rotation is just the product of its angular velocity and its distance from the axis. Therefore, we can rewrite the expression for the total kinetic energy of the object in terms of its angular velocity.

$$K_{system} = \sum \tfrac{1}{2} m_i (r_i\omega)^2$$

Since this angular velocity is a constant, we can take it, and the common factor of ½, outside the sum.

$$K_{system} = \frac{1}{2}\left(\sum m_i r_i^2\right)\omega^2$$

We will define the remaining sum, namely, the sum of the product of each mass with the square of its distance from the axis, to be the moment of inertia of the object about this axis and will denote it with the symbol *I*.

$$I \equiv \sum m_i r_i^2$$

Note that the resulting expression for the kinetic energy of this object has the same form as the kinetic energy of a point particle.

$$K_{system} - \tfrac{1}{2} I\omega^2$$

We have just replaced the velocity by the angular velocity, and mass by the moment of inertia.

In other words, in the same way that the mass of an object tells us how its kinetic energy is related to the square of its velocity, the moment of inertia of a rotating object tells us how its kinetic energy is related to the square of its angular velocity. In a sense, the moment of inertia plays the same role in rotational motion that the mass plays in the simpler motions we have studied up to this point. As we learn more about rotations, we will see this conceptual connection between mass and the moment of inertia appear again and again.

E) Moment of Inertia

We will spend the remainder of this unit exploring the properties of the moment of inertia in more detail. The most obvious difference between mass and the moment of inertia is that the moment of inertial depends not just on the total mass, but also on exactly where that mass is located. Indeed, the moment of inertia even depends on our choice of the rotation axis, since we are measuring all distances relative to this axis.

We will now do a simple example to illustrate these points. Figure 14.4 shows an object made up of four point particles of equal mass M arranged in a square of side $2L$ centered on the origin. The x and y axes are as shown and the z axis points out of the page. We will first calculate the moment of inertia of this object if it is rotated around the x axis. Since the distance of each mass from the x axis is just equal to L, the sum we need to make to determine the moment of inertia is simple.

$$I_x \equiv \sum m_i r_i^2 = 4ML^2$$

Clearly, if we were to calculate the moment of inertia for rotations around the y axis, we would find it to be identical to the moment of inertia for rotations about the x axis

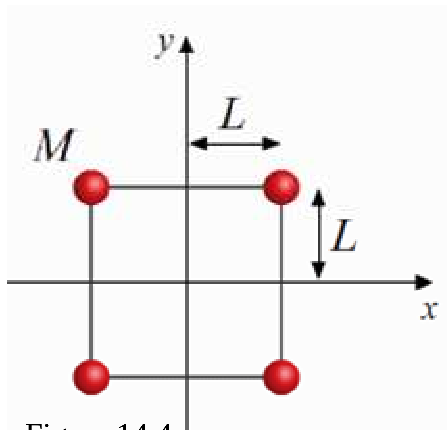


Figure 14.4
A rigid object consists of four point particles, each of mass M, located at the corners of a square of side $2L$.

What about rotations around the z axis? The distance of each particle from the z axis is clearly larger than L. Indeed, using the Pythagorean theorem, we see that the square of the distance to each mass is twice as big as before.

$$I_z \equiv \sum m_i r_i^2 = 8ML^2$$

Therefore, the moment of inertia for rotations about the z axis is twice as big as the moment of inertia for rotations about the x or y axes.

We've seen from this example exactly how to calculate the moment of inertia of an object made up of discrete point particles about any axis. We've learned that the moment of inertia does depend on the choice of the rotation axis. In the next section, we will generalize this calculation to the case of a continuous solid object, rather than a small collection of points.

F) Moment of Inertia of a Solid Object

Figure 14.5 shows a thin rod of mass *M* and length *L* centered along the *x*-axis. How do we go about calculating its moment of inertia for rotations about the *z* axis? Once again, as we have done so often, we will need to replace the discrete sum we used in the last section with an appropriate integral.

We will call the mass per unit length of the rod λ. Each infinitesimal piece of the rod has a length *dx* and a mass equal to the product of this length and the mass per unit length.

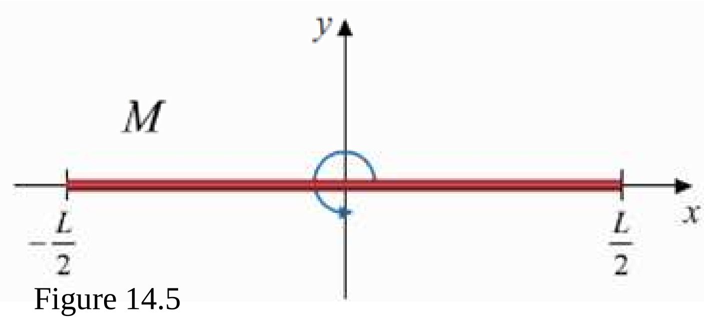


Figure 14.5
A thin rod of mass M and length L is aligned with the *x*-axis.

$$dm = \lambda dx$$

The contribution to the moment of inertia from such an infinitesimal piece of the rod located a distance *x* from the origin is just equal to the product of its mass and the square of its distance from the axis..

$$dI_z = dm x^2$$

To find the total moment of inertia, we integrate over the length of the rod.

$$I_z = \int_{-L/2}^{+L/2} x^2 \lambda dx = \lambda \int_{-L/2}^{+L/2} x^2 dx$$

The mass density can be taken outside the integral and we are left with the integral of x^2 which is just $x^3/3$. Evaluating this expression between the limits, we obtain:

$$I_z = \lambda \left[\frac{1}{3} x^3 \right]_{-L/2}^{L/2} = \lambda \frac{1}{12} L^3$$

We can replace the mass density by the total mass divided by the length of the rod to obtain an expression that is proportional to the product of the total mass and the square of the length of the rod, as expected.

$$I_z = \frac{1}{12} ML^2$$

We know the moment of inertia depends on the choice of the axis. Suppose we want to calculate the moment of inertia about an axis that is parallel to the z axis, but passes through the end of the rod rather than its middle. How does the calculation change? To determine the moment of inertia we just do the integral again, this time shifting the location of the rod to the right so that its left end is at the origin.

$$I_z = \lambda \int_0^L x^2 dx = \frac{1}{3} \lambda L^3 = \frac{1}{3} ML^2$$

Evaluating the integral, we see that the moment of inertia about the end of the rod is four times as big as the moment of inertia about its center. This result is reasonable, since more of the mass is further from the axis when it is located at the end of the rod rather than at its center.

F) Moment of Inertia of a Solid Cylinder

We will now do one more example that will illustrate some general features of moments of inertia of solid objects. Figure 14.6 shows a solid cylinder or mass M and radius R. The axis of the cylinder coincides with the z axis and its end surfaces are located at $z = 0$ and $z = L$.

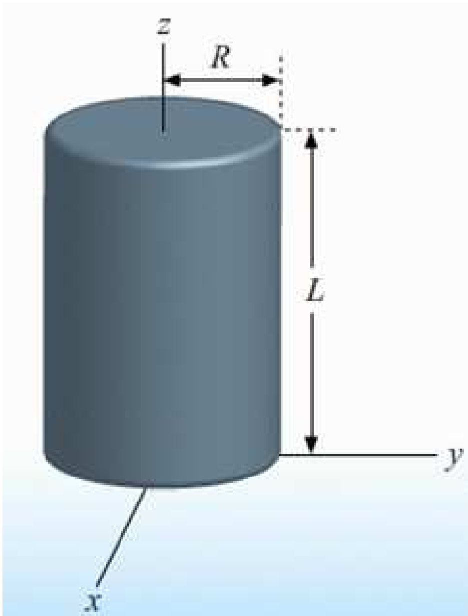


Figure 14.6
A solid cylinder of radius R, length L and mass M.

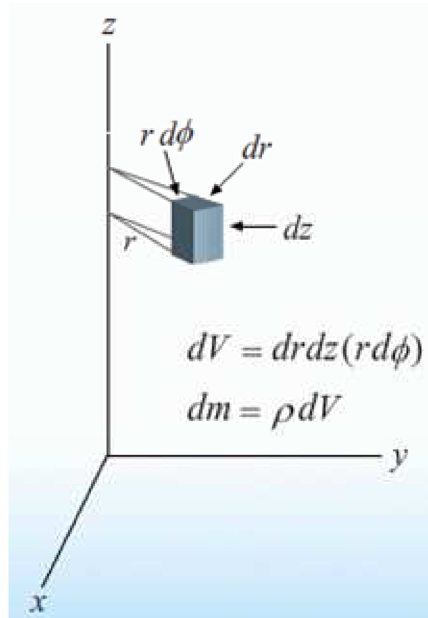


Figure 14.7
A volume element in cylindrical coordinates for the cylinder in Fig 14.6

Since this object has cylindrical symmetry, our integral will be simplified if we use cylindrical coordinates (namely, r, ϕ and z) rather than Cartesian coordinates. Figure 14.7 shows the volume element illustrated as the product of dr, dz and $rd\phi$. To integrate over the mass of the cylinder, we use the mass element dm which is just equal to the product of this volume element and ρ, the mass per unit volume of the cylinder.

$$dm = \rho r dr dz d\phi$$

To evaluate the moment of inertia about the axis of symmetry, the z axis, we just need to integrate $r^2 dm$ over the entire cylinder.

$$I_z = \int r^2 dm = \iiint \rho r^3 dr dz d\phi = \rho \int_0^L dz \int_0^{2\pi} d\phi \int_0^R r^3 dr$$

The z and ϕ integrals are trivial, just being equal to $2\pi L$. We are left then with just the integral of r^3. Evaluating this integral and simplifying, we see that the moment of inertia of the cylinder about its axis is just equal to ½ MR^2.

$$I_z = \frac{M}{\pi R^2 L} 2\pi L \frac{R^4}{4} = \frac{1}{2} MR^2$$

Note that this result does not depend explicitly on the length of the cylinder, only on its mass and radius. For example, if we were to cut the cylinder in half through a plane perpendicular to the z axis, we would have two cylinders, each with half the mass of the original cylinder. Therefore the moment of inertia of each new cylinder is just half of the moment of inertia of the original cylinder.

$$I_{Total} = \sum_i I_i$$

Consequently, we see that if a system is made of two or more parts, and we know the moment of inertia of each part about some axis, then the total moment of inertia about that axis is just the sum of the moments of inertia of the parts. This result may seem somewhat trivial, but it will prove useful later.

G) Moment of Inertia for Solid Objects

We have just determined that the moment of inertia of a solid cylinder about its axis is proportional to the product of its mass and the square of its radius. We can determine the moment of inertia of any other object with cylindrical or spherical symmetry in exactly the same way. We will *always* find this same result, that the moment of inertia will be proportional to the product of its mass and the square of its radius.

$$I_{Total} \propto MR^2$$

The constant of proportionality will be different, of course, for each different shape. The larger this constant, the more mass is located far from the axis. For example, it is easy to see that for a cylindrical shell, this constant of proportionality is just equal to one, since all of the mass is located at the radius of the shell.

Consider a solid sphere and a solid cylinder. We expect the constant of proportionality for the sphere to be smaller than that for the cylinder since more of its mass is concentrated near the axis. When we do the calculation, we see that this constant for the sphere is 2/5 which is indeed smaller than the factor of ½ we have calculated for the solid cylinder. Similarly, we expect that the constant for a spherical shell is smaller than that for a cylindrical shell.

$$I_{solidcylinder} = \frac{1}{2}MR^2 \qquad I_{solidcsphere} = \frac{2}{5}MR^2$$

$$I_{cylindricalshell} = MR^2 \qquad I_{sphericalshell} = \frac{2}{3}MR^2$$

Main Points

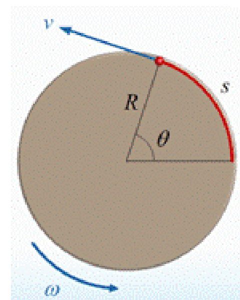
- **Rotational Kinematics**

Rotational motion is described in terms of the (i) angular displacement θ, (ii) the angular velocity ω, and (ii) the angular acceleration α.

$$\omega \equiv \frac{d\theta}{dt} \qquad \alpha \equiv \frac{d\omega}{dt}$$

The displacement, velocity, and acceleration of any point that is rotating is proportional to the corresponding angular parameter

$$s = R\theta$$
$$v = R\omega$$
$$a = R\alpha$$



- **Moment of Inertia & Kinetic Energy**

The Moment of Inertia of a system of particles about an axis is defined to be the sum of the product of the mass and the square of the distance from the axis for all parts of the system

For Discrete Distributions

$$I \equiv \sum m_i r_i^2$$

For Continuous Distributions

$$I = \int r^2 dm$$

The kinetic energy of a system of particles is equal to ½ the product of the square of the angular velocity and the moment of inertia about the axis of rotation.

Rotational Kinetic Energy

$$K_{system} = \frac{1}{2} I \omega^2$$

- **Moment of Inertia of Cylinders and Spheres**

The moment of inertia for any cylindrical or spherical object is proportional to the product of the square of the radius of the object and the total mass of the object.