15. Parallel Axis Theorem and Torque

A) Overview

In this unit we will continue our study of rotational motion. In particular, we will first prove a very useful theorem that relates moments of inertia about parallel axes. We will then move on to develop the equation that determines the dynamics for rotational motion. In so doing, we will introduce a new quantity, the torque, that plays the role for rotational motion that force does for translational motion. We will find, once again, that the rotational analog of mass will be the moment of inertia.

B) Parallel Axis Theorem

We have shown earlier that the total kinetic energy of a system of particles in any reference frame is equal to the kinetic energy of the center of mass of the system, defined to be one half times the total mass times the square of the center of mass velocity, plus the kinetic energy of the motion of all of the parts relative to the center of mass.

$$K_{system,lab} = K_{REL} + K_{CM}$$

For a solid object the only possible motion relative to the center of mass is rotation. Therefore, the kinetic energy relative to the center of mass is just equal to one half times the moment of inertia about a rotation axis through the center of mass times the square of the angular velocity about the center of mass.

$$K_{system,lab} = \tfrac{1}{2} I_{CM} \omega_{CM}^2 + \tfrac{1}{2} M v_{CM}^2$$

We can use this result to calculate the moment of inertia about a chosen axis if the moment of inertia about a parallel axis that passes through the center of mass is known.

For example, Figure 15.1 shows the thin rod from the last unit rotating about an axis perpendicular to the page at a distance D from the center of mass. We can describe this motion, as the center of mass rotating about the axis with angular velocity ω, plus the rod rotating about its center of mass with that same angular velocity ω. Thus, for every rotation the center of mass makes about the axis, the rod makes one revolution about its center of mass.
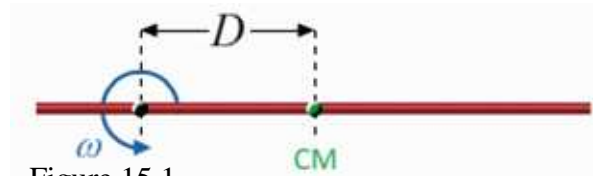
Figure 15.1
A thin rod rotates with angular velocity ω about an axis perpendicular to the page located a distance D from the center of mass of the rod.

We start from our previous result that the total kinetic energy about the chosen axis can be written in terms of the moment of inertia about that axis.

$$K_{total} = \tfrac{1}{2} I_{total} \omega^2$$

Now we know this kinetic energy must be equal to the sum of the kinetic energy of the center of mass and the kinetic energy relative to the center of mass.

$$K_{total} = \tfrac{1}{2} M v_{CM}^2 + \tfrac{1}{2} I_{CM} \omega^2$$

Now the velocity of the center of mass in the lab frame is just equal to the product of D and ω, and the kinetic energy relative to the center of mass is just equal to ½ the product of the moment of inertia about this parallel axis and the square of the angular velocity.

$$K_{total} = \tfrac{1}{2}M(D\omega)^2 + \tfrac{1}{2}I_{CM}\omega^2$$

We can cancel a factor of one half omega squared from each term in the kinetic energy equation to determine that the total moment of inertia about the chosen axis is just the moment of inertia about a parallel axis passing through the center of mass plus the moment of inertia of the center of mass, treated as a point particle, about the chosen axis.

$$I_{total} = MD^2 + I_{CM}$$

This result is known as the parallel axis theorem and it will prove to be very useful in the next few units. You can verify that this parallel axis theorem predicts the relationship we obtained last time for the two moments of inertia for a thin rod.

C) Example: Moment of Inertia of a Dumbbell

We can now use this parallel axis theorem to calculate the moment of inertia of a dumbbell made up of two solid spheres connected by a solid rod about an axis that is perpendicular to the rod and passes through its center as shown in Figure 15.2.

We'll start by using our result from the last unit that the moment of inertia of the dumbbell about the given axis is equal to the sum of the moments of inertia of its components, the rod and the two spheres, about that same axis. To find the moments of inertia of the spheres about the axis through the center of the rod, we will apply the parallel axis theorem we developed in the last section.

Namely, we know that the moment of inertia of a solid sphere about an axis passing through its center is equal to 2/5 the product of its mass and the square of its radius.
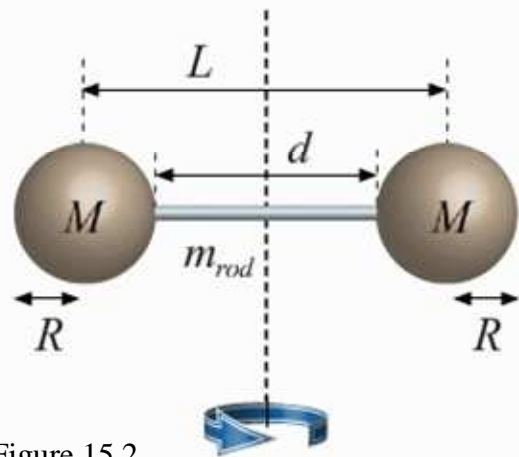
$$I_{sphere,CM} = \frac{2}{5}MR^2$$

Figure 15.2
A dumbbell composed of two spherical masses of radius R and mass M separated by a distance d are connected by a rod with mass m_{rod}.

Applying the parallel axis theorem, we see that the moment of inertia of each sphere about the given axis is just given by:.

$$I_{sphere,axis} = \frac{2}{5}MR^2 + M\left(\frac{L}{2}\right)^2$$

To find the total moment of inertia of the dumbbell, we just add the contributions from the two spheres to the moment of inertia of the thin rod about its center to get the result:

$$I_{dumbbell} = \frac{1}{12}m_{rod}d^2 + 2\left[\frac{2}{5}MR^2 + M\left(\frac{L}{2}\right)^2\right]$$

D) Torque and Angular Acceleration

So far in our study of rotations, we have made explicit connections to the kinematics of one-dimensional linear motion and the concept of mass. We will now use Newton's second law to develop the equation that describes the dynamics of rotations.

Figure 15.3 shows a point mass, constrained to move in a circle being acted upon by an arbitrary force F. We choose to use polar coordinates (r and θ) to describe this motion since r will be constant   Since the mass is moving in a circle, we know there must be a component of the net force acting on the mass that is radial to supply the radial acceleration (v^2/r). In addition, there may be a component of the net force that is tangential which will result in a non-zero tangential acceleration. This tangential acceleration is simply related to the angular acceleration of the rotation.

$$F_\theta = ma_\theta = mr\alpha$$

We know that the rotational analog of mass is the moment of inertia, which in this case is just equal to the product of the mass of the particle and the square of its distance from the axis of rotation.

$$I = mr^2$$

If we now multiply both sides of our force equation by this distance r, we find that the product of r and the tangential component of the force is equal to the product of the moment of inertia of the object about the rotation axis and the angular acceleration.

$$rF_\theta = I\alpha$$

The right hand side now looks like a rotational version of Newton's second law: namely, the product of the rotational mass (the moment of inertia) and the rotational acceleration. It would be natural to identify the left hand side of this equation as the rotational force. Indeed, we will identify the product of the tangential force and the distance between the application point of this force and the rotation axis as the *torque*, the quantity that plays the role of force in rotational dynamics.

$$\tau \equiv rF_\theta$$

IN the next sections we will formally define *torque* as a *vector* quantity and will arrive at a general vector equation that is the rotational version of Newton's second law.

Figure 15.3
A point mass costrained to move on a circle of radius r is acted upon by a force F. The tangential component of this force, F_θ, gives rise to an angular acceleration α.

E) Example: Closing a Door

We've just seen that a torque, the product of a tangential force and the distance between the application point of this force and the rotation axis, produces an angular acceleration. As a concrete example, Figure 15.4 shows the overhead view of an open
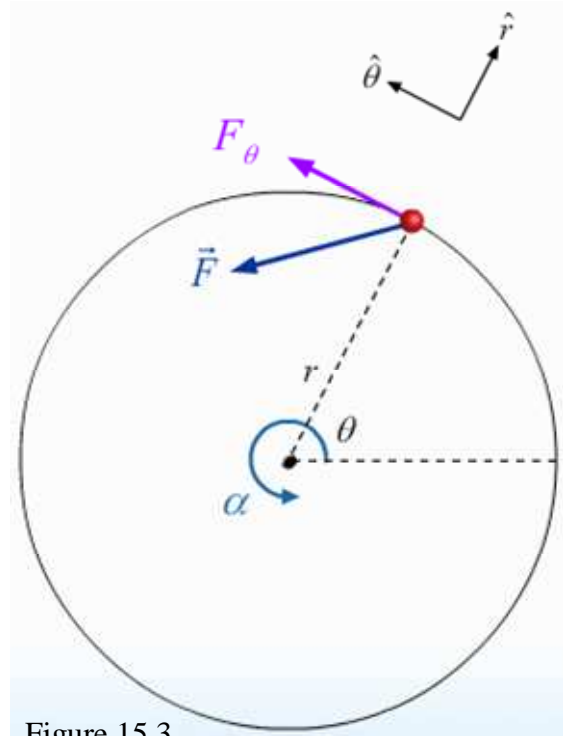
door. To close this door, you need to push on it. Here we see a force **F** being applied a distance *r* from the hinge. The door is heavy and barely moves as you push on it. What would you change about the way you are pushing in order to close it quicker? Your intuition tells you that you would either push harder, or you would push on the door at a point *further* from the hinge. You would certainly not try pushing closer to the hinge, and you would also not change the direction you were pushing to be more parallel to the door. In other words, the biggest effect you can have on the door is to push on it as hard as possible, in a direction perpendicular to the door, at a point far from the hinge.
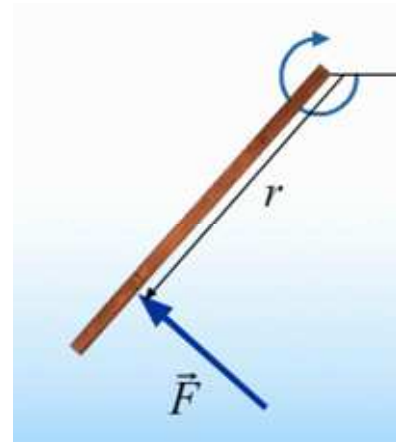
The equations we derived in the previous section tell us exactly the same thing! We expect the angular acceleration to be biggest when the torque is biggest – in other words the door will close fastest when the torque is large. The torque is largest when the distance *r* between the axis and the perpendicular force is biggest, and when the perpendicular force itself is biggest. If we push parallel to the door, there is no perpendicular force component and we don't expect the door to start moving at all!

Figure 15.4
An overhead view of a closing door. The force **F** applies a torque about the hinge which causes an angular acceleration.

In the next section we will generalize this description of rotations by defining torque, angular velocity, and angular acceleration as vector quantities and by defining the cross product of two vectors..

F) Torque and the Cross Product
Figure 15.5 shows a spinning top. So far, we have described this motion in terms of one dimensional variables, the angular displacement, the angular velocity and the
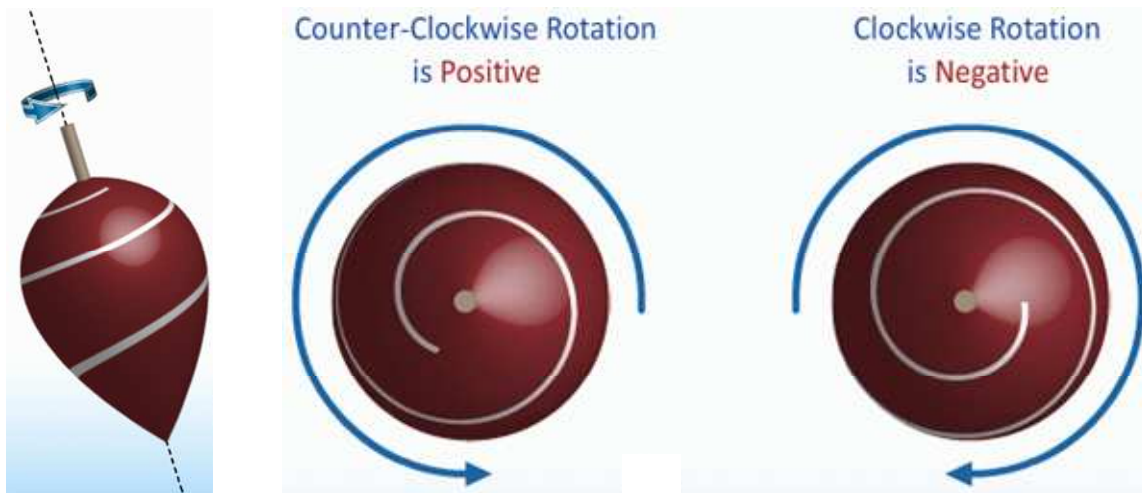

Figure 15.5
A top spins on its axis. A counterclockwise rotation is defined to be positive.

angular acceleration, all defined relative to the axis of rotation. We will need to generalize this description to three dimensions when we allow the direction of the rotation axis itself to change.

If a rotating object is viewed in a reference frame in which the rotation axis is perpendicular to the page, as shown in Figure 15.5, it is conventional to define a counter-clockwise rotation as positive and a clockwise rotation as negative. We adopt this convention in order to match the usual measurement of the angle theta relative to the x-axis in a right-handed Cartesian coordinate system.

There is a simple rule involving the right hand that can be used to define the directions of the angular velocity vector. Namely, if you curl the fingers of your right hand in the direction of rotation of the object, your thumb will point in the direction of the angular velocity vector of the object. In other words, if some object is spinning in the counter-clockwise direction in the x-y plane, curling the fingers of your right hand in this direction results in your thumb pointing in the $+z$ direction which we define to be the direction of the angular velocity vector. Similarly, if the object is rotating in the clockwise direction, the same exercise results in your thumb pointing in the $-z$ direction.

We use the same right hand rule to define the direction of the angular acceleration. For example, if the magnitude of the angular velocity *increases* in time, then the angular acceleration vector has *the same direction* as that of the angular velocity. If the magnitude of the angular velocity *decreases* in time, then the angular acceleration vector has the *opposite direction* as that of the angular velocity.

Figure 15.6
The *right hand rule* for determining the direction of the angular veloicty $\vec{\omega}$. Curl your fingers in the direction of rotation and your thumb will point along the direction of ω.

One nice feature of using the right hand rule to define the direction of vectors in systems involving rotation is that these vectors then do not rely on the choice of a coordinate system at all; the rotation vector becomes a property of the object itself.

Now that we have defined the direction of the angular velocity and angular acceleration vectors, we will now define the torque as a vector quantity and obtain the vector equation that determines rotational dynamics.

We have shown that the magnitude of the torque is given by the product of the length of the line connecting the axis to the point where the force acts, r, and the component of the force F that is perpendicular to this line. If we define θ as the angle

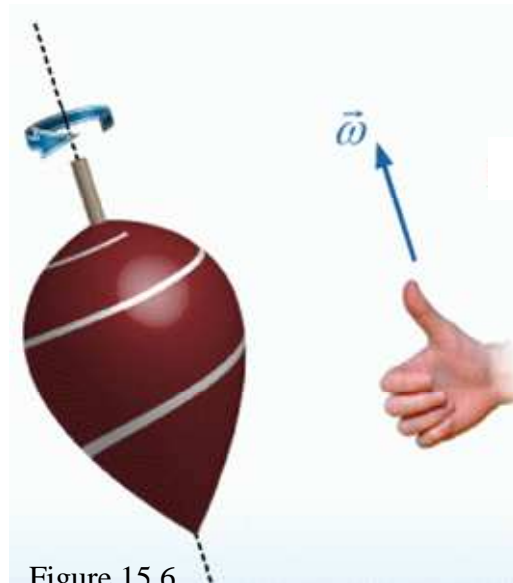between this line and the force as shown in Figure 15.7, then the perpendicular component of the force is just equal to the product of F and $\sin\theta$.

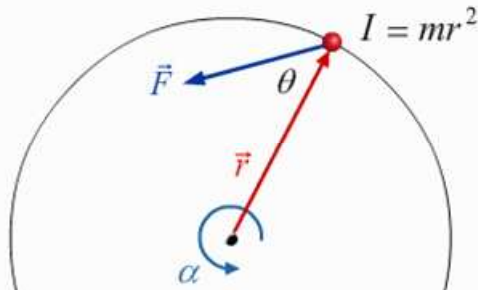

Figure 15.7
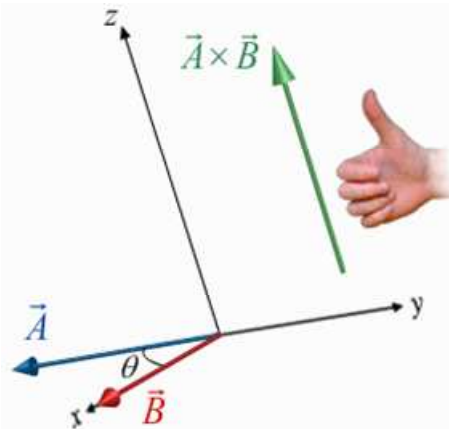The magnitude of the torque exerted by force F is equal to $rF\sin\theta$.

Figure 15.8
The direction of the cross product of two vectors A and B is given by a right hand rule: point the fingers of your right hand in the direction of A and curl them toward the direction of B, then your thumb points in the direction of $A\times B$.

If we consider the line connecting the rotation axis to the force as a vector then the magnitude of the torque is given by the magnitude of the vector F times the magnitude of the vector r times the sin of the angle between them. We write it in this way because there is a mathematical construct called the cross product that has exactly this same property. Namely, given two vectors A and B, the magnitude of $A\times B$ is given by this same form.

$$\left|\vec{A}\times\vec{B}\right| = AB\sin\theta$$

The direction of $A\times B$ is perpendicular to both A and B and is given by the right hand rule shown in Figure 15.8. If you point the fingers of your right hand in the direction of A and curl them toward the direction of B, then your thumb points in the direction of $A\times B$.

Using this definition of the cross product, and what we already know about the right hand rule, we can write a vector equation that neatly summarizes everything we know so far.

$$\vec{\tau}_{Net} = I\vec{\alpha}$$

On the right hand side we have the angular acceleration vector, whose direction is found by applying the right hand rule. On the left hand side we have the torque vector, whose direction is also found using the right hand rule.

In the example shown, we see that the directions of the torque and the angular acceleration are the same; they are both parallel to the rotation axis pointing out of the page. This equation, the net toque about an axis is equal to the product of the moment of inertia about that axis and the angular acceleration, is totally general. This equation is just Newton's second law applied to a system of particles in rotation!

Main Points

- ## Parallel Axis Theorem

The moment of inertia about a chosen axis is equal to the moment of inertia about a parllel axis passing through the center of mass plus the moment of inertia of the center of mass, treated as a point particle, about the chosen axis.
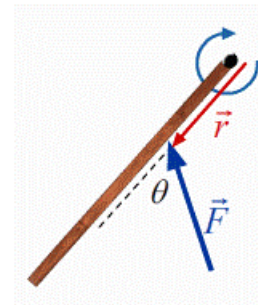
$$I_{Total} = MD^2 + I_{CM}$$
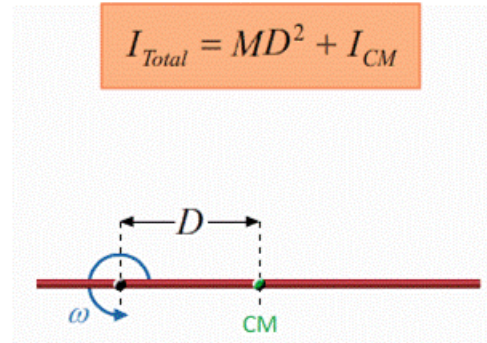
- ## Torque

The concept of torque plays the role for rotational motion that force does fro translational motion.

The direction of the torque vector is determined by a right hand rule: curl fingers from r into F and then thumb points in direction of torque.

$$\vec{\tau} \equiv \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$$

- ## Dynamics Equation for Rotational Motion

The equation that determines the dynamics of rotational motion is derived from Newton's second law.

The net torque about an axis is equal to the product of the moment of inertia about that axis and the angular acceleration.

$$\vec{\tau} = I\vec{\alpha}$$