Method 1: Progressive “collapsing” of circuit elements

In last week’s discussion, we learned how to analyse circuits involving batteries and capacitors. Our method was to progressively collapse groups of capacitors (connected in series or in parallel) into effective capacitors. Once the circuit became simple enough, we could calculate everything about it: charge and voltage. Then we worked backwards, breaking up each combination and calculating the charge and voltage on the individual capacitors.

This week, we will analyze circuits involving resistors. As you know, the formulas for combining resistors in series and in parallel are “opposite” to those for capacitors. We also need an expanded set of rules for “breaking up” combinations of devices:

- For devices connected in parallel: the voltage across them is always the same.
- For devices connected in series: the charge and current is always the same.

But otherwise, the procedure is the same!

(a) Can you explain the rules stated above in terms of physical principles? They have simple physical origins, so they’re easy to remember.

(b) Consider the circuit shown at right. All the resistors R_i have the same value R. Find the current I_4 through resistor R_4 (including its direction) and the electric potential V_A at the indicated point A. Be sure to express your answers in terms of the given parameters $\mathcal{E}$ (battery “EMF” = voltage) and R (resistance of each resistor).
The equivalent resistor would be \( R_{eq} = R + \frac{R \times R}{(R + R)} + R = \frac{5}{2} R \)

The current is then \( I_1 = \frac{E}{R_{eq}} = \frac{2E}{5R} \). The \( \Delta V \) across \( R_1 \) is then

\[ \Delta V = I_1 R = \frac{2}{5} E \]

\[ V_A = E - \Delta V = E - \frac{2}{5} E \rightarrow V_A = \frac{3}{5} E \]

The currents \( I_3 = I_4 \) since their voltage drops are equal \( \Delta V = R I_3 = R I_4 \)

From KCL \( I_1 = I_3 + I_4 = 2I_4 \rightarrow I_4 = \frac{E}{5R} \)

Method 2: Kirchoff’s Laws

This week, we will also practice a new and more powerful method for circuit analysis. The new method is based on Kirchoff’s two Laws:

- **Kirchoff’s Current Law (KCL):** The sum of the currents flowing into any node (junction) in the circuit equals the sum of the currents flowing out of that node.

- **Kirchoff’s Voltage Law (KVL):** The sum of the potential changes \( \Delta V \) around any loop is zero. And here are the rules for how you sum up those \( \Delta V \)’s as you go around a loop in the circuit:
  - Crossing a battery from the negative to the positive terminal = a rise → add +\( E \)
  - Crossing a battery from the positive to the negative terminal = a drop → add -\( E \)
  - Crossing a resistor in the same direction as the current flow = a drop → add -\( IR \)
  - Crossing a resistor against the direction of the current flow = a rise → add +\( IR \)

(c) Can you explain these rules on physical principles? (They’re relatively obvious, actually …)

The first is conservation of charge. The second is that the voltage is a function of position in the circuit hence the net change of voltage around a complete loop must be zero.

Now, let’s apply these laws to analyze the same circuit as on the previous page. Once more, your job is to find the current \( I_4 \) and the potential \( V_A \). Let’s go through the procedure in detail:

(d) Identify how many different currents there are in the circuit. These are often called branch currents. Label them on the figure, from \( I_1 \) to \( I_n \). Be sure to label them with arrows, indicating the direction of each current! If the directions you chose are wrong, no problem, your currents will simply come out with minus signs.
(e) Next, apply KCL to every node in the circuit. How many nodes are there in this circuit? Write down a KCL equation for each one.

2 nodes: \( I_1 = I_3 + I_4, \) \( I_3 + I_4 = I_2 \)

Also note that \( I_1 = I_2 \)

(f) Examine the equations you just wrote down … how many of them are independent? Here’s the rule: if you have \( N_{\text{nodes}} \) nodes in a circuit, KCL will only give you \( N_{\text{nodes}} - 1 \) independent equations. In other words, one of the equations gives you no new information. Use all but one of your equations to ‘get rid’ of as many of the unknown currents as possible, by writing expressions for them in terms of the remaining currents.

\( I_1 = I_2 \)
\( I_1 = I_3 + I_4 \)

(g) How many unknown currents are left? Let’s call this number \( n \). To determine these \( n \) currents, we will use KVL. Apply KVL to \( n \) loops in the circuit … this will give you the \( n \) equations you need to determine all the branch currents in the circuit.

Inner left loop \( E - I_1R_1 - I_3R_3 - I_2R_2 = 0 \) (Eq A)
Outside loop: \( E - I_1R_1 - I_4R_4 - I_2R_2 = 0 \) (Eq B)

(h) Solve your complete set of KCL and KVL equations! Remember, your job is to find \( I_4 \) and \( V_A \).

Subtract the two loop Eq. of previous part:

\[ \text{Eq A - Eq B: } \rightarrow 0 = I_1R_1 - I_3R_3 = (I_4 - I_3)R \rightarrow \frac{I_4}{2} = \frac{I_3}{2} \text{ from part (f)} \]

Substitute all currents in terms of \( I_4 \) into \( E - I_1R_1 - I_3R_3 - I_2R_2 = 0 \)

\( E = 2I_4R + I_4R + 2I_4R = 5I_4R \rightarrow \frac{I_4}{5} = \frac{E}{5R} \)

\( V_A = E - I_1R = E - 2I_4R = E - 2 \frac{E}{5R} \frac{R}{5} \rightarrow \frac{V_A}{5} = \frac{3E}{5} \)

For reference, here is a summary of the steps to take in analyzing a circuit using Kirchhoff’s Laws:

1. Identify all the different branch currents in the circuit, and label them with arrows to indicate direction. Let \( N_I \) be this number of currents.
2. Apply **KCL** to *all but one* of the *nodes* in the circuit. This will give you \((N_{\text{nodes}} - 1)\) equations.

3. Apply **KVL** to \(N_l - (N_{\text{nodes}} - 1)\) *loops* in the circuit. Together with your KCL equations, this gives you a total of \(N_l\) equations \(\rightarrow\) enough to determine all the different currents.

4. *Solve* your complete set of \(N_l\) equations to determine all the branch currents. Knowing that, and good old “\(V = IR\)”, you can calculate *everything* about your resistor network!
Kirchoff's Law Analysis of a Resistor Network

So we now have two methods for analyzing resistor networks. Which one should we use? (i) It depends on the particular problem, and (ii) often a mixture of methods is the best.

In general, method 1 (successive combination of elements) is easier, but method 2 (Kirchhoff's Laws) is more powerful ... and some circuits simply cannot be analyzed with method 1 alone.

The circuit shown at right is a typical example: here we have three batteries and they prevent us from "collapsing" any of the resistors in groups, as none of them are in simple series or parallel combinations. Our only chance is to use Kirchhoff's Laws!

Let all the resistors $R_i$ have resistance 10 $\Omega$, and let the battery EMF's be $E_1 = 10$ V, $E_2 = 20$ V, $E_3 = 30$ V.

Your tasks are:

(a) Find the current $I_3$ through resistor $R_3$. 

![Resistor Circuit Diagram]
We begin by writing all currents in terms of $I_1$ and $I_3$ and inserting all of the resistor and battery values.

We next write the loop equation for the left loop. Start at neg terminal and go clockwise.

$10 - 10I_1 - 10(I_1 - I_3) - 20 - 10I_1 = 0$

$\rightarrow -30I_1 + 10I_3 = 10$

Next the right loop: (neg term of 20 and clockwise)

$20 + 10(I_1 - I_3) - 10I_3 - 30 - 10I_3 = 0$

$\rightarrow 10I_1 - 30I_3 = 10$

Multiply right loop eq. by 3 and add to left loop eq.

$30I_1 - 90I_3 = 30$

$-30I_1 + 10I_3 = 10$

Hence $I_3 = -0.5A$

(b) Find the electric potential difference, $V_A - V_B$, for the points A and B indicated on the circuit.

Looking at the right most branch, we see $V_A$ is the 30V battery potential plus the voltage drop across $R_5$

or $V_A = 30 + 10I_3 = 30 - 10(0.5) = 25V$
In the previous question, you analyzed a “2-loop” circuit. In such cases, the linear algebra is relatively easy. But circuits of 3 loops or more are another story, the linear algebra can become really time consuming! Consider the circuit below. Your tasks are:

(a) Find the current $I_4$ through resistor $R_4$.

(b) Find the potential difference $V_B - V_A$.

This circuit looks like a truly horrible 3-loop affair … but it isn’t! With two batteries arranged as shown, we will need to use method 2 = Kirchoff’s Laws. But we can greatly simplify our lives by first applying method 1 = the “collapse” of series and parallel resistor combinations as much as possible.

Go for it! Solve for the quantities requested in (a) and (b) using a combination of methods 1 and 2.
In fact this circuit and is very simple. There are two unknown currents and they are easy to solve for. Lower loop from neg term of 20 V battery (clockwise)

\[ 20 - 5I_c + 10 = 0 \rightarrow I_c = 6 \]

Outside loop from neg term of 10 V (clockwise)

\[ 10 - 30 \cdot I_3 = 0 \rightarrow I_3 = 1/3 \]

\[ I_c = 6 = I_4 + I_5 = 2I_4 \rightarrow I_4 = 3 A \]

Moving from B to A we see that \( V_B - V_A \) is the voltage drop across \( R_3 \)

\[ \rightarrow V_B - V_A = 10 \Omega \times I_3 = \frac{10}{3} \text{ V} \]
Finally, for practice, here are two examples of real “3-loop” circuits. In these rather complex cases, your task is merely to (c) identify all the different branch currents and (d) write down enough equations to solve for all of them. You may assume that all resistors $R_i$ have the same value $R$. Solving the equations is just algebra … we won’t bother with that part here.

(c)

Answers will vary a lot for these last two problems.

An answer:

KCL equations:

$I_1 = I_2 + I_4$

$I_2 + I_6 = I_3$

$I_4 = I_6 + I_5$

with $I_1$ defined to go up, $I_6$ going from right to left, and the rest to flow downward.

KVL equations:

$E_1 - I_1R_1 - I_2R_2 - I_3R_3 = 0$

$E_1 - I_1R_1 - I_4R_4 - I_5R_5 = 0$

$E_1 - I_1R_1 - I_4R_4 + E_2 - I_6R_6 - I_3R_3 = 0$
(d)

KCL:
\[ I_1 = I_2 + I_4 \]
\[ I_2 = I_5 + I_3 \]
\[ I_6 = I_3 \]
\[ I_6 = I_9 \]
\[ I_8 = I_9 + I_5 \]
\[ I_7 = I_8 + I_4 \]

Currents on top go from left to right on bottom, from right to left downward for vertical connections

KVL:
\[ E_1 - I_1R_1 - I_4R_4 - E_2 - I_7R_7 = 0 \]
\[ E_1 - I_1R_1 - I_2R_2 - I_5R_5 - E_3 - I_8R_8 - I_7R_7 = 0 \]
\[ E_1 - I_1R_1 - I_2R_2 - I_3R_3 - I_6R_6 - E_4 - I_9R_9 - I_8R_8 - I_7R_7 = 0 \]