Lecture 12

Examples and Problems:
Law of Atmospheres, Global Warming

Boltzmann Distribution

If we have a system that is coupled to a heat reservoir at temperature T:

- The entropy of the reservoir decreases when the small system extracts energy E_n from it.
- Therefore, this will be less likely (fewer microstates).
- The probability for the small system to be in a particular state with energy E_n is given by the Boltzmann factor:

$$P_n = \frac{e^{-E_n/kT}}{Z}$$

where, $Z = \sum_n e^{-E_n/kT}$ to make $P_{tot} = 1$.

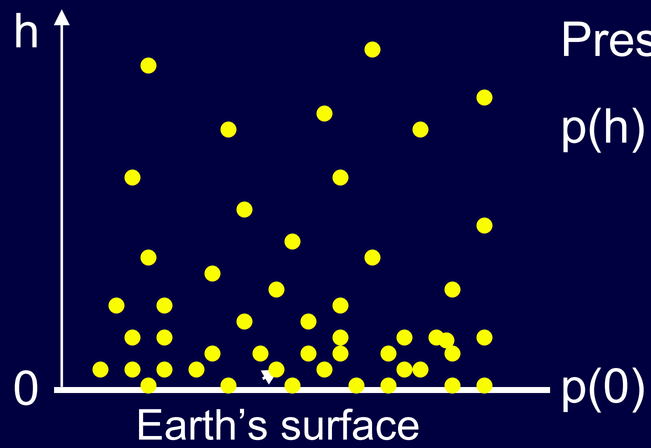
Z is called the "partition function".

$$P_n = \frac{d_n e^{-E_n/kT}}{\sum_n d_n e^{-E_n/kT}}$$

d_n = degeneracy of state n

The Law of Atmospheres

How does atmospheric pressure vary with height?



Quick Act: In equilibrium, how would T vary with height?

a) increase b) decrease c) constant

For every state of motion of a molecule at sea level, there's one at height h that's identical except for position. Their energies are the same except for mgh.

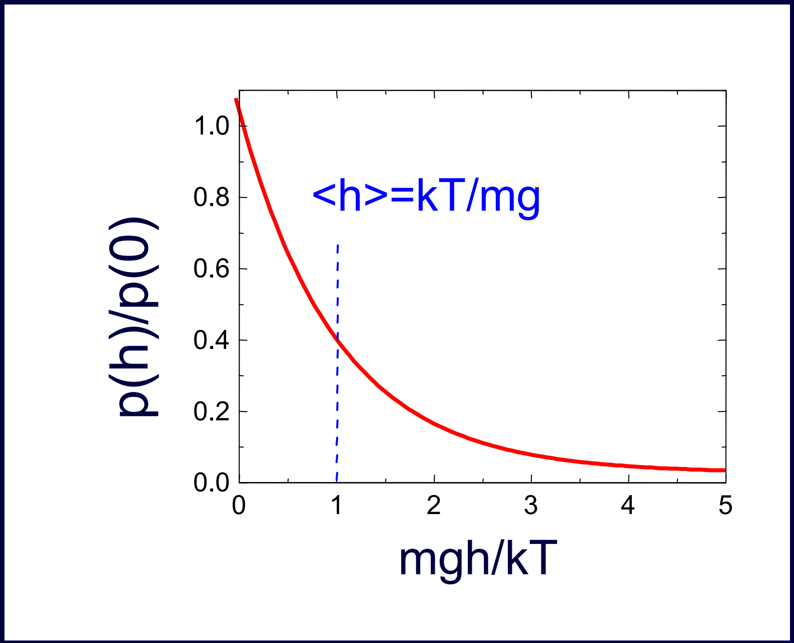
Therefore, the ratio of probabilities for those two states is just the Boltzmann factor.

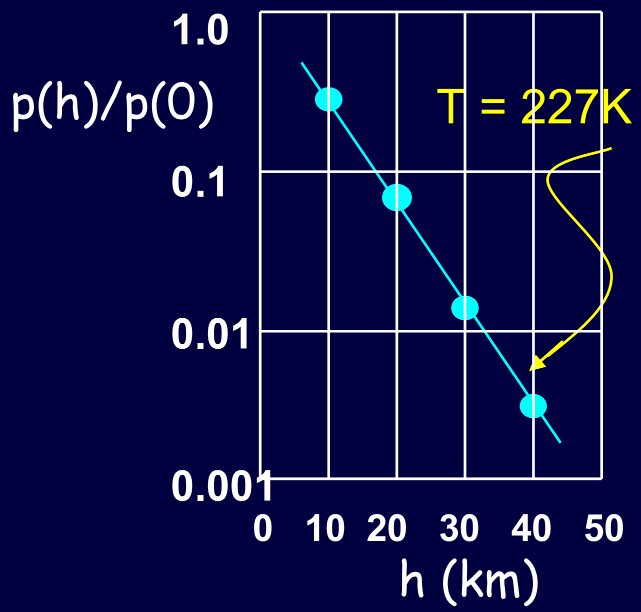
$$\frac{P(h)}{P(0)} = e^{-mgh/kT}$$

The ideal gas law, pV = NkT, tells us that this is also the ratio of pressures.
This is called the "law of atmospheres".

$$\frac{p(h)}{p(0)} = e^{-mgh/kT}$$

Atmosphere (2)





Actual data
(from Kittel, *Thermal Physics*)

Define a characteristic height, h_c:

$$\frac{p(h)}{p(0)} = e^{-mgh/kT} \equiv e^{-h/h_c}$$

where, h_c = kT/mg.

Note: m is the mass of one molecule.

From this semi-log plot,
$h_c \approx 7$ km is the height at which the atmospheric pressure drops by a factor of e.

Act 1

What is the ratio of atmospheric pressure in Denver (elevation 1 mi = 1609 m) to that at sea level? (Assume the atmosphere is N_2.)

a) 1.00 b) 1.22 c) 0.82

$$m = \frac{\text{molecular weight}}{N_A}$$

$$= \frac{28 \text{ g/mol}}{6.022x10^{23} \text{ molecules/mol}}$$

$$= 4.7x10^{-26} \text{ kg/molecule}$$

$$k = 1.38x10^{-23} \text{ J/K}$$

$$T = 273 \text{ K}$$

Solution

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$$\frac{p(1\,mile)}{p(sea\ level)} = \exp\left\{-\frac{4.7\times10^{-26}\ \text{kg}\cdot 9.8\ \text{m/s}^2\cdot 1600\ \text{m}}{1.38\times10^{-23}\,\text{J/K}\cdot 273\ \text{K}}\right\} = 0.822$$

Law of Atmospheres - Discussion

We have now quantitativeliy answered one of the questions that arose earlier in the course:

Which of these will "fly off into the air" and how far?

- O_2 about 7 km
- virus a few cm (if we ignore surface sticking)
- baseball much less than an atomic size

In each case, $h_c = kT/mg$.

Note: h_c is the average height $\langle h \rangle$ of a collection in thermal equilibrium.

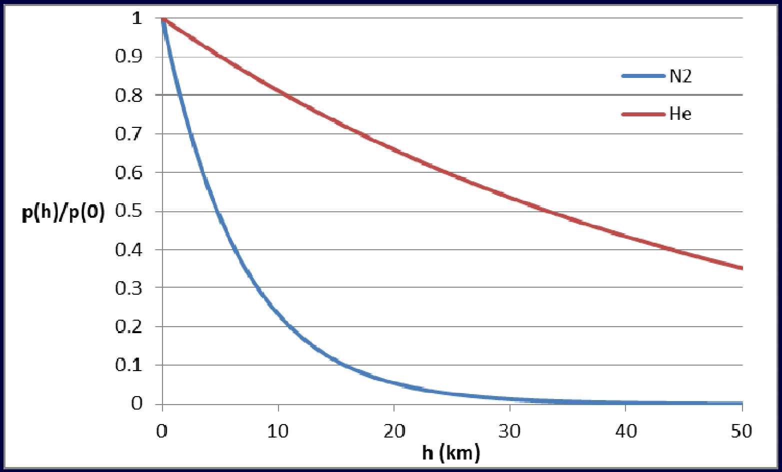
MicroACT:

Explain why the water in a glass won't just spontaneously jump out of the glass as a big blob, but does in fact spontaneously jump out, molecule by molecule.

Law of Atmospheres: Practical Implication – Helium shortage!!

Helium Shortage Has Balloon Sales Dropping - NationalJournal.com

MRI's need liquid He to cool the superconducting magnets…



He gas extends much farther up in the atmosphere. Although it's still gravitationally bound to earth, it does get high enough to be ionized by the sun's UV radiation, and then other processes sweep it away…

Exercise: Spin alignment

The magnetic moment of the electron is $\mu_B = 9.28\times10^{-24}$ J/T.

1) At room temperature (300 K), what magnetic field is required to make 2/3 of the electrons have their magnetic moments point along B (that's the low energy state)? Note: This is called the "up" state.

2) What B is needed for 2/3 of the protons ($\mu_p = 1.41\times10^{-26}$ J/T) to be that way?

Solution

The magnetic moment of the electron is $\mu_B = 9.28\times10^{-24}$ J/T.

1) At room temperature (300 K), what magnetic field is required to make 2/3 of the electrons have their magnetic moments point along B (that's the low energy state)? Note: This is called the "up" state.

If you take ratios of probabilities, the normalization factor cancels.

$$P_{up} / P_{down} = 2 = e^{-(E_{up}-E_{down})/kT} = e^{2\mu_B B/kT}$$

$$\ln 2 = 2\mu_B B / kT$$

$$B = kT \ln 2 / 2\mu_B = 150 \text{ T}$$

Typical MRI field ~ 2 T.

2) What B is needed for 2/3 of the protons ($\mu_p = 1.41\times10^{-26}$ J/T) to be that way?

μ_p is smaller than μ_B by a factor of 658, so B must be larger by that factor:

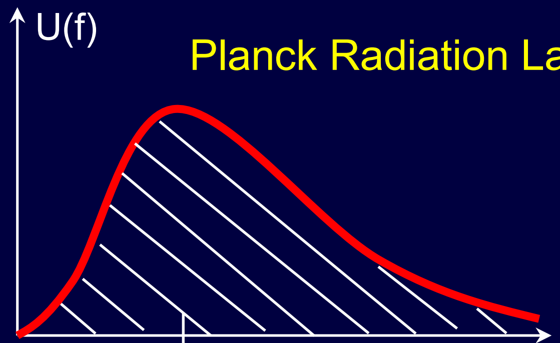
$B = 9.9\times10^4$ T.

We can make 150 T fields (not easily), but not 10^5 T.
That's as large as the field of some neutron stars.

Blackbody Radiation

The Planck law gives the spectrum of electromagnetic energy contained in modes with frequencies between f and f + Δf:

$$U(f) \propto \frac{f^3}{e^{hf/kT} - 1}$$



$$U(\lambda) \propto \frac{1}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

The peak wavelength:

$\lambda_{max} T = 0.0029$ m-K

This relation is known as Wien's Displacement law.

Integrating over all frequencies gives the total radiated energy per unit surface area.

The power radiated per unit surface area by a perfect radiator is:

$$J = \sigma_{SB} T^4$$

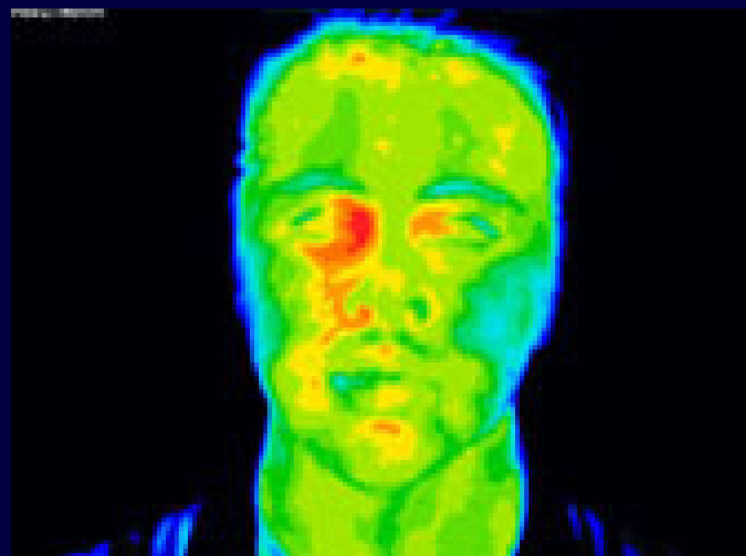
Stefan-Boltzmann Law of Radiation

$\sigma_{SB} = 5.670 \times 10^{-8}$ W m^{-2} K^{-4} "Stefan-Boltzmann constant"

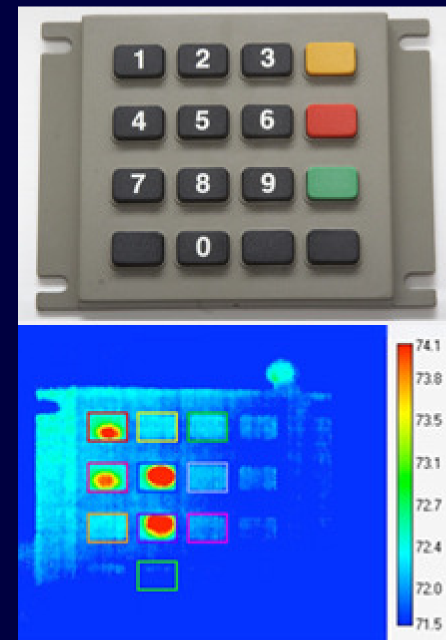
The total power radiated = J$\times$Area

Applications

Heat loss through windows



Infection of right eye and sinus



The code was 1485

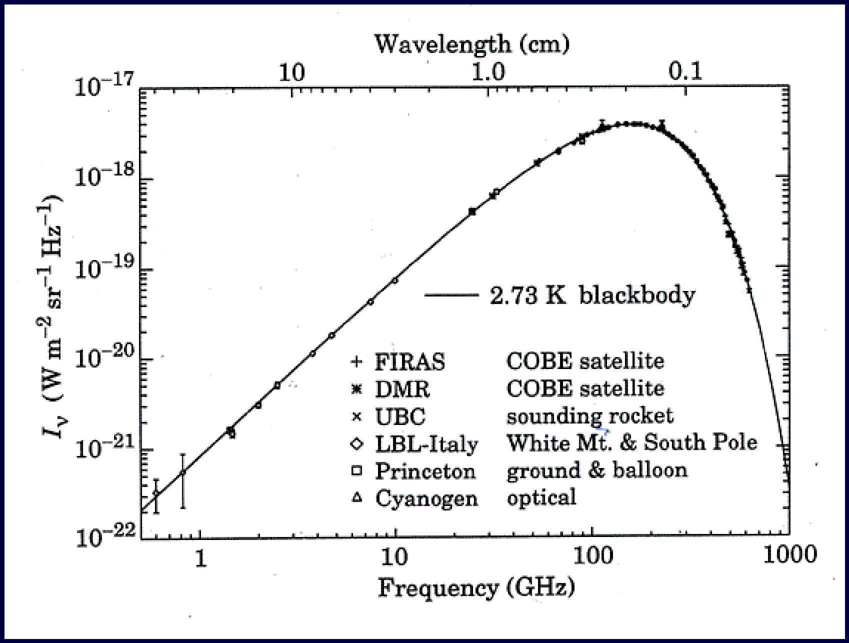
FYI:
What is the biggest black body? The entire Universe!

The universe started with a big bang – an incredibly rapid expansion involving immense densities of very hot plasma. After about 400,000 years, the plasma cooled and became transparent (ionized hydrogen becomes neutral when T ~ 3000 K). We can see the thermal radiation that was present at that time.

The universe has expanded and cooled since then, so what we see a lower T.

In 1965 Bell Labs researchers Penzias and Wilson found some unexplained *microwave noise* on their RF antenna. This noise turned out to be the cooled remnants of the black-body radiation. It has T = 2.73 K (f_{max} ~ 160 GHz).

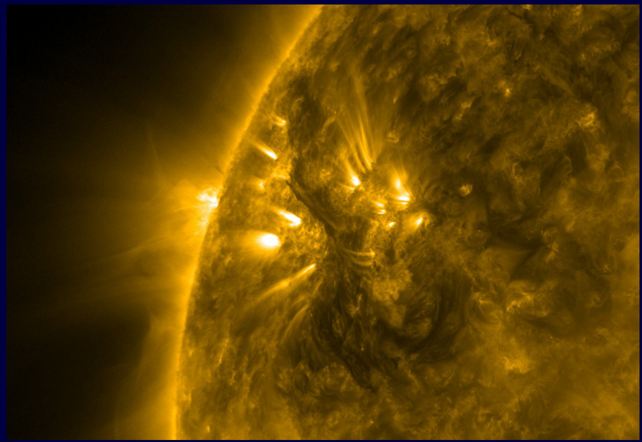
The Cosmic Microwave Background has the best black-body spectrum ever observed.



Act 2

The surface temperature of the sun is T ~ 6000 K. What is the wavelength of the peak emission?

a) 970 nm (near infrared)
b) 510 nm (green)
c) 485 nm (blue)



http://apod.nasa.gov/apod/ap100522.html

Solution

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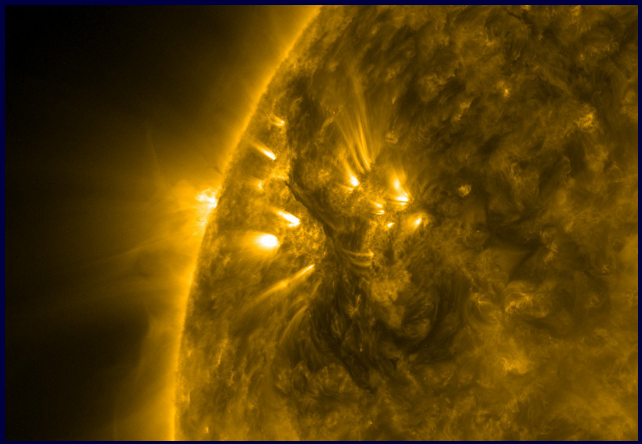


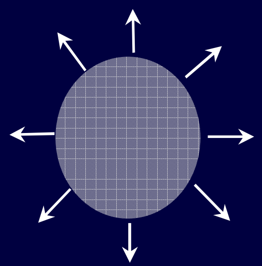
http://apod.nasa.gov/apod/ap100522.html

λ_{max} = 0.0029 m-K / 6000 K
= 4.83 x 10^{-7} m
= 483 nm

Note: If you can measure the spectrum, you can infer the temperature of distant stars.

Exercise: Thermal Radiation

Calculate the power radiated by a 10-cm-diameter sphere of aluminum at room temperature (20° C). (Assume it is a perfect radiator.)



Solution

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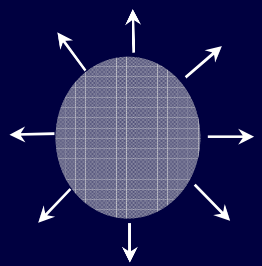


$$J = \sigma_{SB} T^4 = (5.670 \times 10^{-8}\ \text{W m}^{-2}\ \text{K}^{-4})(293\ \text{K})^4$$

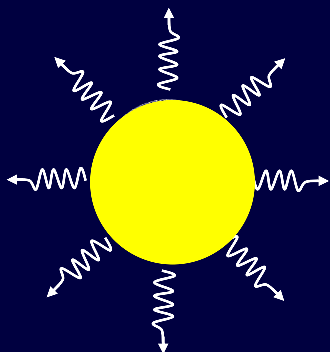
$$= 418\ \text{W/m}^2 \quad \text{(power radiated per area)}$$

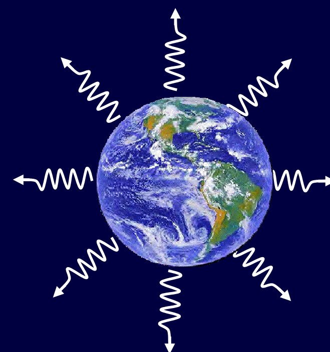
$$A = 4\pi r^2 = 4\pi(5 \times 10^{-2}\ \text{m})^2 = 3.14 \times 10^{-2}\ \text{m}^2$$

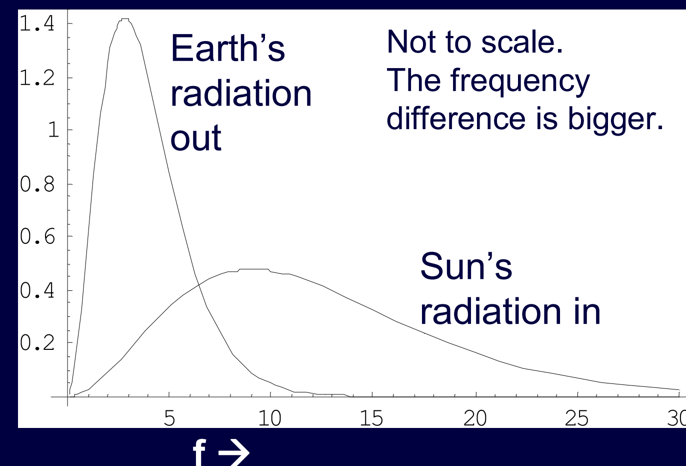
$$\text{Power} = J \times A = (418\ \text{W/m}^2)(3.14 \times 10^{-2}\ \text{m}^2) = 13\ \text{Watts}$$

Home exercise: Calculate how much power your body (at T = 310 K) is radiating. If you ignore the inward flux at T = 293 K from the room, the answer is roughly 1000 W! (For comparison, a hair dryer is about 2000 W.) However, if you subtract off the input flux, you get a net of about 200 W radiated power.

Application:
The Earth's Temperature



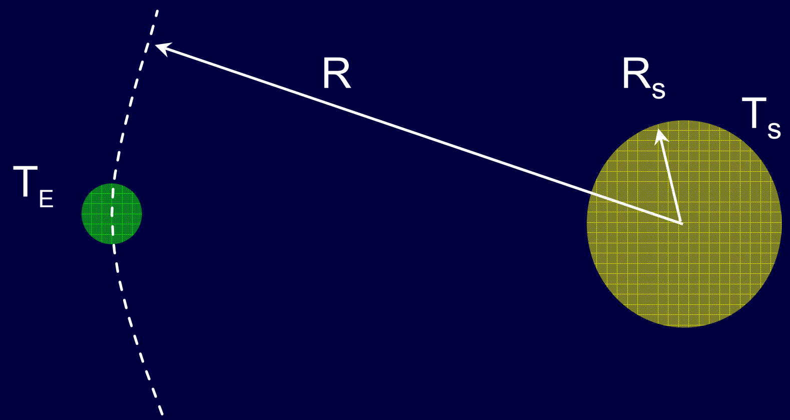




The Earth's temperature remains approximately constant, because the thermal radiation it receives from the Sun is balanced by the thermal radiation it emits.

This works because, although the Sun is much hotter (and therefore emits much more energy), the Earth only receives a small fraction.

The Earth's Temperature (2)



J_S = Sun's flux at its surface = $\sigma_{SB}T_S^4$

J_R = Sun's flux at the Earth = $\sigma_{SB}T_S^4(R_S/R)^2$

J_E = Earth's flux at its surface = $\sigma_{SB}T_E^4$

$R_S = 7\times10^8$ m

$R = 1.5\times10^{11}$ m

$T_S = 5800$ K

Balance the energy flow:

Power absorbed from sun = Power radiated by earth

$$J_R\left(\pi R_E^2\right) = J_E\left(4\pi R_E^2\right)$$

$$J_R = 4J_E$$

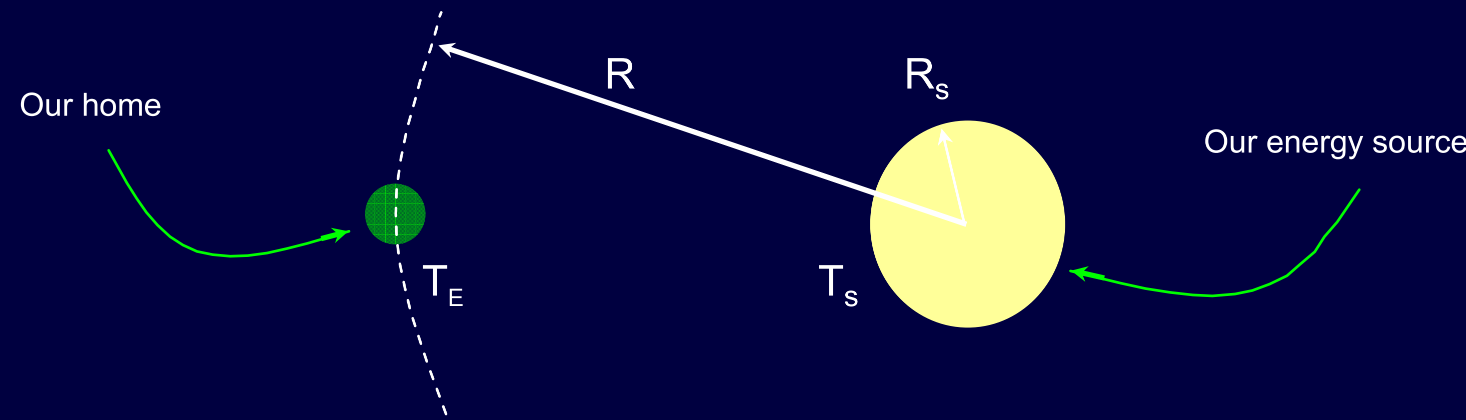
$$\sigma_{SB}T_S^4\left(\frac{R_S}{R}\right)^2 = 4\sigma_{SB}T_E^4$$

$$T_E = \left(\frac{R_S}{2R}\right)^{1/2}T_S$$

$$= 280\ \text{K}$$

~ room temperature!

The Earth's Temperature (3)



Last lecture, we did not account for the fact that about 30% of the sun's radiation reflects off our atmosphere! (The planet's "albedo".)

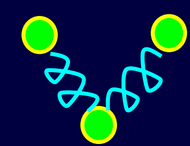
30% less input from the sun means about 8% lower temperature because $T \propto P^{1/4}$. This factor reduces our best estimate of the Earth's temperature by about 30 K. Now T_E = 250 K, or 0˚ F. Brrr!

In fact the average surface temperature of the Earth is about 290 K, or about 60° F, just right for Earthly life. (coincidence? I think not.)

The extra 60° F of warming is mainly due to the "greenhouse effect", the fact that some of the radiation from the Earth is reflected back by the atmosphere. What exactly is the "greenhouse effect"?

Remember Molecular vibrations:

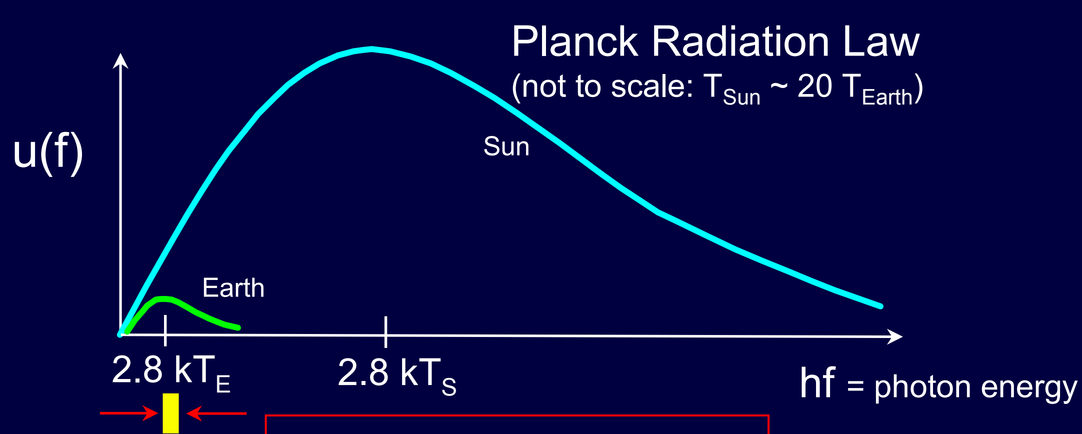
Polyatomic molecules (e.g., CO_2, H_2O, CH_4) have rotational and vibrational modes that correspond to photon wavelengths (energies) in the infrared. These molecules absorb (and emit) IR radiation much more effectively than O_2 and N_2.



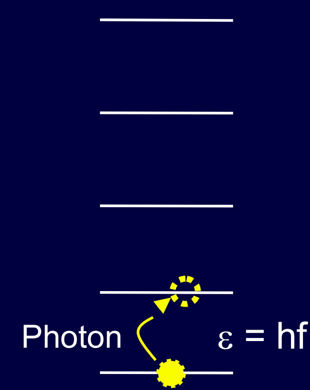
Strong absorption in the infrared.
(rotational and vibrational motions)

This absorption is in the middle of the Earth's thermal spectrum, but in the tail of the Sun's.

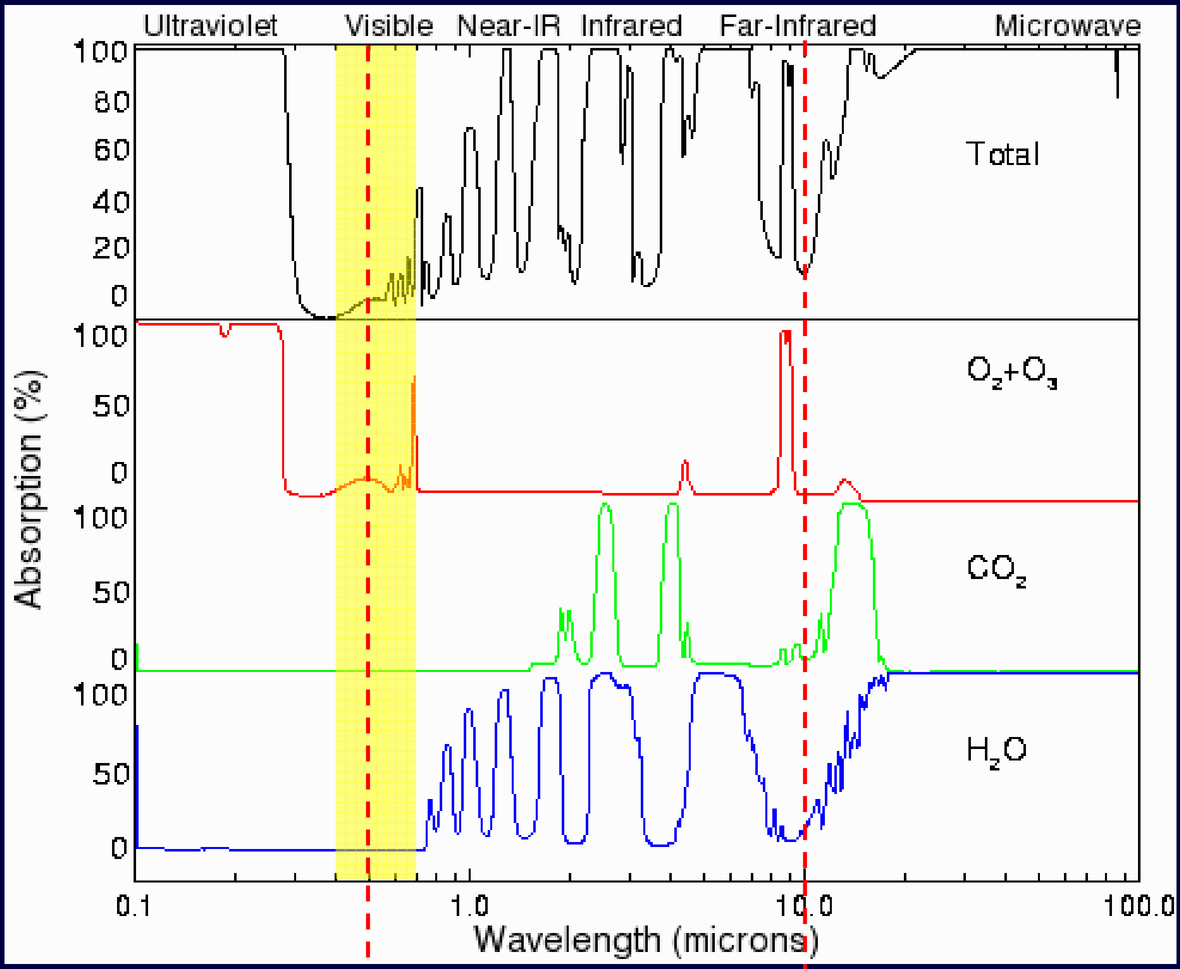
The result is that our atmosphere lets most of the sunlight through, but absorbs a larger fraction of the radiation that the Earth emits.



Greenhouse gas absorption
(Infrared)



Some Absorption Spectra

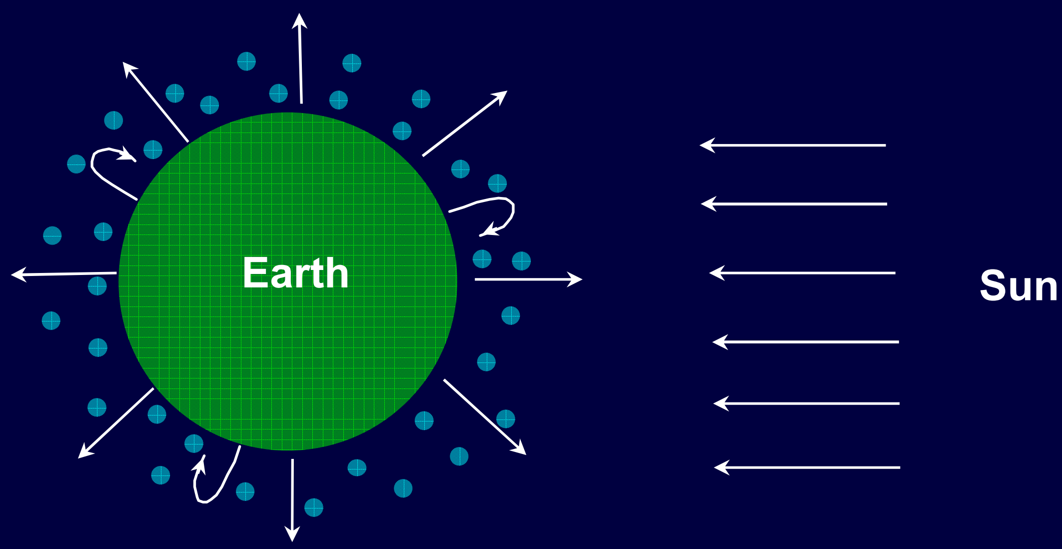


Water is the most important greenhouse gas.

Peak wavelength of Sun's thermal spectrum

Peak wavelength of Earth's thermal spectrum

The Greenhouse Effect



Thermal radiation from the earth (infrared) is absorbed by certain gases in our atmosphere (such as CO_2) and redirected back to earth.

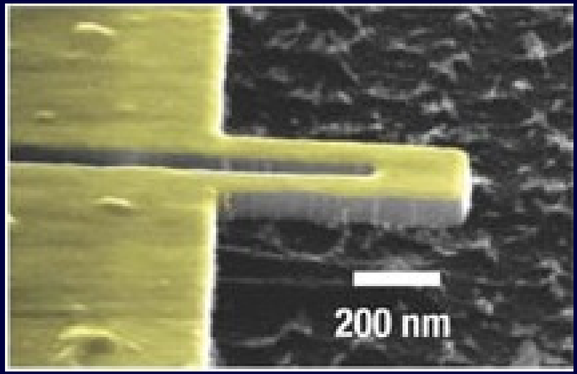
These 'greenhouse gases' provide additional warming to our planet - essential for life as we know it.

Radiation from the sun is not affected much by the greenhouse gases because it has a much different frequency spectrum.

Mars has a thin atmosphere with few greenhouse gases: 70˚ F in the day and -130° F at night. Venus has lots of CO_2 : T = 800° F

Act 3

Very sensitive mass measurements (10^{-18} g sensitivity) can be made with nanocantilevers, like the one shown. This cantilever vibrates with a frequency, f = 127 MHz. FYI: h = 6.6×10^{-34} J-s and k = 1.38×10^{-23} J/K



Li, *et al.*, Nature Nanotechnology **2,** p114 (2007)

1) What is the spacing, ε, between this oscillator's energy levels?

a) $\varepsilon = 6.6\times10^{-34}$ J b) $\varepsilon = 8.4\times10^{-26}$ J c) $\varepsilon = 1.4\times10^{-23}$ J

2) At what temperature, T, will equipartition fail for this oscillator?

a) T = 8.4×10^{-26} K b) T = 6.1×10^{-3} K c) T = 295 K

Solution

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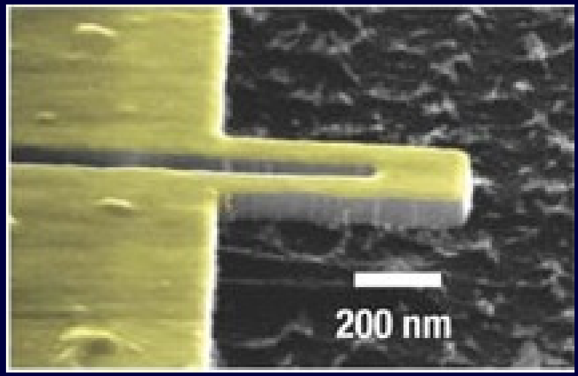

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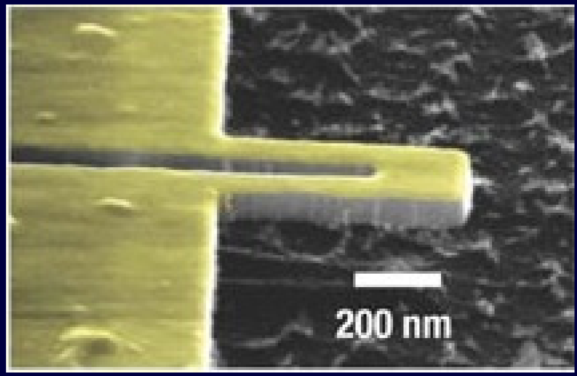

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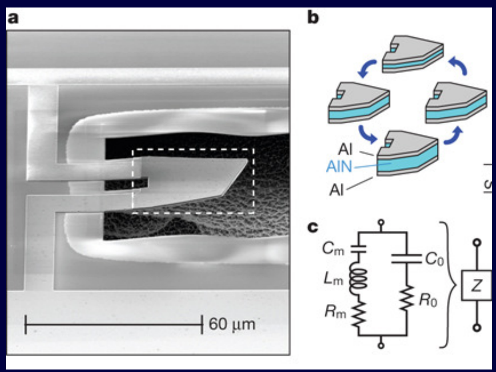
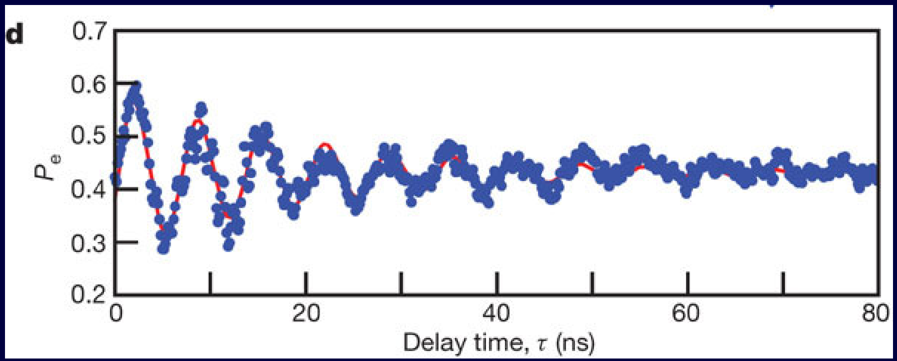
$$T = \varepsilon/k = (8.38\times10^{-26}\text{ J})/(1.38\times10^{-23}\text{ J/K}) = 6.1\times10^{-3}\text{ K}$$

FYI: Recent Physics Milestone!

There has been a race over the past ~20 years to put a ~macroscopic object into a quantum superposition. The first step is getting the object into the ground state, below all thermal excitations. This was achieved for the first time in 2010, using vibrations in a small “drum” :





“Quantum ground state and single-phonon control of a mechanical resonator”,
A. D. O’Connell, et al., *Nature* **464**, 697-703 (1 April 2010)

Quantum mechanics provides a highly accurate description of a wide variety of physical systems. However, a demonstration that quantum mechanics applies equally to macroscopic mechanical systems has been a long-standing challenge... Here, using conventional cryogenic refrigeration, we show that **we can cool a mechanical mode to its quantum ground state** by using a microwave-frequency mechanical oscillator—a ‘quantum drum’... We further show that **we can controllably create single quantum excitations (phonons) in the resonator**, thus taking the first steps to complete quantum control of a mechanical system.