Lecture 12

Examples and Problems:
Law of Atmospheres, Thermal Radiation

Reading: Elements Ch. 9

Boltzmann Distribution

If we have a system that is coupled to a heat reservoir at temperature T:

- The entropy of the reservoir decreases when the small system extracts energy E_n from it.
- Therefore, this will be less likely (fewer microstates).
- The probability for the small system to be in a particular state with energy E_n is given by the Boltzmann factor:

$$\boxed{P_n = \frac{e^{-E_n/kT}}{Z}}$$

where, $Z = \sum_n e^{-E_n/kT}$ to make $P_{tot} = 1$.

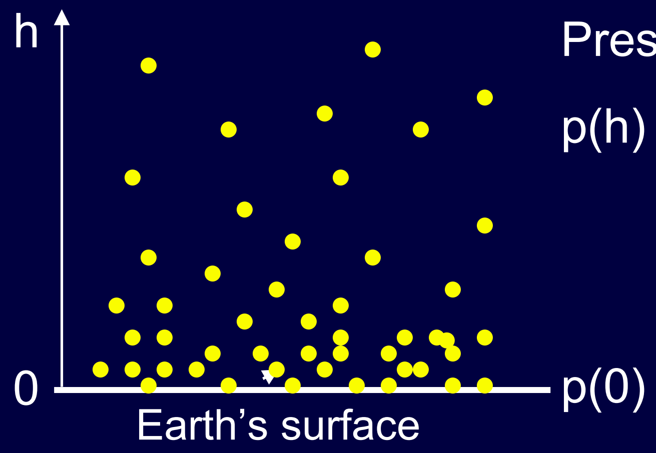
Z is called the “partition function”.

$$\boxed{P_n = \frac{d_n e^{-E_n/kT}}{\sum_n d_n e^{-E_n/kT}}}$$

d_n = degeneracy of state n

The Law of Atmospheres

How does atmospheric pressure vary with height?



Quick Act: In equilibrium,
how would T vary with height?

a) increase b) decrease c) constant

For every state of motion of a molecule at sea level,
there's one at height h that's identical except for position.
Their energies are the same except for mgh.

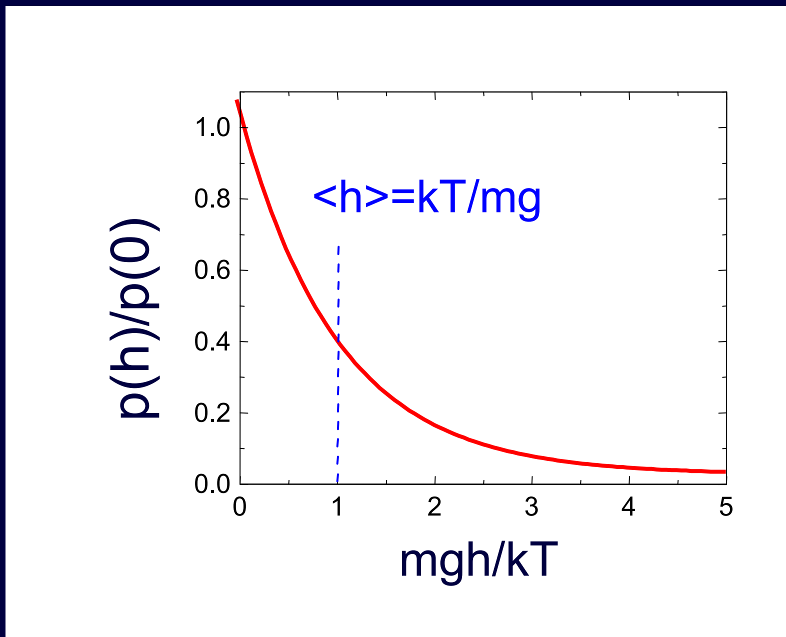
Therefore, the ratio of probabilities for those two states is just the Boltzmann factor.

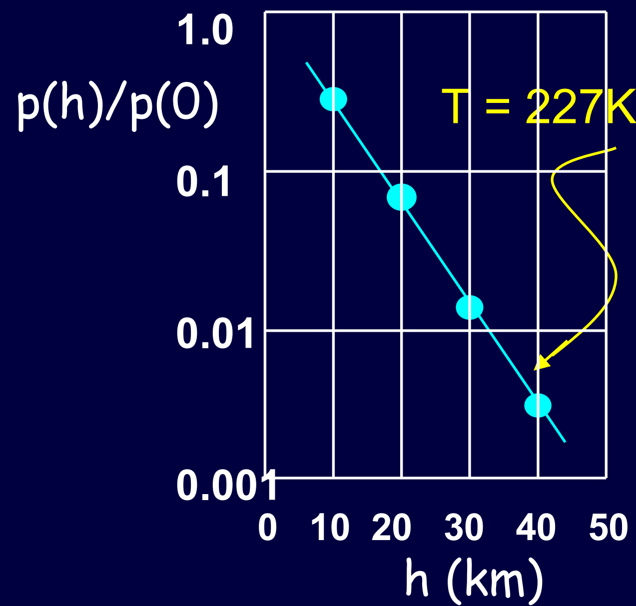
$$\frac{P(h)}{P(0)} = e^{-mgh/kT}$$

The ideal gas law, pV = NkT, tells us that this is also the ratio of pressures.
This is called the "law of atmospheres".

$$\frac{p(h)}{p(0)} = e^{-mgh/kT}$$

Atmosphere (2)





Actual data
(from Kittel, *Thermal Physics*)

Define a characteristic height, h_c:

$$\frac{p(h)}{p(0)} = e^{-mgh/kT} \equiv e^{-h/h_c}$$

where, $h_c = kT/mg$.

Note: m is the mass of one molecule.

From this semi-log plot,
$h_c \approx 7$ km is the height at which the atmospheric pressure drops by a factor of e.

Act 1

What is the ratio of atmospheric pressure in Denver (elevation 1 mi = 1609 m) to that at sea level? (Assume the atmosphere is N_2.)

a) 1.00 b) 1.22 c) 0.82

$$m = \frac{\text{molecular weight}}{N_A}$$

$$= \frac{28\ \text{g/mol}}{6.022x10^{23}\ \text{molecules/mol}}$$

$$= 4.7x10^{-26}\ \text{kg/molecule}$$

$$k = 1.38x10^{-23}\ \text{J/K}$$

$$T = 273\ \text{K}$$

Solution

What is the ratio of atmospheric pressure in Denver (elevation 1 mi = 1609 m) to that at sea level? (Assume the atmosphere is N_2.)

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$$k = 1.38x10^{-23}\ \text{J/K}$$

$$T = 273\ \text{K}$$

$$\frac{p(1\,mile)}{p(sea\ level)} = \exp\left\{-\frac{4.7\times10^{-26}\ \text{kg}\cdot 9.8\ \text{m/s}^2\cdot 1600\ \text{m}}{1.38\times10^{-23}\ \text{J/K}\cdot 273\ \text{K}}\right\} = 0.822$$

Law of Atmospheres - Discussion

We have now quantitativeliy answered one of the questions that arose earlier in the course:

Which of these will “fly off into the air” and how far?

- O_2 about 7 km
- virus a few cm (if we ignore surface sticking)
- baseball much less than an atomic size

In each case, $h_c = kT/mg$.

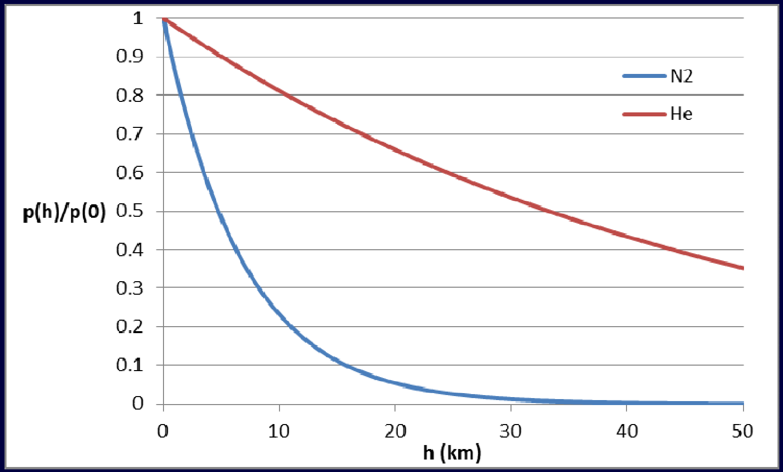
Note: h_c is the average height $\langle h \rangle$ of a collection in thermal equilibrium.

MicroACT:
Explain why the water in a glass won’t just spontaneously jump out of the glass as a big blob, but does in fact spontaneously jump out, molecule by molecule.

Law of Atmospheres: Practical Implication - Helium shortage!!

Helium Shortage Has Balloon Sales Dropping - NationalJournal.com

MRI's need liquid He to cool the superconducting magnets…

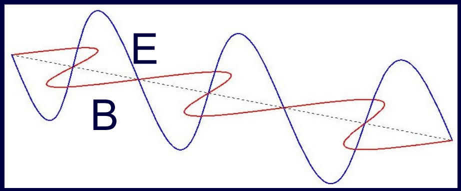


He gas extends much farther up in the atmosphere. Although it's still gravitationally bound to earth, it does get high enough to be ionized by the sun's UV radiation, and then other processes sweep it away…

MIDTERM MATERIAL
ENDS HERE

Basics of Thermal Radiation

Every object in thermal equilibrium emits (and absorbs) electromagnetic (EM) waves from its surface. (It glows.) How much? What colors?

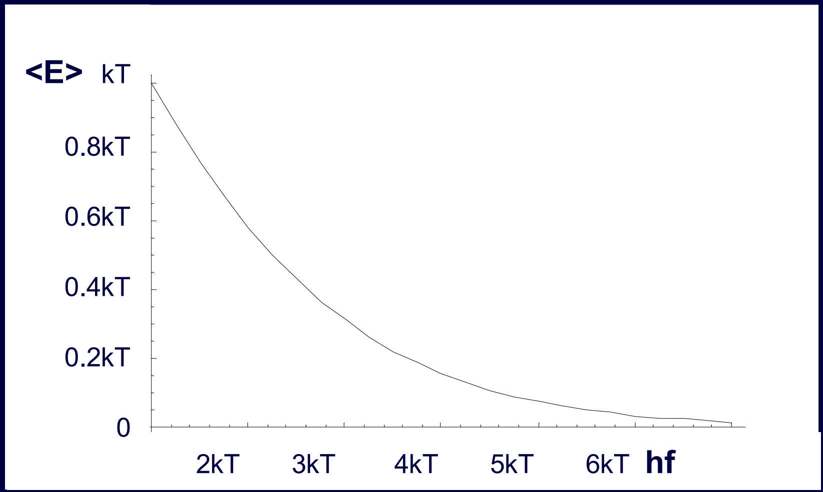


For our purposes, the important feature of EM waves is that they oscillate with a frequency, f, just like a mechanical oscillator. Therefore, the energy of an EM wave is a multiple of ε = hf, just like a mechanical oscillator.
Note: Each of these packets of energy ε=hf is called a “photon”.

This means that in thermal equilibrium:

The average energy of an EM wave of frequency f is the same as the average energy of a mechanical oscillator with the same f:

$$\langle E \rangle = \frac{\varepsilon}{e^{\varepsilon / kT} - 1} = \frac{hf}{e^{hf/kT} - 1}$$



Low frequency modes (such that ε = hf << kT) satisfy equipartition. They have <E> = kT. High frequency modes do not.

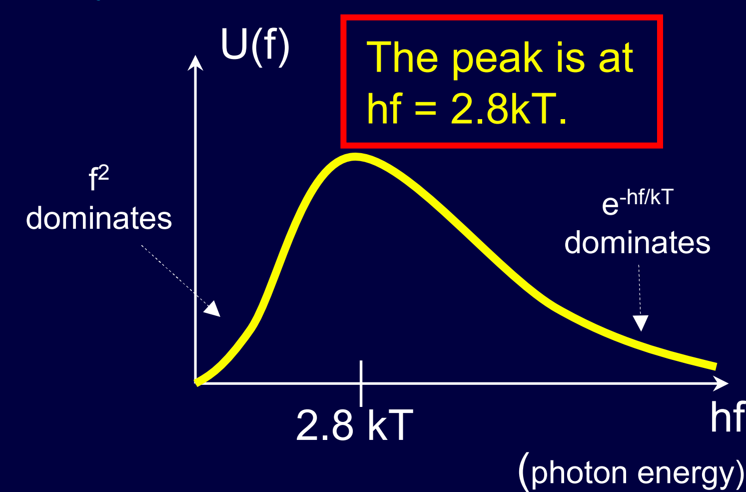
Planck Radiation Law
“Black Body Radiation”

The calculation of <E> on the previous slide is for each mode (specific f). However, what we really want to know is how much energy there is per frequency interval. The more frequency modes there are near a particular frequency, the brighter the object is at that frequency. This is similar to the degeneracy effect from last lecture.

The density of frequency modes is proportional to f^2. (See “Elements” for the derivation)
So, the EM radiation intensity as a function of frequency is:

$$U(f) \propto \frac{hf}{e^{hf/kT} - 1} f^2$$

Energy per frequency interval — Average energy per mode — Number of modes per frequency interval



This formula applies to almost any hot object, *i.e.*, it doesn’t matter if it’s hot gas on the sun, or the filament of a tungsten lamp.

Dependence of Color on Temperature

You can also write the Planck distribution in terms of power/unit wavelength, U(λ), instead of power/unit frequency, U(f). The energy distribution varies as:

$$U(\lambda) \propto \frac{1}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}$$

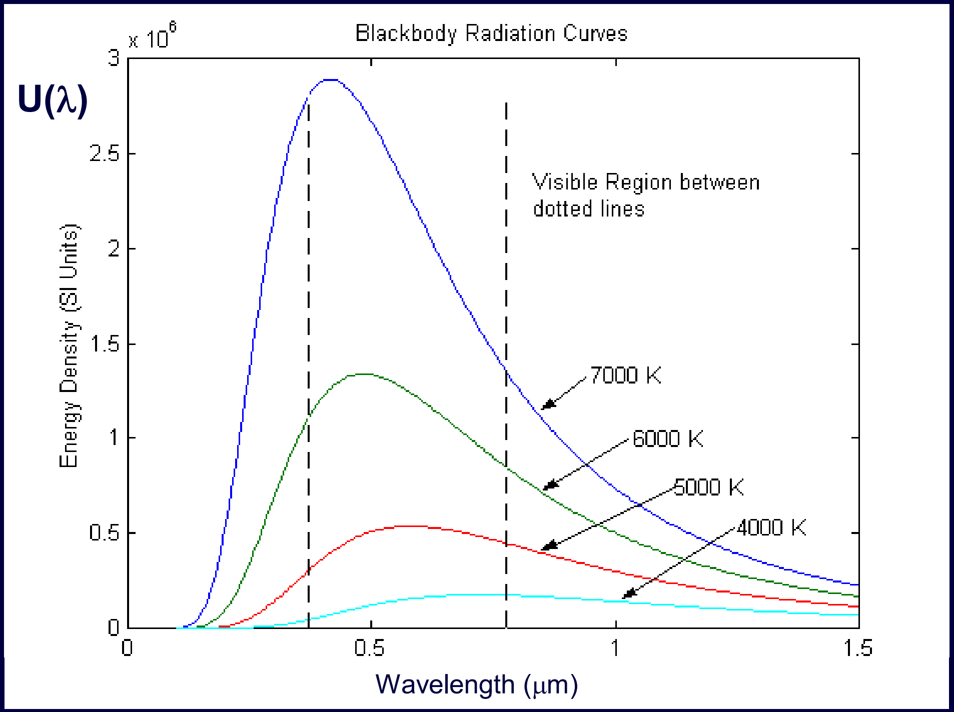
$f = c/\lambda \Rightarrow df \propto d\lambda/\lambda^2$

(The - sign doesn't matter.)

The peak wavelength:

$$\boxed{\lambda_{max} T = 0.0029 \text{ m-K}}$$

This relation is known as Wien's Displacement law.

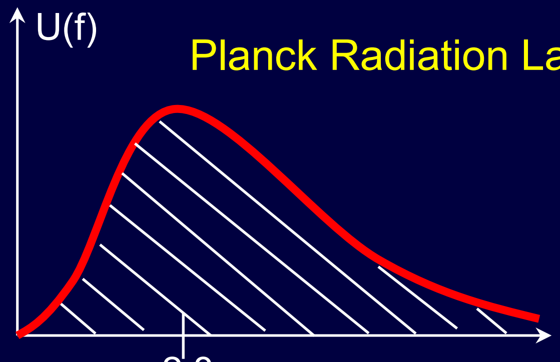


Where most burning is occurring, the fire is white, the hottest color possible for organic material in general, or yellow. Above the yellow region, the color changes to orange, which is cooler, then red, which is cooler still.

Blackbody Radiation

The Planck law gives the spectrum of electromagnetic energy contained in modes with frequencies between f and f + Δf:

$$U(f) \propto \frac{f^3}{e^{hf/kT} - 1}$$



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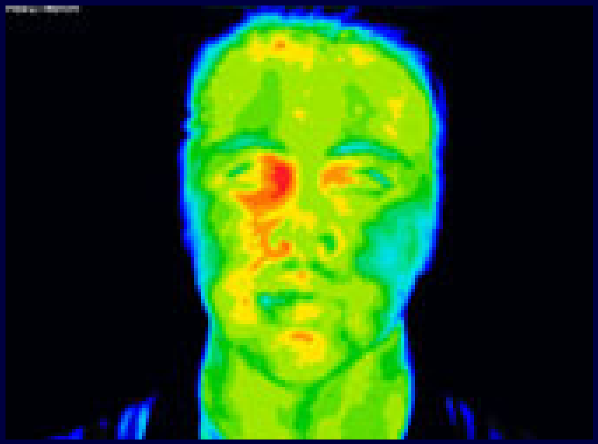
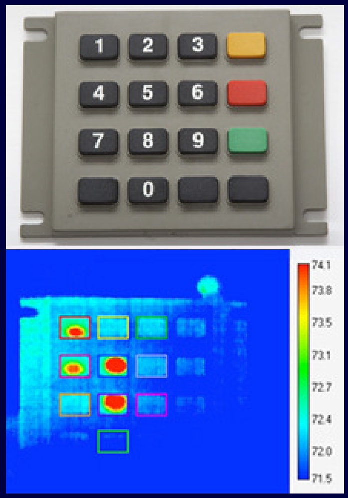
Heat loss through windows



Infection of right eye and sinus



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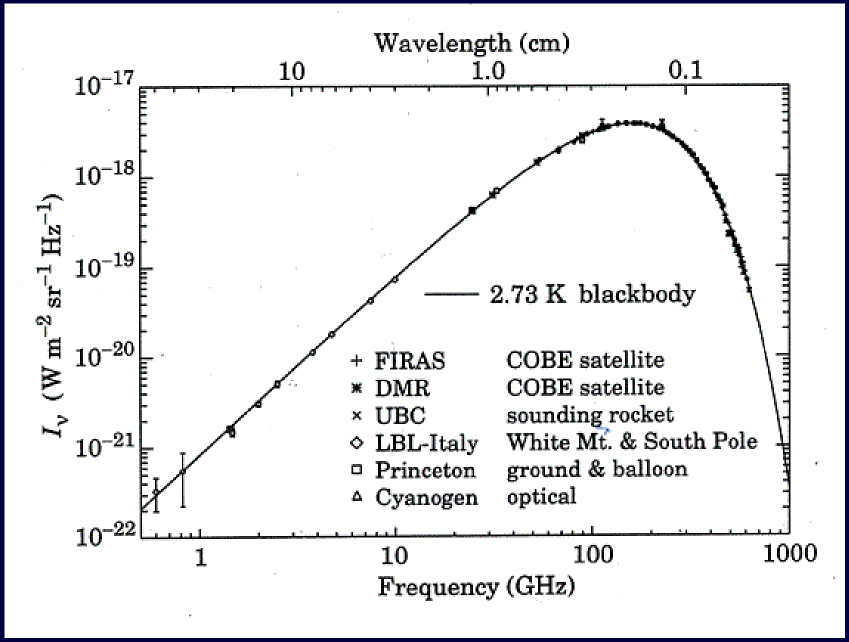
FYI:
What is the biggest black body? The entire Universe!

The universe started with a big bang – an incredibly rapid expansion involving immense densities of very hot plasma. After about 400,000 years, the plasma cooled and became transparent (ionized hydrogen becomes neutral when $T \sim 3000$ K). We can see the thermal radiation that was present at that time.

The universe has expanded and cooled since then, so what we see a lower T.

In 1965 Bell Labs researchers Penzias and Wilson found some unexplained *microwave noise* on their RF antenna. This noise turned out to be the cooled remnants of the black-body radiation. It has $T = 2.73$ K ($f_{max} \sim 160$ GHz).

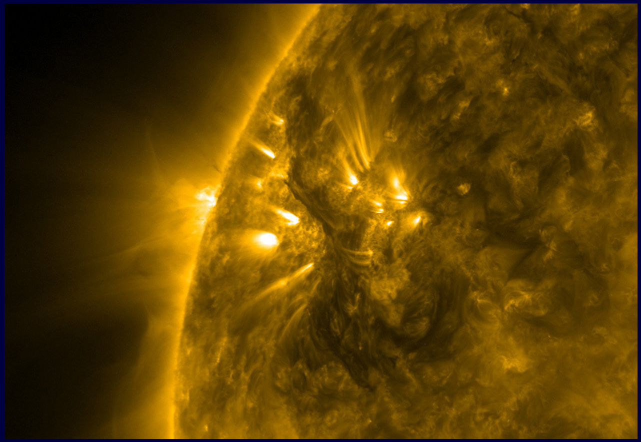
The Cosmic Microwave Background has the best black-body spectrum ever observed.



Act 2

The surface temperature of the sun is T ~ 6000 K. What is the wavelength of the peak emission?

a) 970 nm (near infrared)
b) 510 nm (green)
c) 485 nm (blue)



http://apod.nasa.gov/apod/ap100522.html

Solution

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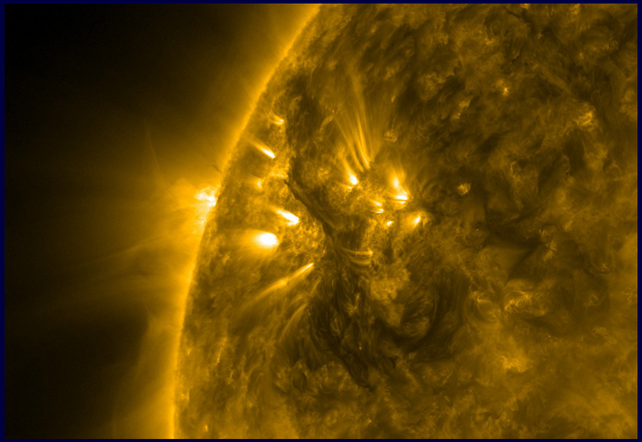


http://apod.nasa.gov/apod/ap100522.html

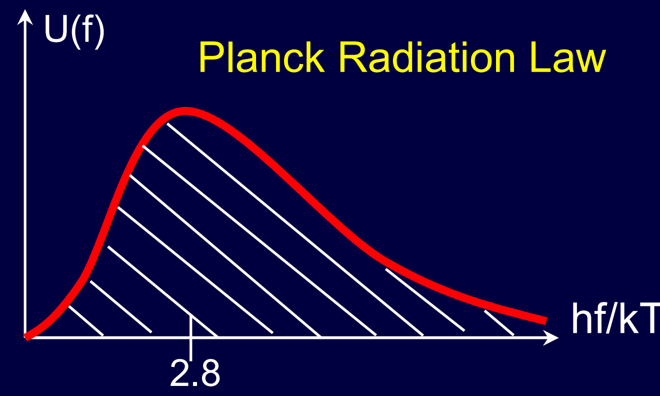
$$\lambda_{max} = 0.0029 \text{ m-K} / 6000 \text{ K}$$
$$= 4.83 \times 10^{-7} \text{ m}$$
$$= 483 \text{ nm}$$

Note: If you can measure the spectrum, you can infer the temperature of distant stars.

What is the Total Energy Radiated?

The Planck law gives the spectrum of electromagnetic energy contained in modes with frequencies between f and f + Δf:

$$U(f) \propto \frac{f^3}{e^{hf/kT} - 1}$$



Integrating over all frequencies gives the total radiated energy per unit surface area:

$$\int_0^\infty U(f)\,df \propto \int_0^\infty \frac{f^3}{e^{hf/kT} - 1}\,df = \left(\frac{kT}{h}\right)^4 \boxed{\int_0^\infty \frac{x^3}{e^x - 1}\,dx} \qquad x = hf/kT$$

Just a number: $\pi^4/15$

The power radiated per unit surface area by a perfect radiator is:

$$\boxed{J = \sigma_{SB} T^4}$$

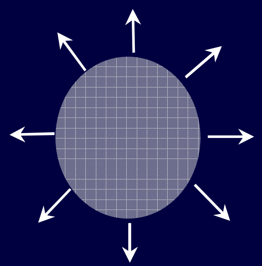
Stefan-Boltzmann Law of Radiation

$\sigma_{SB} = 5.670\times10^{-8}$ W m^{-2} K^{-4} Stefan-Boltzmann constant

The total power radiated = J×Area

Exercise: Thermal Radiation

Calculate the power radiated by a 10-cm-diameter sphere of aluminum at room temperature (20° C). (Assume it is a perfect radiator.)



Solution

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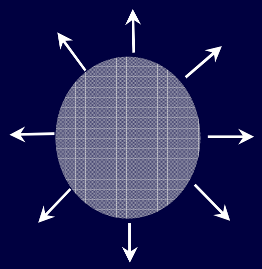


$$J = \sigma_{SB} T^4 = (5.670 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4})(293 \text{ K})^4$$

$$= 418 \text{ W/m}^2 \quad \text{(power radiated per area)}$$

$$A = 4\pi r^2 = 4\pi (5 \times 10^{-2} \text{ m})^2 = 3.14 \times 10^{-2} \text{ m}^2$$

$$\text{Power} = J \times A = (418 \text{ W/m}^2)(3.14 \times 10^{-2} \text{ m}^2) = 13 \text{ Watts}$$

Home exercise: Calculate how much power your body (at T = 310 K) is radiating. If you ignore the inward flux at T = 293 K from the room, the answer is roughly 1000 W! (For comparison, a hair dryer is about 2000 W.) However, if you subtract off the input flux, you get a net of about 200 W radiated power.

Not all Bodies are Black

Real materials are not truly “black” (*i.e.*, they don’t completely absorb all wavelengths). The fraction absorbed is called absorbance, which is equal* to its emissivity, *e*, a dimensionless number $0 \leq e \leq 1$ that depends on the properties of the surface.

$e = 1$ for an ideal emitter (an ideal blackbody absorber).

$e = 0$ for something that doesn’t emit (or absorb) at all, i.e., a perfect reflector.

Modified Stefan-Boltzmann Law of Radiation:

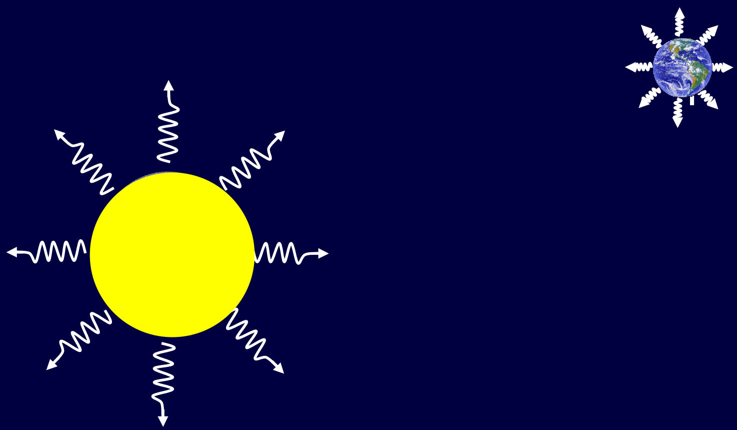
$$J = e\sigma_{SB}T^4$$

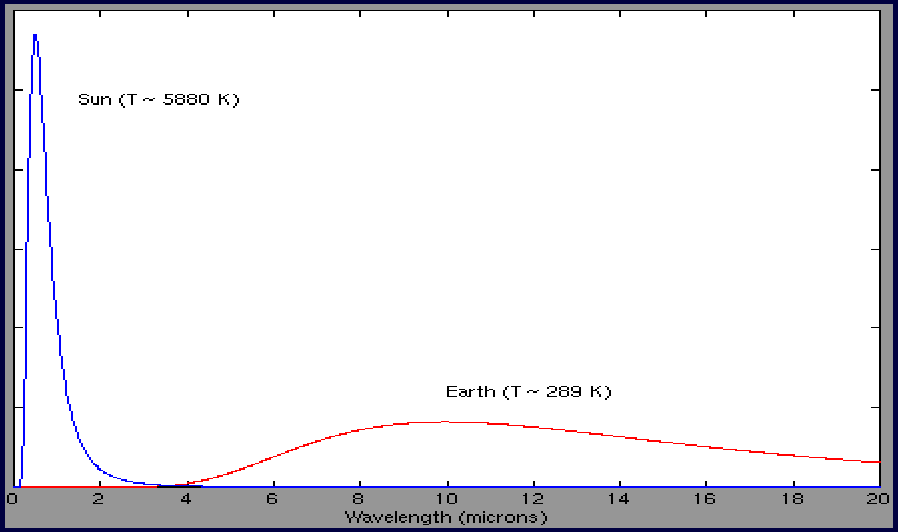
<u>Typical emissivities (300 K):</u>

gold, polished	0.02
aluminum, anodized	0.55
white paper	0.68
brick	0.93
soot	0.95
skin (!)	0.98

*If this equality didn’t hold, we wouldn’t have thermal equilibrium.

Application:
The Earth's Temperature



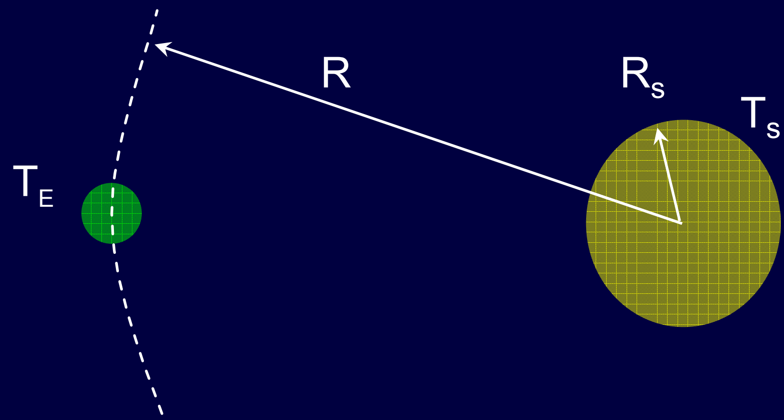


Note: The magnitude of the Earth curve has been magnified 500,000 times

The Earth's temperature remains approximately constant, because the thermal radiation it receives from the Sun is balanced by the thermal radiation it emits.

This works because, although the Sun is much hotter (and therefore emits much more energy), the Earth only receives a small fraction.

The Earth's Temperature (2)



$R_S = 7\times10^8$ m

$R = 1.5\times10^{11}$ m
$T_S = 5800$ K

J_S = Sun's flux at its surface = $\sigma_{SB} T_S^4$

J_R = Sun's flux at the Earth = $\sigma_{SB} T_S^4 (R_S/R)^2$

J_E = Earth's flux at its surface = $\sigma_{SB} T_E^4$

Balance the energy flow:
Power absorbed from sun = Power radiated by earth → T_E = 280 K (~room temperature!)

But wait! We did not account for the fact that about 30% of the sun's radiation reflects off our atmosphere! (The planet's "albedo".)

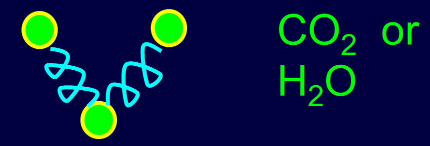
30% less input from the sun means about 8% lower temperature because $T \propto P^{1/4}$. This factor reduces our best estimate of the Earth's temperature by about 30 K. Now T_E = 250 K, or 0˚ F. Brrr!

In fact the average surface temperature of the Earth is about 290 K, or about 60° F, just right for Earthly life. (coincidence? I think not.)

The extra 60° F of warming is mainly due to the "greenhouse effect", the fact that some of the radiation from the Earth is reflected back by the atmosphere. What exactly is the "greenhouse effect"?

Remember Molecular vibrations:

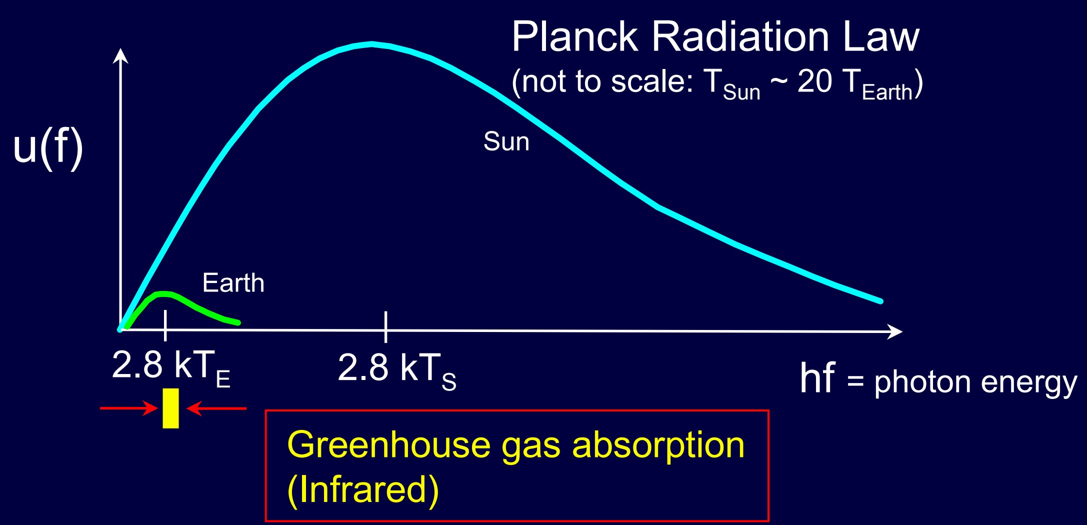
Polyatomic molecules (e.g., CO_2, H_2O, CH_4) have rotational and vibrational modes that correspond to photon wavelengths (energies) in the infrared. These molecules absorb (and emit) IR radiation much more effectively than O_2 and N_2.

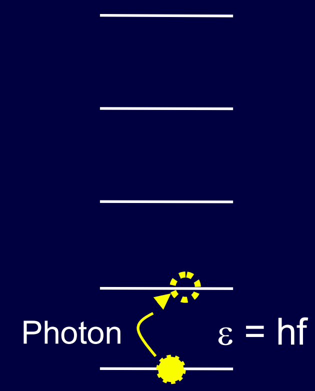


Strong absorption in the infrared.
(rotational and vibrational motions)

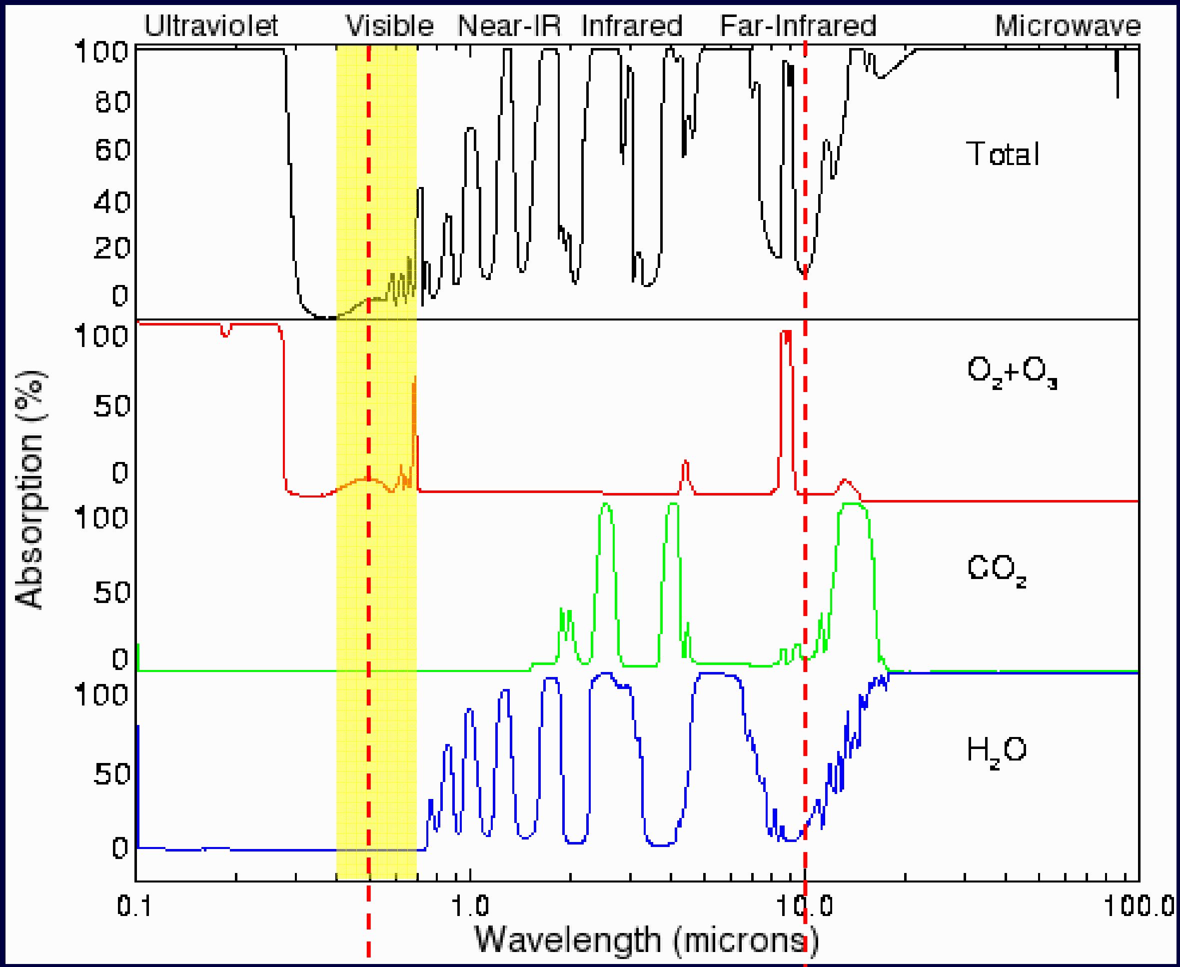
This absorption is in the middle of the Earth's thermal spectrum, but in the tail of the Sun's.

The result is that our atmosphere lets most of the sunlight through, but absorbs a larger fraction of the radiation that the Earth emits.





Some Absorption Spectra

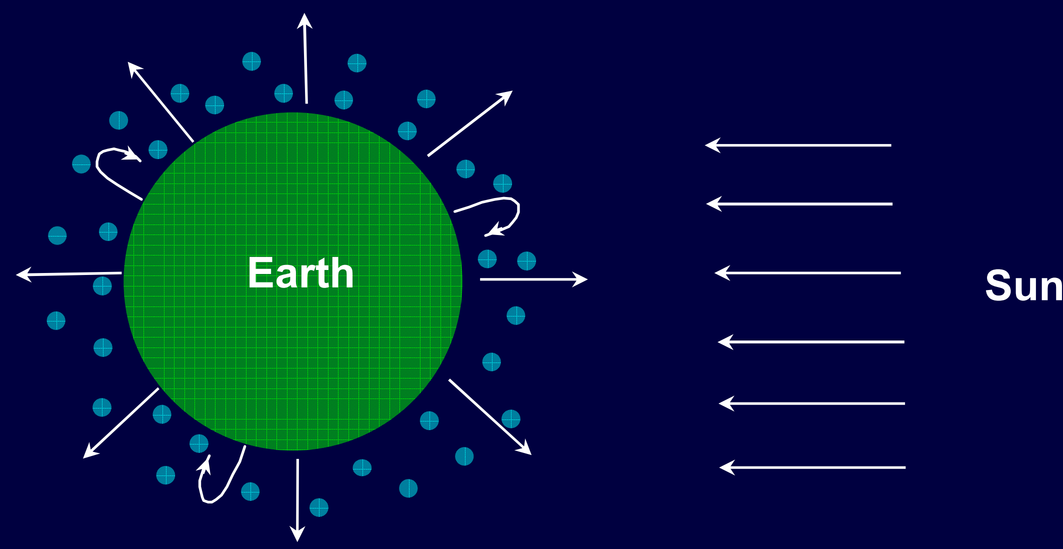


Peak wavelength of
Sun's thermal spectrum

Peak wavelength of
Earth's thermal spectrum

Water is the most important greenhouse gas.

The Greenhouse Effect



Thermal radiation from the earth (infrared) is absorbed by certain gases in our atmosphere (such as CO_2) and redirected back to earth.

These ‘greenhouse gases’ provide additional warming to our planet - essential for life as we know it.

Radiation from the sun is not affected much by the greenhouse gases because it has a much different frequency spectrum.

Mars has a thin atmosphere with few greenhouse gases: 70˚ F in the day and -130° F at night. Venus has lots of CO_2 : T = 800° F