"A vast time bubble has been projected into the future to the precise moment of the end of the universe.

This is, of course, impossible."

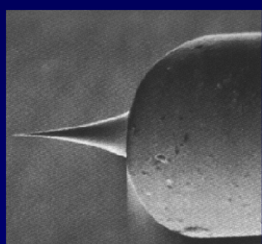
--D. Adams, *The Hitchhiker's Guide to the Galaxy*

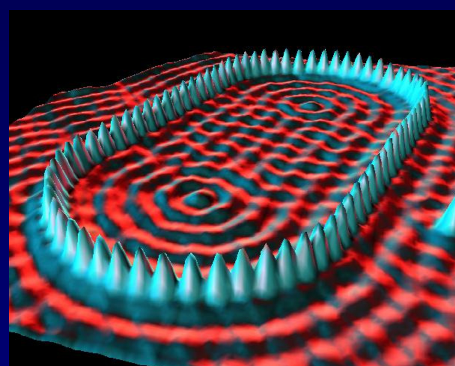
"There is light at the end of the tunnel." -- proverb

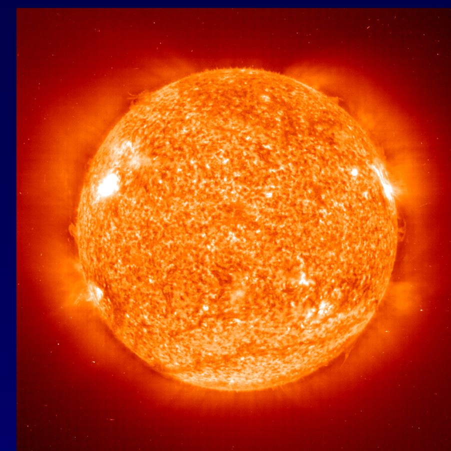
"The light at the end of the tunnel is just the light of an oncoming train."

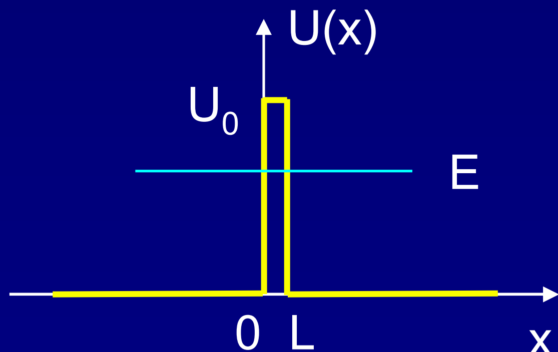
--R. Lowell

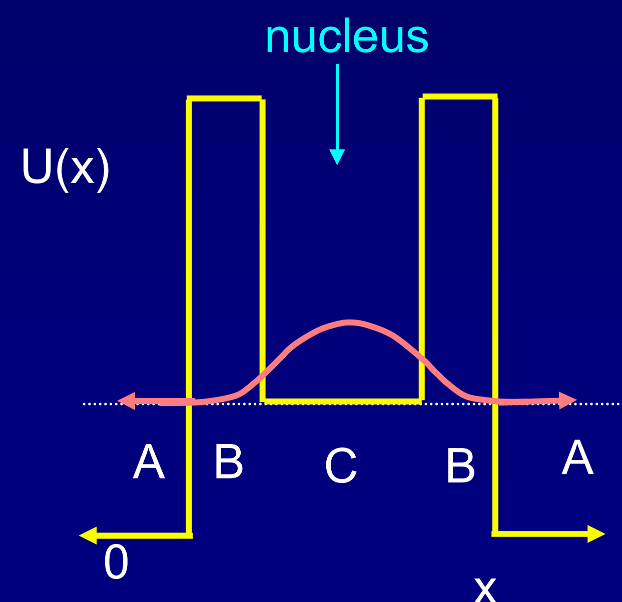
Lecture 13:
Barrier Penetration and Tunneling











Today

Tunneling of quantum particles

- Scanning Tunneling Microscope (STM)
- Nuclear Decay
- Solar Fusion

Next time: Time-*dependent* quantum mechanics

- Oscillations
- Measurements in QM
- Time-Energy Uncertainty Principle

The rest of the course:

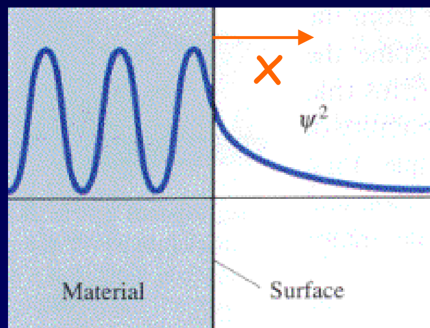
Next week: 3 dimensions - orbital and spin angular momentum
H atom, exclusion principle, periodic table

Last week: Molecules and solids.
Metals, insulators, semiconductors, superconductors, lasers, . .

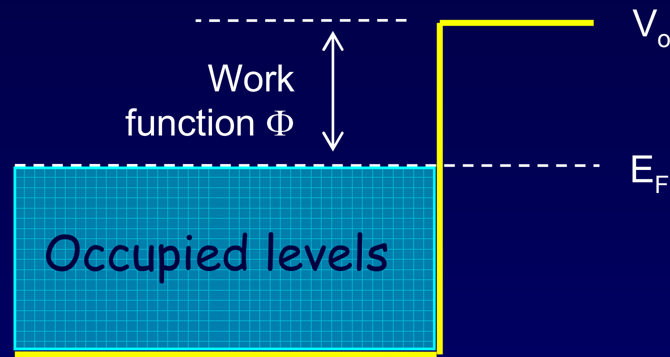
Good web site for animations http://www.falstad.com/qm1d/

"Leaky" Particles

Due to "barrier penetration", the electron density of a metal actually extends outside the surface of the metal!



x = 0



Assume that the work function (i.e., the energy difference between the most energetic conduction electrons and the potential barrier at the surface) of a certain metal is Φ = 5 eV. Estimate the distance x outside the surface of the metal at which the electron probability density drops to 1/1000 of that just inside the metal.

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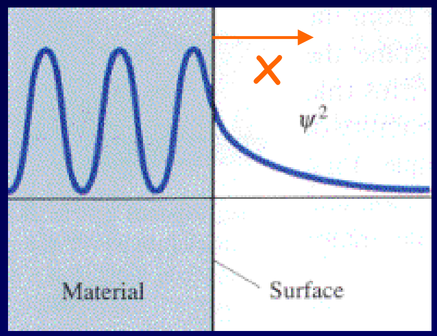

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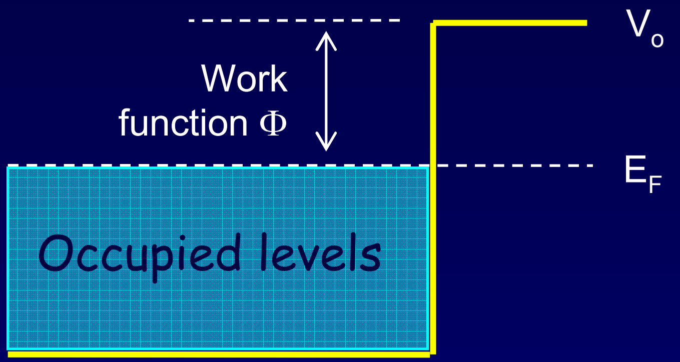


Assume that the work function (i.e., the energy difference between the most energetic conduction electrons and the potential barrier at the surface) of a certain metal is Φ = 5 eV. Estimate the distance x outside the surface of the metal at which the electron probability density drops to 1/1000 of that just inside the metal.

$$\frac{|\psi(x)|^2}{|\psi(0)|^2} = e^{-2Kx} \approx \frac{1}{1000}$$

$$\Rightarrow x = -\frac{1}{2K}\ln\left(\frac{1}{1000}\right) \approx 0.3nm$$

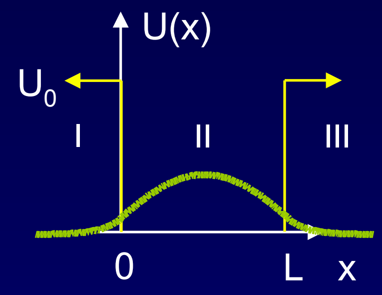
$$K = \sqrt{\frac{2m_e}{\hbar^2}(V_0 - E)} = 2\pi\sqrt{\frac{2m_e}{h^2}\Phi} = 2\pi\sqrt{\frac{5eV}{1.505\ \text{eV}\cdot\text{nm}^2}} = 11.5\ \text{nm}^{-1}$$

Tunneling: Key Points

In quantum mechanics a particle can penetrate into a barrier where it would be classically forbidden.

The finite square well:

In region III, $E < U_0$, and $\psi(x)$ has the exponential form D_1e^{-Kx}. We did not solve the equations – too hard! You will do this using the computer in Lab #3.



The probability of finding the particle in the barrier region decreases as e^{-2Kx}.

The finite-width barrier:

Today we consider a related problem – a particle approaching a finite-width barrier and "tunneling" through to the other side.

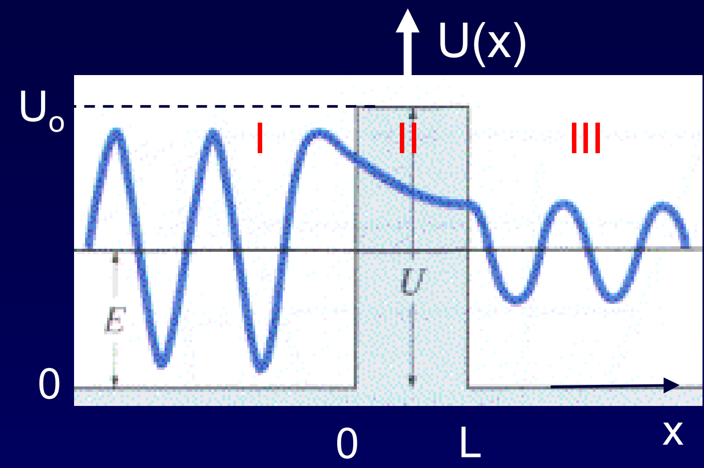
The result is very similar, and again the problem is too hard to solve exactly here:

The probability of the particle tunneling through a finite width barrier is approximately proportional to e^{-2KL} where L is the width of the barrier.

Tunneling Through a Barrier (1)

What is the the probability that an incident particle tunnels through the barrier?
It's called the "Transmission Coefficient, T".

Consider a barrier (II) of height U_0.
U = 0 everywhere else.



Getting an exact result requires applying the boundary conditions at x = 0 and x = L, then solving six transcendental equations for six unknowns:

$$\psi_I(x) = A_1 \sin kx + A_2 \cos kx$$

$$\psi_{II}(x) = B_1 e^{Kx} + B_2 e^{-Kx}$$

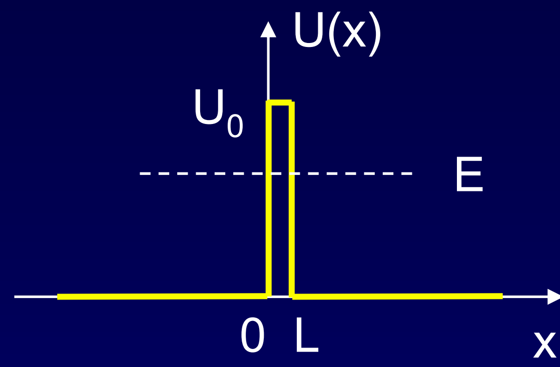
$$\psi_{III}(x) = C_1 \sin kx + C_2 \cos kx$$

A_1, A_2, B_1, B_2, C_1, and C_2 are unknown. K and k are known functions of E.

This is more complicated than the infinitely wide barrier, because we can't require that $B_1 = 0$. (Why not?)

Tunneling Through a Barrier (2)

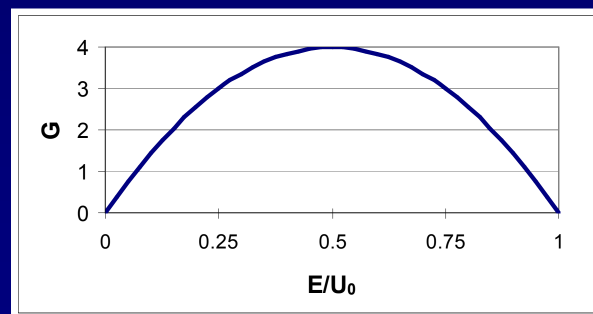
In many situations, the barrier width L is much larger than the 'decay length' 1/K of the penetrating wave (KL >> 1). In this case $B_1 \approx 0$ (why?), and the result resembles the infinite barrier. The tunneling coefficient simplifies:



$$T \approx Ge^{-2KL} \text{ where } G = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right)$$

$$K = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)}$$

This is nearly the same result as in the "leaky particle" example! Except for G:



We will often ignore G.
(We'll tell you when to do this.)

The important result is e^{-2KL}.

Act 1

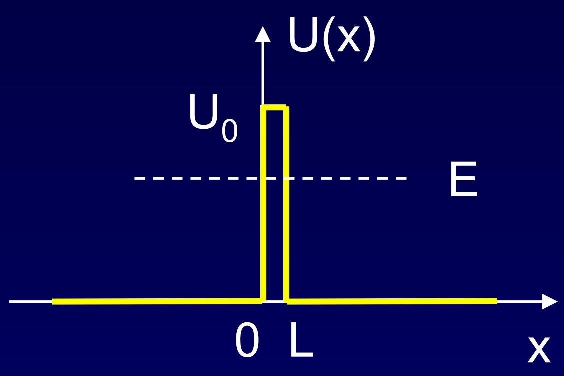
Consider a particle tunneling through a barrier.

1. Which of the following will increase the likelihood of tunneling?

   a. decrease the height of the barrier

   b. decrease the width of the barrier

   c. decrease the mass of the particle



2. What is the energy of the emerging particles?

   a. < initial energy b. = initial energy c. > initial energy

Solution

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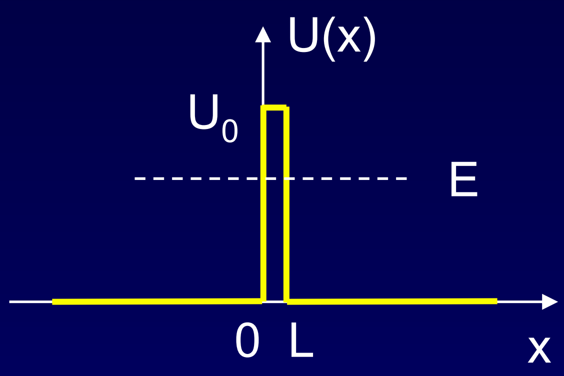

$T \approx e^{-2KL}$ Decreasing U_0 or m_e will decrease *K*.

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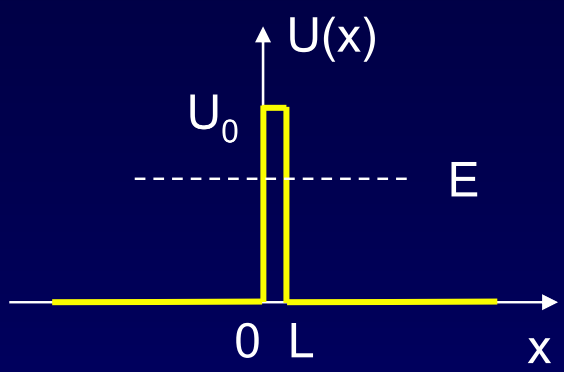

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2. What is the energy of the emerging particles?

   a. < initial energy b. = initial energy c. > initial energy

The barrier does not absorb energy from the particle. The amplitude of the outgoing wave is smaller, but the wavelength is the same. E is the same everywhere.

Probability $\neq$ Energy

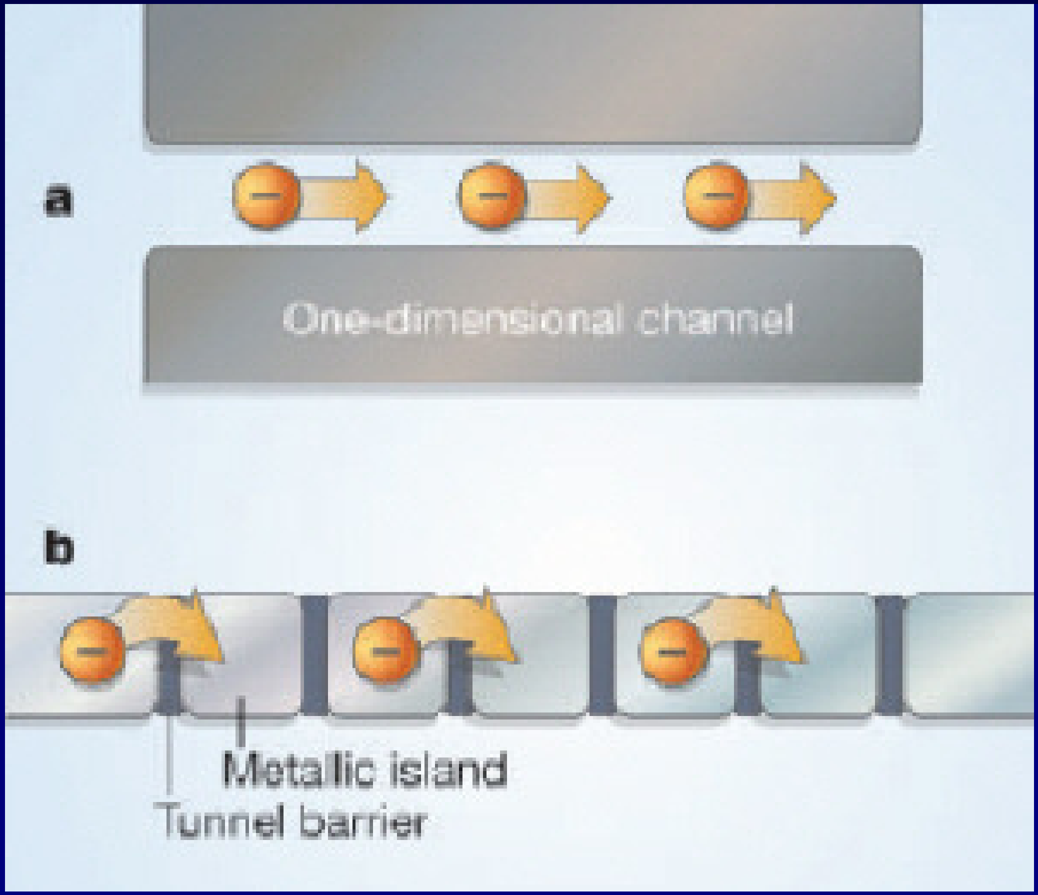
Example: Electrons in Nanoscale devices

letters to nature

Nature 434, 361 - 364 (17 March 2005)

Current measurement by real-time counting of single electrons

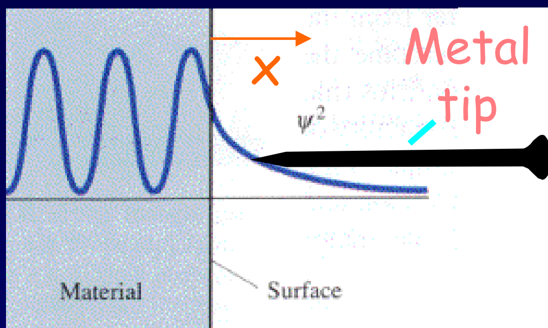
JONAS BYLANDER, TIM DUTY & PER DELSING

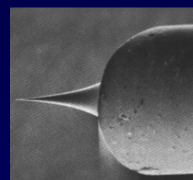


Electrons that successfully tunnel through the 50 junctions are detected using a fast single-electron transistor (SET).

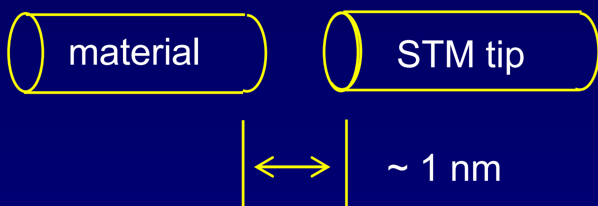
Application: Tunneling Microscopy

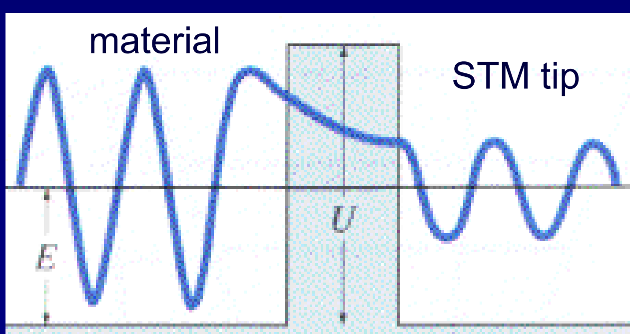
One can use barrier penetration to measure the electron density on a surface.





Real STM tip

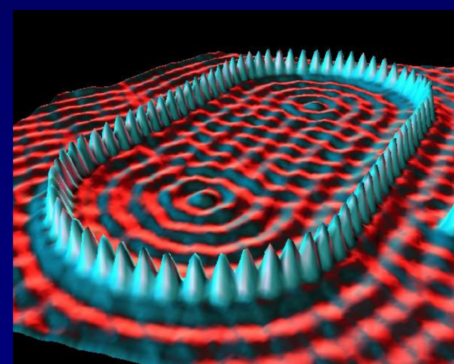




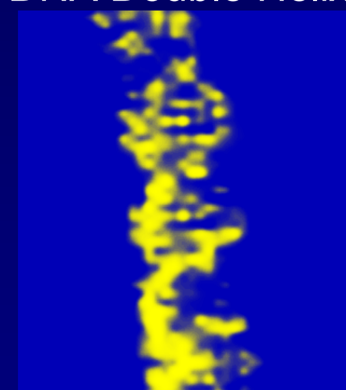
STM demo:
http://www.quantum-physics.polytechnique.fr/en/

Scanning
Tunneling
Microscope
images

Na atoms on metal:



DNA Double Helix:

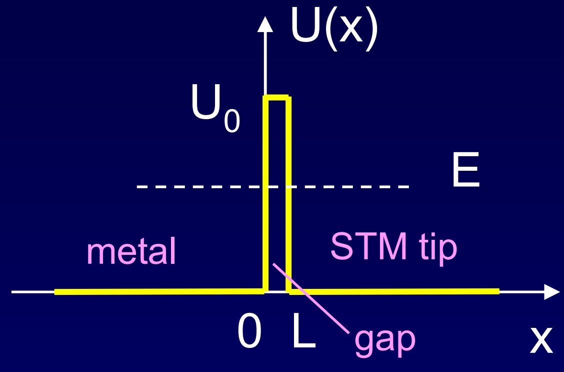


Barrier penetration is a wave phenomenon, not only QM.
It is used in optical microscopes also. See:
http://en.wikipedia.org/wiki/Total_internal_reflection_fluorescence_microscope

The STM

The STM (scanning tunneling microscope) tip is L = 0.18 nm from a metal surface.

An electron with energy of E = 6 eV in the metal approaches the surface. Assume the metal/tip gap is a potential barrier with a height of U_o = 12 eV. What is the probability that the electron will tunnel through the barrier?



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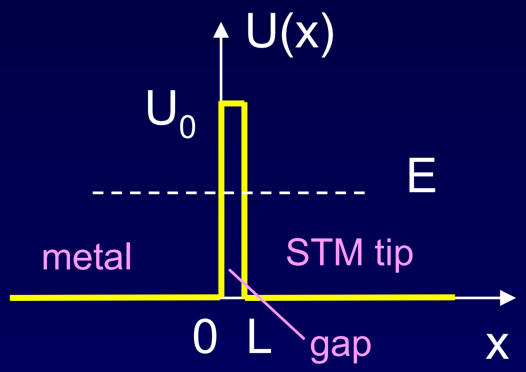


$$T \approx Ge^{-2KL} = 4\,e^{-2(12.6)(0.18)}$$
$$= 4(0.011) = \boxed{4.3\%}$$

$$G = 16\frac{E}{U_0}\left(1-\frac{E}{U_0}\right) = 16\frac{1}{2}\left(1-\frac{1}{2}\right) = 4$$

$$K = \sqrt{\frac{2m_e}{\hbar^2}(U_0-E)} = 2\pi\sqrt{\frac{2m_e}{h^2}(U_0-E)}$$
$$= 2\pi\sqrt{\frac{6\,\text{eV}}{1.505\ \text{eV-nm}^2}} \approx 12.6\ \text{nm}^{-1}$$

T << 1, so our use of the KL >> 1 approximation is justified.

Q: What will T be if we double the width of the gap?

ACT 2

What effect does a barrier have on probability?

Suppose T = 0.05. What happens to the other 95% of the probability?

a. It's absorbed by the barrier.

b. It's reflected by the barrier.

c. The particle "bounces around" for a while, then escapes.

Solution

What effect does a barrier have on probability?

Suppose T = 0.05. What happens to the other 95% of the probability?

a. It's absorbed by the barrier.

b. It's reflected by the barrier.

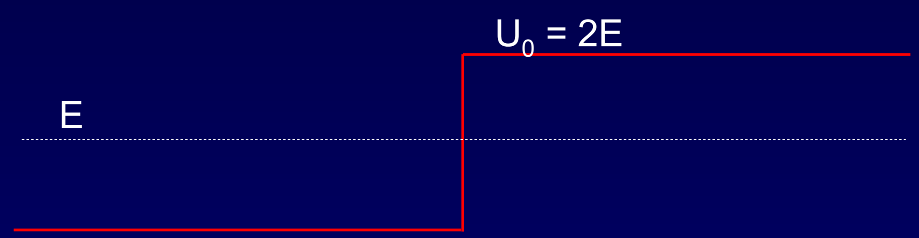
c. The particle "bounces around" for a while, then escapes.

Absorbing probability would mean that the particles disappear.
We are considering processes on which this can't happen.
The number of electrons remains constant.

Escaping after a delay would contribute to T.

Electron Approaching a Step

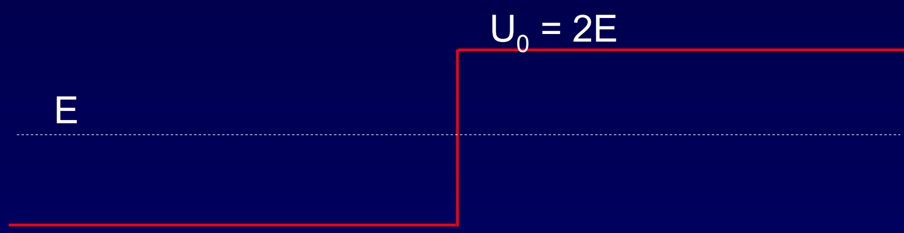
Suppose an electron with energy E approaches a step, effectively an infinitely wide barrier of height 2E. (I picked this ratio to simplify the math.)



What does the wave function look like, and what is happening?

Solution

Suppose an electron with energy E approaches an infinitely wide barrier of height 2E. (I picked this ratio to simplify the math.)



What does the wave function look like, and what is happening?

Here's the solution:

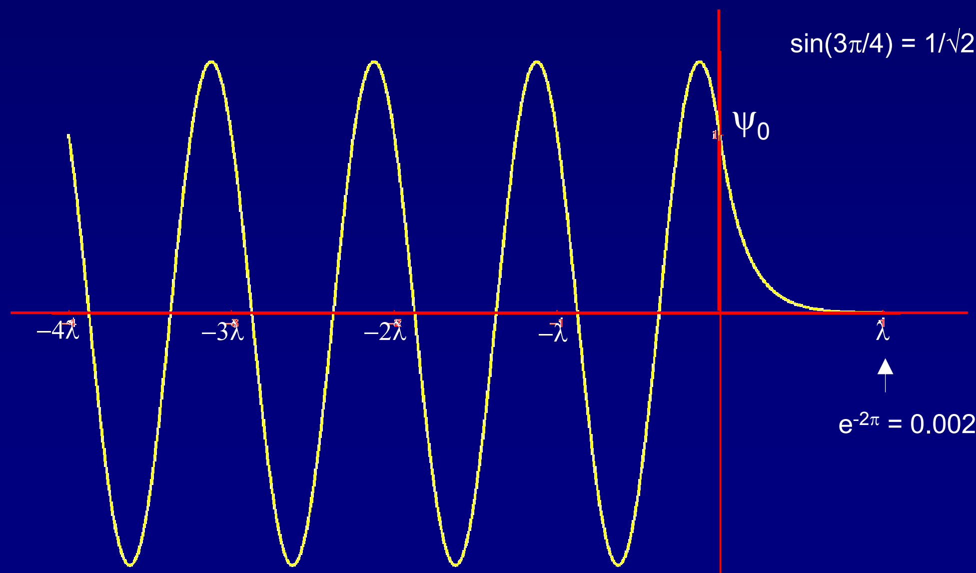
For x < 0: $\psi(x) = \psi_0 \sqrt{2}\sin\left(kx + \frac{3\pi}{4}\right)$

For x > 0: $= \psi_0 e^{-kx}$

K = k, because U_0-E = E.

The constants √2 and 3π/4 come from the boundary conditions.

What is this graph telling us?



Solution

For legibility, I'm ignoring the $3\pi/4$ phase shift.

Sin(kx) is a standing wave.
It has nodes every $\lambda/2$.

So, what do I mean when I say that the electron approaches the barrier?

Remember two things:

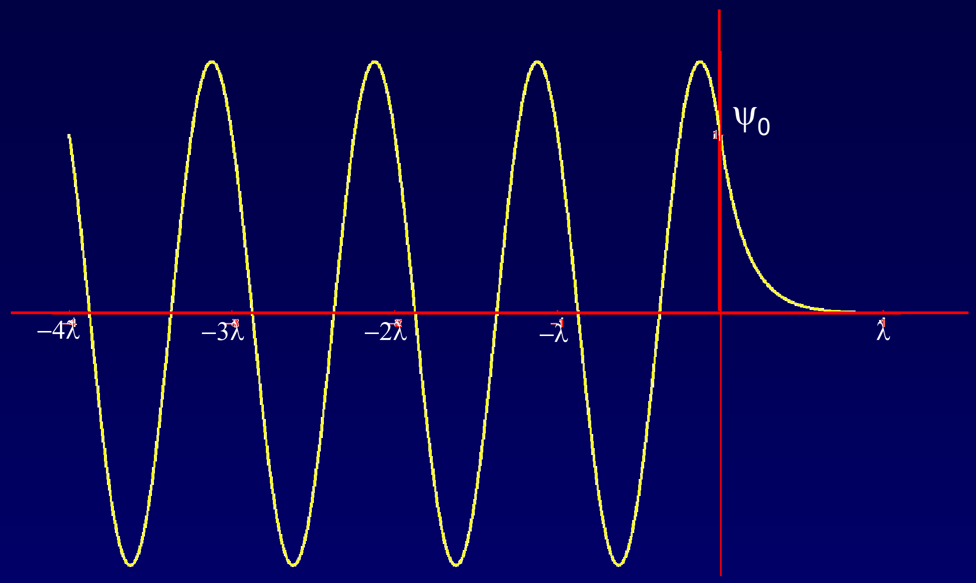
- The wave oscillates: $e^{-i\omega t}$.
- We can write: $\sin(kx) = (e^{ikx} - e^{-ikx}) / (2i)$

Thus, this standing wave is actually a superposition of two traveling waves:

$e^{i(kx-\omega t)}$ and $e^{i(-kx-\omega t)}$.

The incoming wave, traveling to the right.

The reflected wave, traveling to the left.



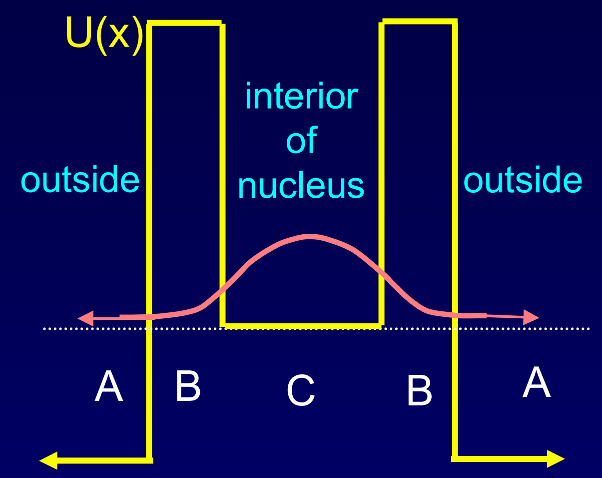
For $x < 0$: $\psi(x) = \psi_0\sqrt{2}\sin\left(kx + \frac{3\pi}{4}\right)$

For $x > 0$: $= \psi_0 e^{-kx}$

The wave is entirely reflected. None is absorbed by the barrier.
It penetrates a short distance, but then bounces out.

Tunneling and Radioactivity

In large atoms (e.g., Uranium), the nucleus can be unstable to the emission of an alpha particle (a He nucleus). This form of radioactivity is a tunneling process, involving transmission of the alpha particle from a low-energy valley through a barrier to a lower energy outside.



Why do we observe exponential decay?

- ψ leaks out from C through B to A – the particle "tunnels" out.
- The leakage is slow (T << 1), so ψ just outside the barrier stays negligible.
- The shape of ψ remaining in B-C shows almost no change: Its amplitude slowly decreases. That is, P_{inside} is no longer 1.
- The rate at which probability flows out is proportional to P_{inside} (by linearity) $\Rightarrow$ exponential decay in time.

$$\frac{dx}{dt} = -Ax \quad \Rightarrow \quad x = e^{-At} = e^{-t/\tau}$$

$t_{1/2} = (\tau \ln 2)$ is the "half life" of the substance

α-Radiation: Illustrations of the enormous range of decay rates in different nuclei

Consider a very simple model of a-radiation:

Assume the alpha particle ($m = 6.64 \times 10^{-27}$ kg) is trapped in a nucleus which presents a square barrier of width L and height U_o. To find the decay rate we consider:

(1) the "attempt rate" at which the alpha particle of energy E inside the nucleus hits the barrier

Rough estimate with E ~ 5 to 10 MeV: the alpha particle makes about 10^{21} "attempts" per second (~velocity/nuclear diameter)

(2) the tunneling probability for an alpha particle with energy E each time the particle hits the barrier. [For this order of magnitude calculation you may neglect G.] Here we use

$$T \approx e^{-2KL} \qquad K = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)}$$

Because of the exponential this factor can vary enormously!

Act 3

Polonium has an effective barrier width of ~10 fermi, leading to a tunneling probability of ~10^{-15}. Now consider Uranium, which has a similar barrier height, but an effective width of about ~20 fermi.

Estimate the tunneling probability in Uranium:

a. 10^{-30}

b. 10^{-14}

c. 10^{-7}

Solution

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Estimate the tunneling probability in Uranium:

a. 10^{-30}

b. 10^{-14}

c. 10^{-7}

Think of it this way – there is a 10^{-15} chance to get through the first half of the barrier, and a 10^{-15} chance to then get through the second half.

Alternatively, when we double L in

$$T \approx e^{-2KL}$$

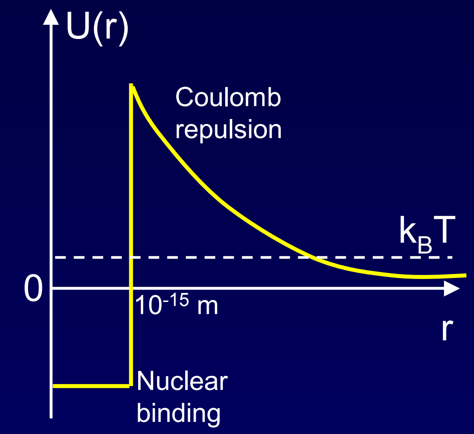
this is equivalent to squaring the transmission T.

Polonium: Using 10^{21} "attempts" at the barrier per second, the probability of escape is about 10^{6} per second→ decay time ~1 μs.

Uranium: Actually has a somewhat higher barrier too, leading to P(tunnel) ~ 10^{-40}→ decay time ~10^{10} years!

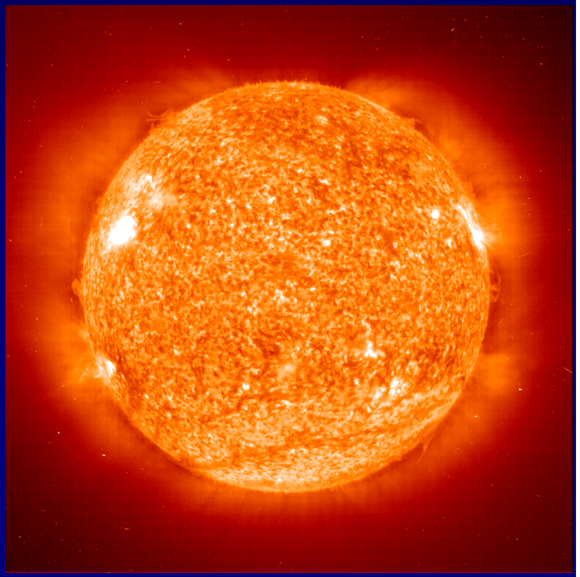
Tunneling Example: The Sun

The solar nuclear fusion process starts when two protons fuse together. In order for this reaction to proceed, the protons must "touch" (approach to within 10^{-15} m of each other).
The potential energy, U(r), looks something like this:



The temperature of the sun's core is T ~ 1.3×10^7 K.
This corresponds to an average kinetic energy:
$k_BT = 2 \times 10^{-16}$ J   (k_B = Boltzman's constant)
At r = 10^{-15} m the height of the Coulomb barrier is:

$$U(r) = (1/4\pi\varepsilon_0)e^2/r = (9\times10^9)\times(1.6\times10^{-19}\ C)^2/10^{-15}\ m$$
$$= 2 \times 10^{-13}\ J$$



Thus, the protons in the sun very rarely have enough thermal energy to go over the Coulomb barrier.

How do they fuse then? By tunneling through the barrier!

Next Lectures

Tunneling of quantum particles

- Scanning Tunneling Microscope (STM)
- Nuclear Decay
- Solar Fusion
- The Ammonia Maser