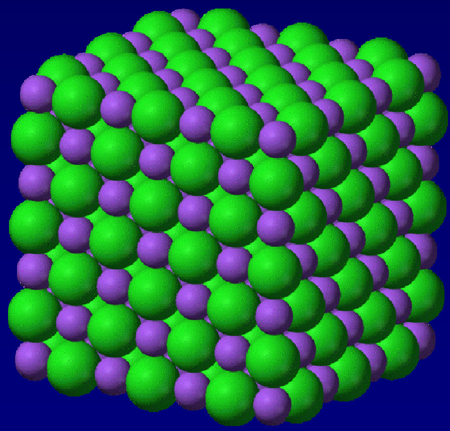
Lecture 6:
Waves Review, Crystallography, and Examples



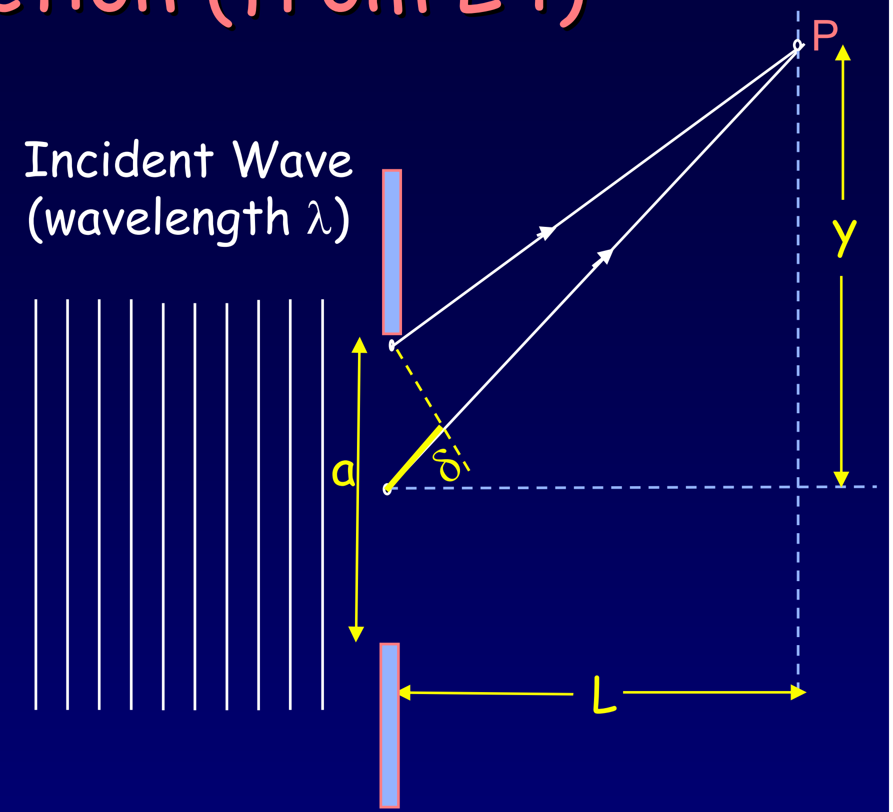
Single-Slit Diffraction (from L4)

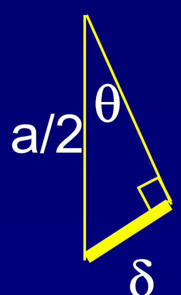
Slit of width a. Where are the minima?

Use Huygens' principle: treat each point across the opening of the slit as a wave source.

The first minimum is at an angle such that the light from the top and the middle of the slit destructively interfere.

This works, because for every point in the top half, there is a corresponding point in the bottom half that cancels it.





$$\delta = \frac{a}{2}\sin\theta_{\min} = \frac{\lambda}{2}$$

$$\Rightarrow \boxed{\sin\theta_{\min} = \frac{\lambda}{a}}$$

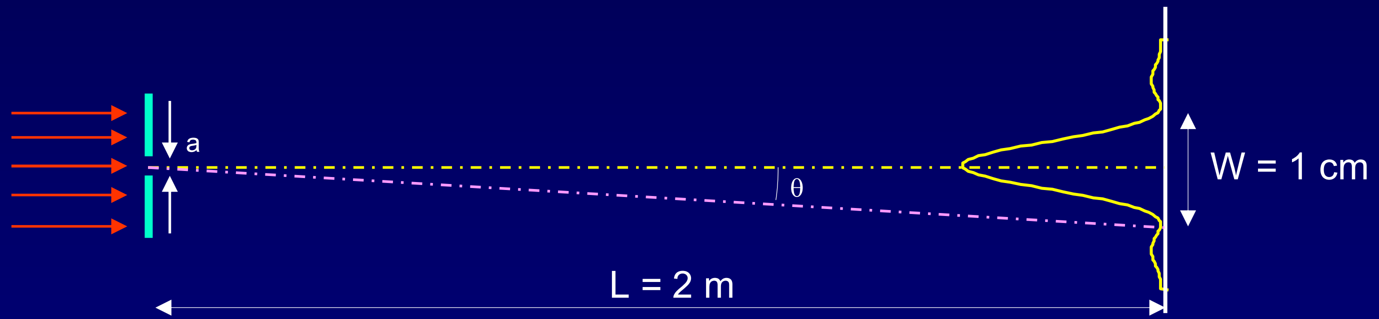
The second minimum is at an angle such that the light from the top and a point at a/4 destructively interfere:

$$\delta = \frac{a}{4}\sin\theta_{\min,2} = \frac{\lambda}{2} \qquad \Rightarrow \sin\theta_{\min,2} = \frac{2\lambda}{a}$$

Location of nth-minimum: $\sin\theta_{\min,n} = \frac{n\lambda}{a}$

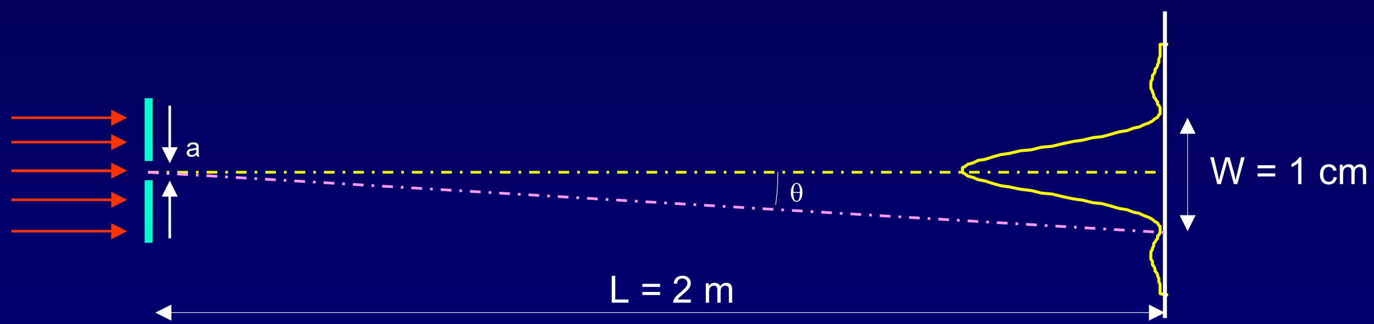
Single-Slit Diffraction Example

Suppose that when we pass red light (λ = 600 nm) through a slit of unknown width a, the width of the spot (the distance between the first zeros on each side of the bright peak) is W = 1 cm on a screen that is L = 2 m behind the slit. How wide is the slit?



Solution

Suppose that when we pass red light (λ = 600 nm) through a slit of unknown width a, the width of the spot (the distance between the first zeros on each side of the bright peak) is W = 1 cm on a screen that is L = 2 m behind the slit. How wide is the slit?

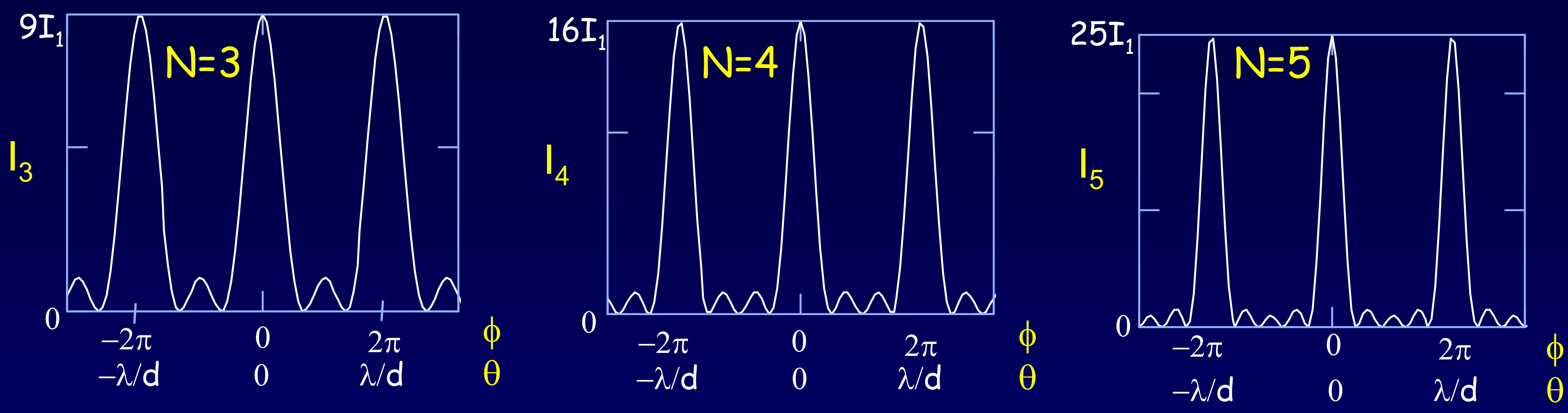


The angle to the first zero is: $\theta = \pm\lambda/a$

$W = 2L\tan\theta \cong 2L\theta = 2L\lambda/a$

$a = 2L\lambda/W = (4\text{m})(6\times10^{-7}\text{ m})/(10^{-2}\text{ m}) = 2.4\times10^{-4}\text{ m} = 0.24\text{ mm}$

Multiple Slit Interference (from L4)



The positions of the principal maxima occur at $\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$

where ϕ is the phase between adjacent slits. $\theta = 0, \pm\lambda/d, \pm 2\lambda/d, \ldots$

The intensity at the peak of a principal maximum goes as N^2.

3 slits: $A_{tot} = 3A_1 \Rightarrow I_{tot} = 9I_1$. N slits: $I_N = N^2 I_1$.

Between two principal maxima there are N-1 zeros and

N-2 secondary maxima $\Rightarrow$ The peak width $\propto 1/N$.

The total power in a principal maximum is proportional to $N^2(1/N) = N$.

Act 1

Light interfering from 10 equally spaced slits initially illuminates a screen. Now we double the number of slits, keeping the spacing constant.

1. What happens to the intensity I at the principal maxima?

   a. stays same (I) b. doubles (2I) c. quadruples (4I)

2. What happens to the net power on the screen?

   a. stays same b. doubles c. quadruples

Solution

Light interfering from 10 equally spaced slits initially illuminates a screen. Now we double the number of slits, keeping the spacing constant.

1. What happens to the intensity I at the principal maxima?

   a. stays same (I)   b. doubles (2I) $\boxed{\text{c. quadruples (4I)}}$

   $I_N = N^2 I_1$. 10 $\rightarrow$ 20 means 100 $\rightarrow$ 400.

2. What happens to the net power on the screen?

   a. stays same   b. doubles   c. quadruples

Solution

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   $I_N = N^2 I_1$. $10 \rightarrow 20$ means $100 \rightarrow 400$.

2. What happens to the net power on the screen?

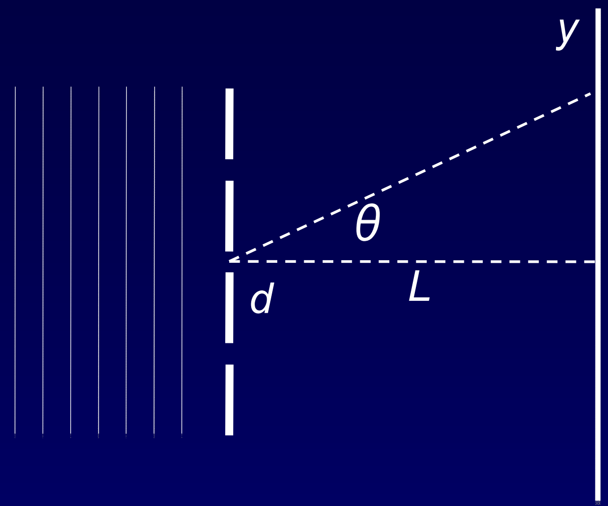
   a. stays same $\boxed{\text{b. doubles}}$ c. quadruples

If we double the number of slits, we expect the power on the screen to double. How does this work?

- The number of principal maxima (which have most of the power) does *not* change.
- The principal maxima become 4x brighter.
- But they also become only half as wide.
- Therefore, the net power (integrating over all the peaks) increases two-fold, as we would expect.

Multiple-slit Example

Three narrow slits with equal spacing *d* are at a distance L = 1.4 m away from a screen. The slits are illuminated at normal incidence with light of wavelength λ = 570 nm. The first principal maximum on the screen is at y = 2.0 mm.



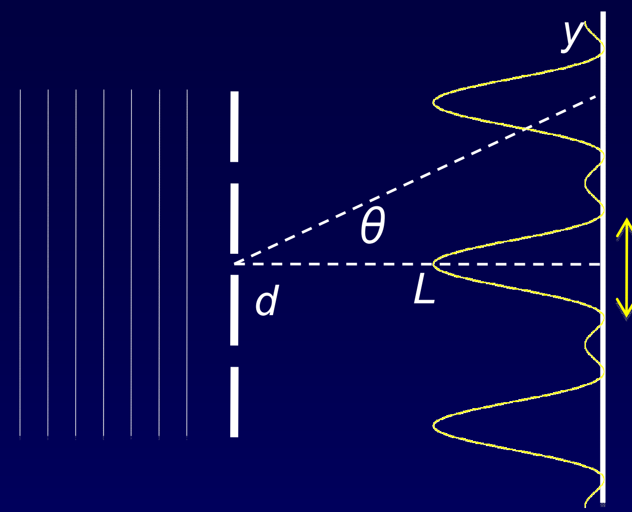
1. What is the slit spacing, *d*?

2. If the wavelength, λ, is increased, what happens to the width of the principal maxima?

3. If the intensity of each slit alone is I_1, what is the intensity of the secondary maximum?

Solution

Three narrow slits with equal spacing d are at a distance L = 1.4 m away from a screen. The slits are illuminated at normal incidence with light of wavelength λ = 570 nm. The first principal maximum on the screen is at y = 2.0 mm.

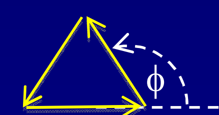


1. What is the slit spacing, d?

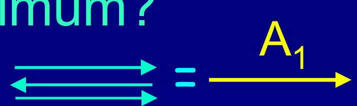
   The first maximum occurs when the path difference between adjacent slits is λ. This happens at $\sin\theta = \lambda/d$. We are told that $\tan\theta = y/L = 1.43\times10^{-3}$, so the small angle approximation is OK. Therefore, $d \approx \lambda/\theta$ = 0.40 mm.

2. If the wavelength, λ, is increased, what happens to the width of the principal maxima?

   The relation between θ and ϕ is $\phi/2\pi = \delta/\lambda = d\sin\theta / \lambda$.
   Therefore, for every feature that is described by ϕ (peaks, minima, *etc.*) $\sin\theta$ is proportional to λ. The width increases.



3. If the intensity of each slit alone is I_1, what is the intensity of the secondary maximum?

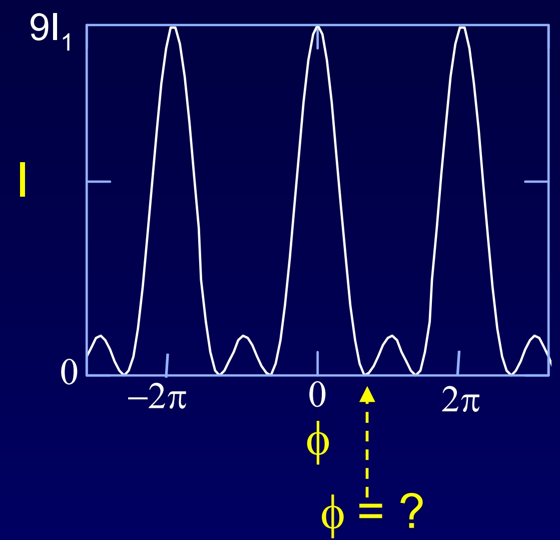
Phasor diagram: 

Two phasors cancel, leaving only one → I_1

ACT 2: Multiple Slits

1. What value of ϕ corresponds to the first *zero* of the 3-slit interference pattern?

   a) $\phi=\pi/2$ b) $\phi=2\pi/3$ c) $\phi=3\pi/4$

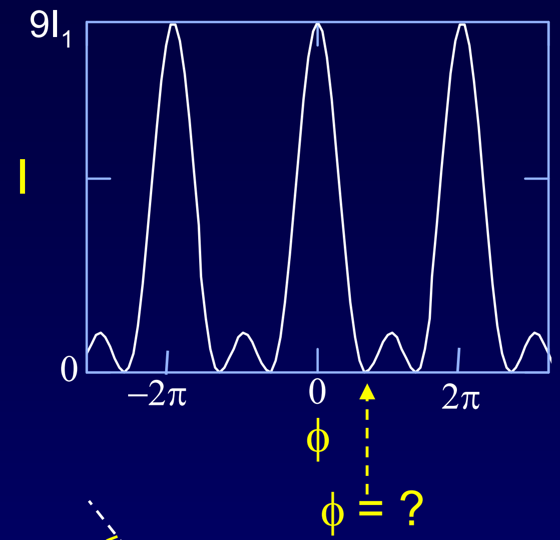


2. What value of ϕ corresponds to the first zero of the 4-slit interference pattern?

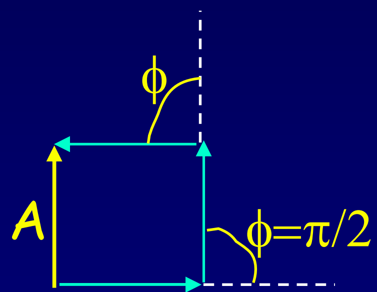
   a) $\phi=\pi/2$ b) $\phi=2\pi/3$ c) $\phi=3\pi/4$

Solution

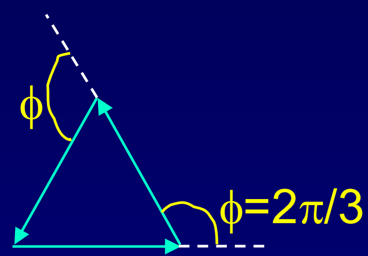
1. What value of ϕ corresponds to the first *zero* of the 3-slit interference pattern?



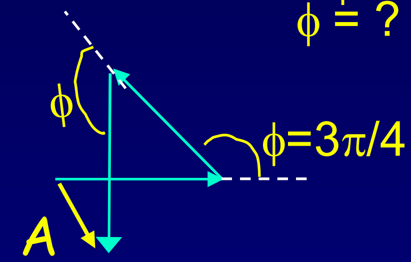
a) $\phi=\pi/2$ b) $\boxed{\phi=2\pi/3}$ c) $\phi=3\pi/4$



No.
A is not zero.



Yes!
Equilateral triangle gives A = 0.



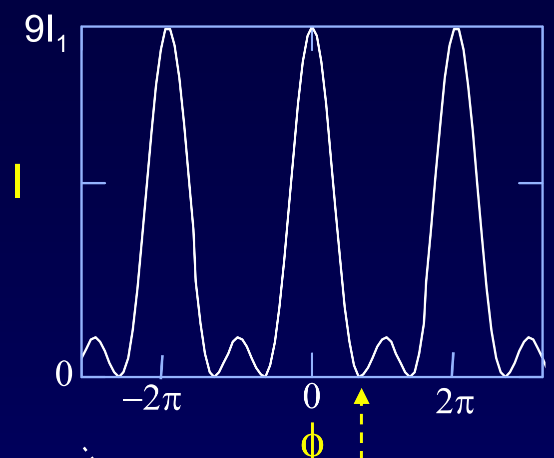
No.
Triangle does not close.

2. What value of ϕ corresponds to the first zero of the 4-slit interference pattern?

a) $\phi=\pi/2$ b) $\phi=2\pi/3$ c) $\phi=3\pi/4$

Solution

1. What value of ϕ corresponds to the first *zero* of the 3-slit interference pattern?



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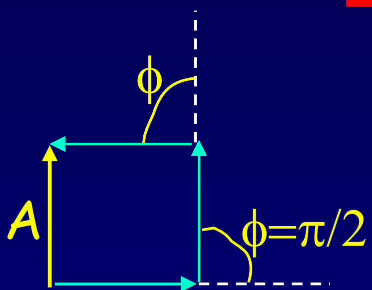
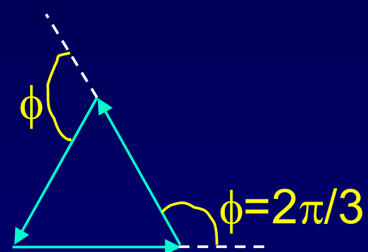


No.
A is not zero.



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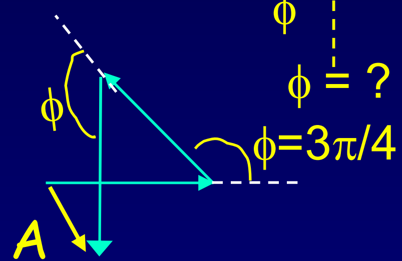
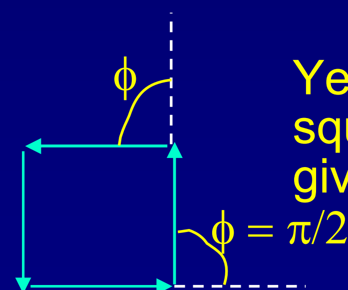


No.
Triangle does not close.

2. What value of ϕ corresponds to the first zero of the 4-slit interference pattern?

$\boxed{\text{a) } \phi=\pi/2}$ b) $\phi=2\pi/3$ c) $\phi=3\pi/4$

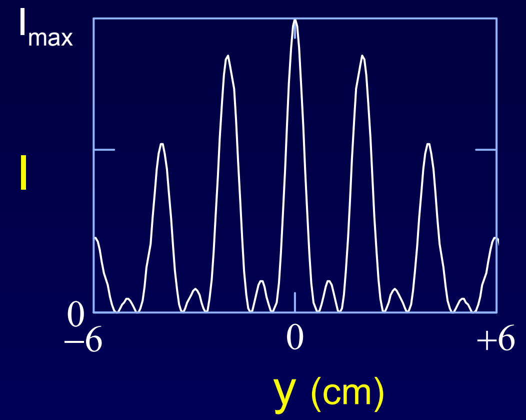


Yes. The square gives A = 0.

To get a zero, we need a closed figure. $N\phi$ must be a multiple of 2π, so the first zero is at $2\pi/N$.

Interference & Diffraction Exercise

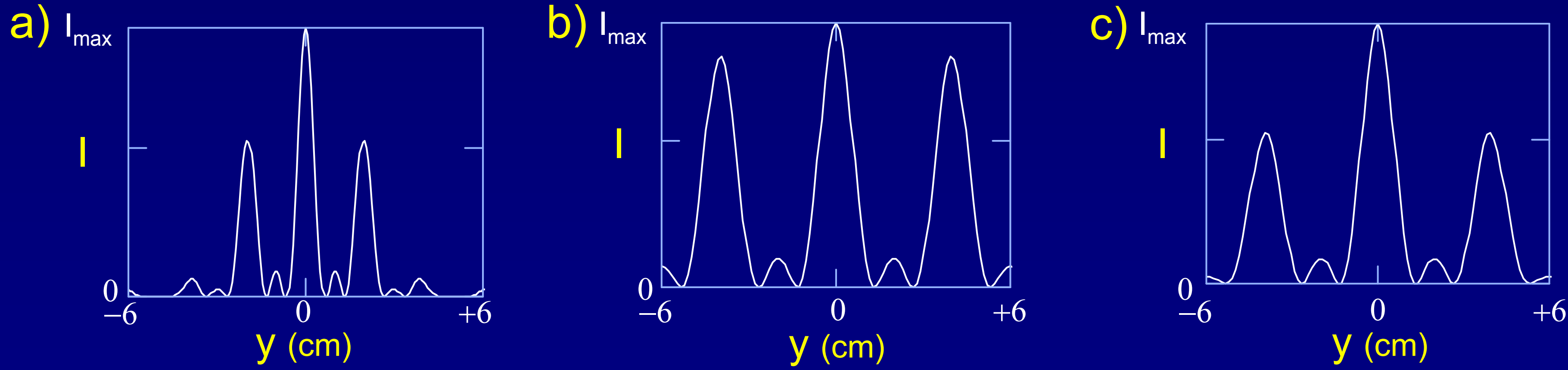
Light of wavelength λ is incident on an N-slit system with slit width a and slit spacing d.



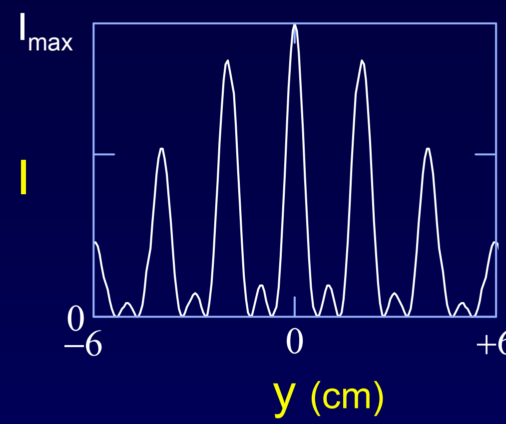
1. The intensity I as a function of y at a viewing screen located a distance L from the slits is shown to the right. L >> d, y, a. What is N?

a) N = 2 b) N = 3 c) N = 4

2. Now the slit spacing d is halved, but the slit width a is kept constant. Which of the graphs best represents the new intensity distribution?



Solution



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N is determined from the number of minima between two principal maxima.
N = $\#_{minima}$+1 Therefore, N = 3 .

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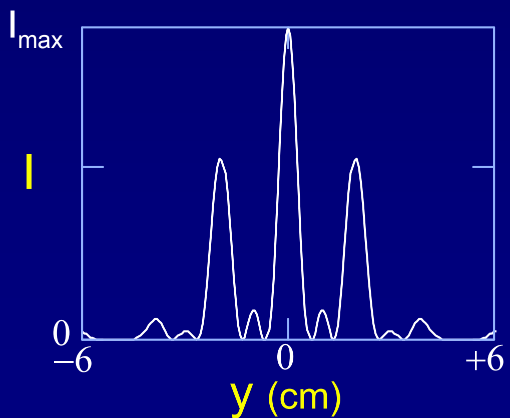
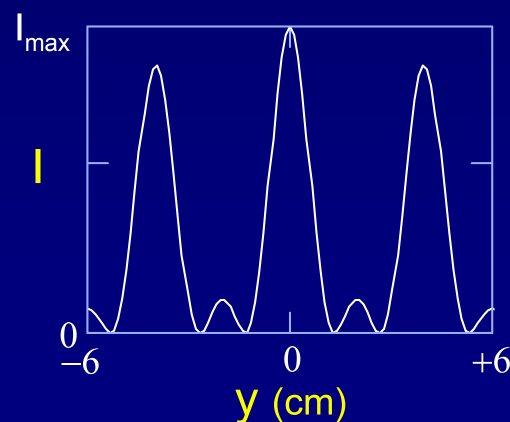
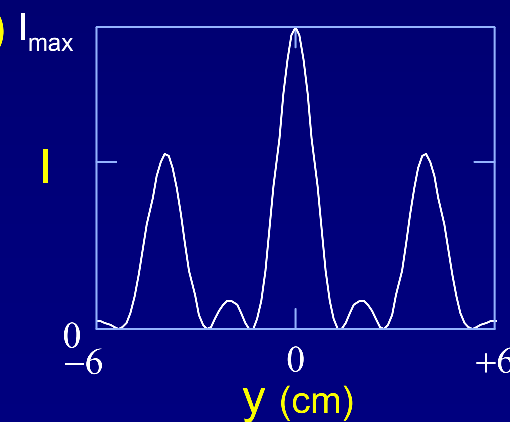
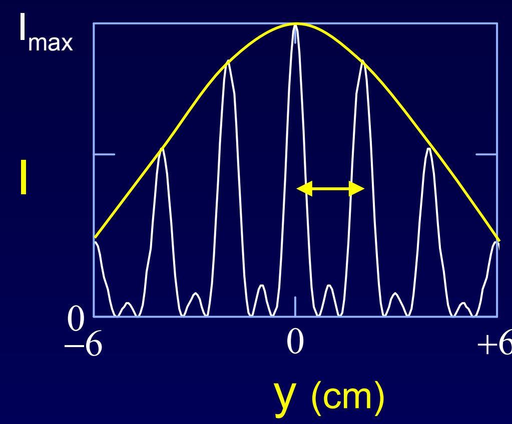
a) 

b) 

c) 

Solution

Light of wavelength λ is incident on an N-slit system with slit width a and slit spacing d.



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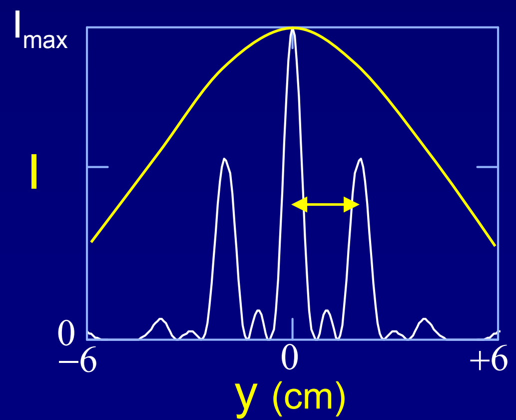
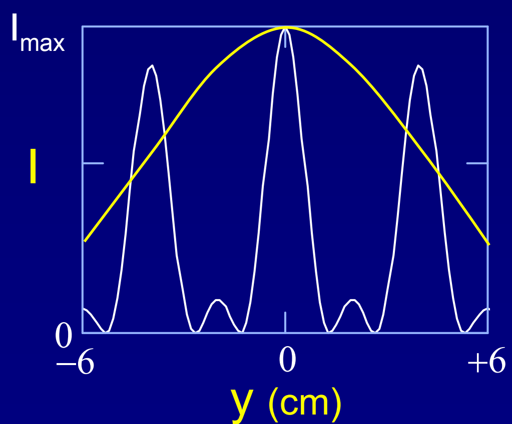
a)



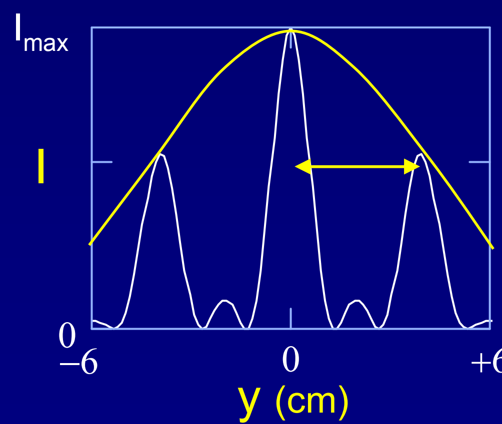
Interference spacing should change.

b)



Diffraction profile shouldn't change.

c)



Interference spacing doubles. Diffraction profile is unchanged.

Diffraction from gratings

$$I_N = I_1\left(\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right)^2$$

The slit/line spacing determines the location of the peaks (and the angular dispersing power $\theta(\lambda)$ of the grating:

The positions of the principal interference maxima are the *same* for any number of slits!

$$\boxed{d \sin\theta = m\lambda}$$

The number of slits/beam size determines the *width* of the peaks (narrower peaks easier to resolve).

$$\delta\theta \approx \lambda/Nd$$

Resolving power of an N-slit grating: The Rayleigh criterion

Diffraction Grating Example

Angular splitting of the Sodium doublet:

Consider the two closely spaced spectral (yellow) lines of sodium (Na), λ_1 = 589 nm and λ_2 = 589.6 nm. If light from a sodium lamp fully illuminates a diffraction grating with 4000 slits/cm, what is the angular separation of these two lines in the second-order (m=2) spectrum?

Hint: First find the slit spacing d from the number of slits per centimeter.

Solution

Angular splitting of the Sodium doublet:

Consider the two closely spaced spectral (yellow) lines of sodium (Na), λ_1 = 589 nm and λ_2 = 589.6 nm. If light from a sodium lamp fully illuminates a diffraction grating with 4000 slits/cm, what is the angular separation of these two lines in the second-order (m=2) spectrum?

Hint: First find the slit spacing d from the number of slits per centimeter.

$$d = \frac{1\,\text{cm}}{4000} = 2.5 \times 10^{-4}\,cm = 2.5\,\mu m$$

$$\theta_1 = \sin^{-1}\left(m\frac{\lambda_1}{d}\right) = 28.112^\circ$$

Small angle approximation is not valid.

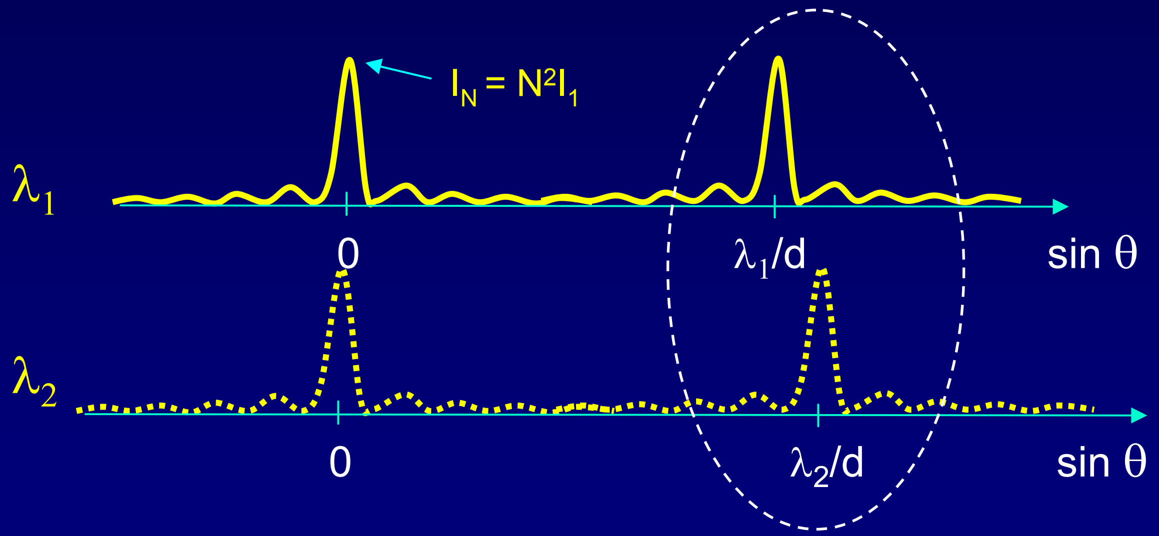
$$\Delta\theta = \sin^{-1}\left(m\frac{\lambda_2}{d}\right) - \sin^{-1}\left(m\frac{\lambda_1}{d}\right) = \boxed{0.031}^\circ$$

Diffraction Gratings (1)

Diffraction gratings rely on N-slit interference.
They consist of a large number of evenly spaced parallel slits.

An important question:

How effective are diffraction gratings at resolving light of different wavelengths (*i.e.* separating closely-spaced 'spectral lines')?



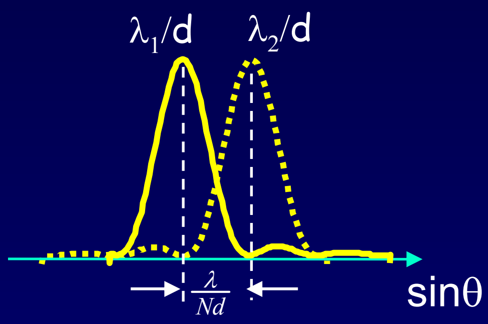
sinθ depends on λ.

Example: Na has a spectrum with two yellow "lines" very close together: λ_1 = 589.0 nm, λ_2 = 589.6 nm ($\Delta\lambda$ = 0.6 nm)

Are these two lines distinguishable using a particular grating?
We need a "resolution criterion".

Diffraction Gratings (2)

We use Rayleigh's criterion:
The minimum wavelength separation we can resolve occurs when the λ_2 peak coincides with the first zero of the λ_1 peak:



So, the Raleigh criterion is $\Delta(\sin\theta)_{min} = \lambda/Nd$.

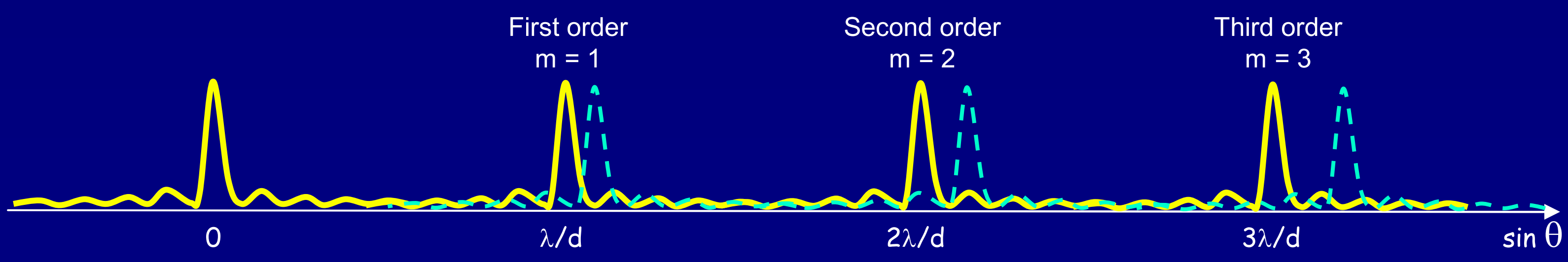
However, the location of the peak is $\sin\theta = m\lambda/d$.

Thus, $(\Delta\lambda)_{min} = (d/m)\Delta(\sin\theta)_{min} = \lambda/mN$:

$$\boxed{\frac{\Delta\lambda_{min}}{\lambda} = \frac{1}{Nm}}$$

Comments:

- It pays to use a grating that has a large number of lines, N. However, one must illuminate them all to get this benefit.
- It also pays to work at higher order (larger m): The widths of the peaks don't depend on m, but they are farther apart at large m.



ACT 2

1. Suppose we fully illuminate a grating for which $d = 2.5\ \mu m$. How big must it be to resolve the Na lines (589 nm, 589.6 nm), if we are operating at second order ($m = 2$)?

   a. 0.12 mm b. 1.2 mm c. 12 mm

2. How many interference orders can be seen with this grating?

   a. 2 b. 3 c. 4

3. Which will reduce the maximum number of interference orders?

   a. Increase λ b. Increase d c. Increase N

Solution

1. Suppose we fully illuminate a grating for which d = 2.5 μm. How big must it be to resolve the Na lines (589 nm, 589.6 nm), if we are operating at second order (m = 2)?

a. 0.12 mm $\quad$ $\boxed{\text{b. 1.2 mm}}$ $\quad$ c. 12 mm

We need to make N large enough to satisfy Raleigh's criterion.

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{Nm} \qquad \text{So: } N \geq \frac{\lambda}{m\,\Delta\lambda} = \frac{589\text{nm}}{2\,(0.6\text{nm})} = 491$$

Size = Nd $\geq$ 491$\times$2.5 μm $\approx$ 1.2 mm

2. How many interference orders can be seen with this grating?

a. 2 $\quad$ b. 3 $\quad$ c. 4

Solution

1. Suppose we fully illuminate a grating for which $d = 2.5\ \mu m$. How big must it be to resolve the Na lines (589 nm, 589.6 nm), if we are operating at second order ($m = 2$)?

a. 0.12 mm $\qquad$ b. $\boxed{1.2\text{ mm}}$ $\qquad$ c. 12 mm

2. How many interference orders can be seen with this grating?

a. 2 $\qquad$ b. 3 $\qquad$ c. $\boxed{4}$

The sin of the diffraction angle can never be larger than 1: $\sin\theta \leq 1$.

From $\sin\theta = m\lambda/d$, we obtain $m \leq d/\lambda = 2.5\ \mu m/0.589\ \mu m = 4.2$.

So, $m = 4$.

3. Which will reduce the maximum number of interference orders?

a. Increase λ $\qquad$ b. Increase d $\qquad$ c. Increase N

Solution

1. Assuming we fully illuminate the grating from the previous problem ($d = 2.5\ \mu m$), how big must it be to resolve the Na lines (589 nm, 589.6 nm)?

a. 0.12 mm $\boxed{\text{b. 1.2 mm}}$ c. 12 mm

2. How many interference orders can be seen with this grating?

a. 2 b. 3 $\boxed{\text{c. 4}}$

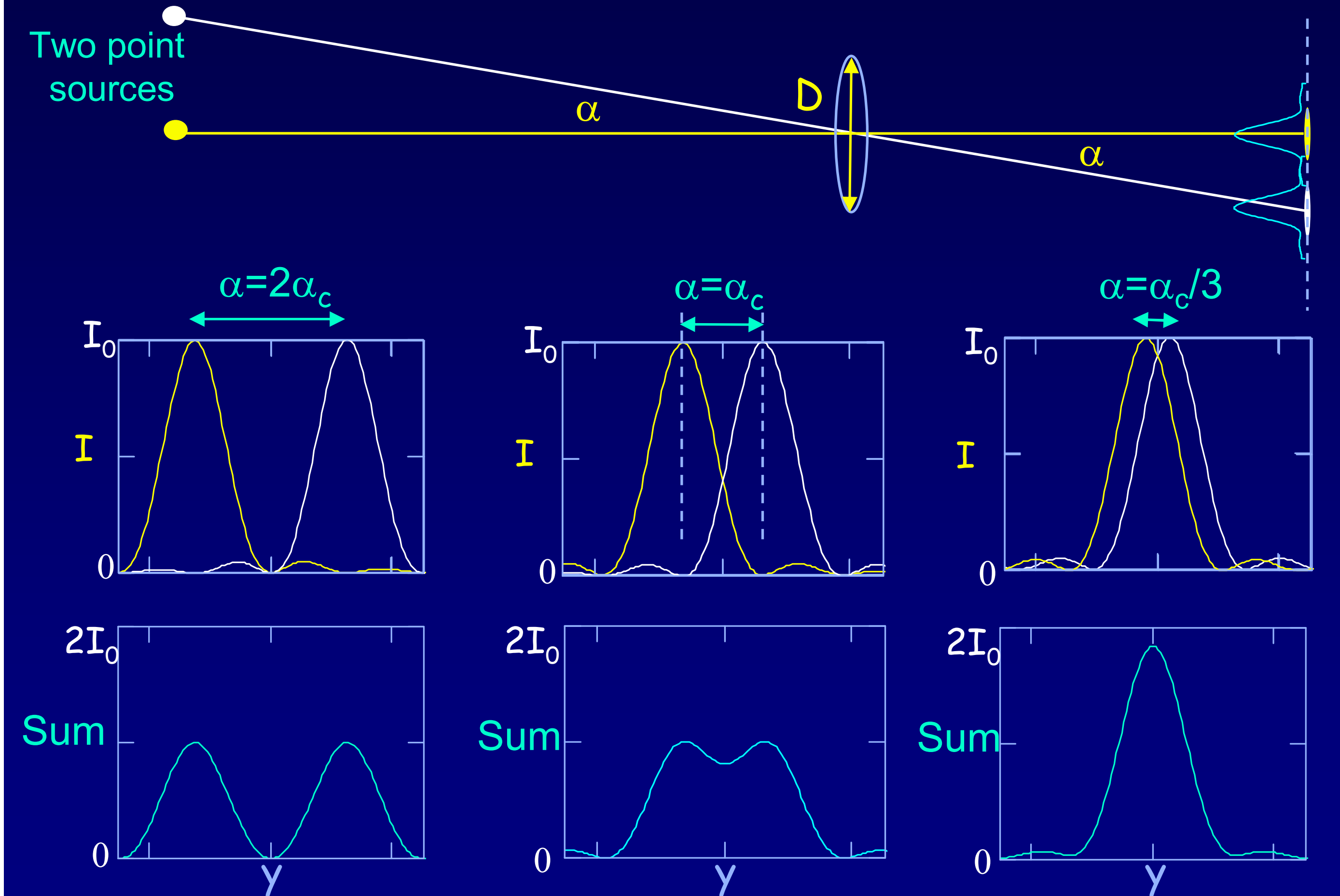
3. Which will reduce the maximum number of interference orders?

$\boxed{\text{a. Increase } \lambda}$ b. Increase d c. Increase N

$m \leq d/\lambda$, so increase λ, or decrease d.
Changing N does not affect the number of orders.

Angular Resolution (from L5)

Diffraction also limits our ability to "resolve" (*i.e.*, distinguish) two point sources. Consider two point sources (e.g., stars) with angular separation α viewed through a circular aperture or lens of diameter D.



Two images resolvable

'Diffraction limit' of resolution

Two images not resolvable

Just as before, Rayleigh's Criterion define the images to be resolved if the central maximum of one image falls on the first minimum of the second image.

$$\alpha_c = 1.22\frac{\lambda}{D}$$

NOTE: No interference!! Why not?

Exercise: Angular resolution

Car headlights in the distance:

What is the maximum distance L you can be from an oncoming car at night, and still distinguish its two headlights, which are separated by a distance d = 1.5 m? Assume that your pupils have a diameter D = 2 mm at night, and that the wavelength of light is λ = 550 nm.

Solution

Car headlights in the distance:

What is the maximum distance L you can be from an oncoming car at night, and still distinguish its two headlights, which are separated by a distance d = 1.5 m? Assume that your pupils have a diameter D = 2 mm at night, and that the wavelength of light is λ = 550 nm.

Use Rayleigh's criterion: $\alpha_c = 1.22\dfrac{\lambda}{D} = 3.4 \times 10^{-4}$ (radians)

Then, $L \approx d/\alpha_c$ = 4500 m = 2.8 miles (assuming perfect eyes).

The small angle approximation is valid.