A Brief Introduction to Diffraction Gratings

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In the Physics 214 laboratories we make extensive use of diffraction gratings: plastic films inscribed with thousands of fine grooves. These useful devices make use of the same concepts that we've discussed with two-slit interference. Below I give a brief discussion of these tools that may prove useful for the laboratory and pre-lab exercises. Note that we do not discuss gratings extensively in lecture, and you will not be tested on the additional material below.

1 Reminder: two-slit interference

In this course we've discussed in some detail the workings of a 2-slit diffraction experiment. Light of wavelength λ is normally incident upon an opaque screen inscribed with a pair of identical parallel slits, spaced apart by a distance d. If we project the transmitted light from this assembly onto a distant screen we will see a pattern of light and dark fringes. This occurs because of the wave nature of light: if observed from an angle θ relative to the central axis of the beam, the light transmitted through the two slits will arrive with a relative phase given by $\phi = 2\pi\frac{d\sin\theta}{\lambda}$. This leads to fully constructive interference (bright fringes) at angles for which $d\sin\theta = m\lambda$ for any integer m, and complete destructive interference (dark fringes) at angles for which $d\sin\theta = (m + 1/2)\lambda$. We often describe the integer m as the **order** of the fringes, *e.g.* the maxima at $m = \pm 1$ are the "first-order" fringes, $m = \pm 2$ are the "second-order" fringes, etc.

Using phasors we can derive the complete intensity pattern projected onto the distant screen:

$$I(\theta) = 4I_1 \cos^2(\phi/2), \tag{1}$$

where I_1 is the intensity transmitted through each slit on its own.

Note that the above discussion assumes ideal, infinitely-narrow slits. Since real slits have some finite width, the sinusoidal pattern above is further modulated by the effects of single-slit **diffraction**. This adds an "envelope" to the interference pattern that suppresses the higher-order fringes. More orders will thus be visible with narrow slits, fewer orders with wide slits. We study this effect qualitatively in Lab 1; a full discussion is beyond the scope of this course, and leads toward important topics such as convolution and Fourier optics.

2 A two-slit spectrometer

Now imagine that we illuminate our two-slit apparatus with a beam containing two wavelengths of light, λ_1 and λ_2. Each wavelength will produce its own pattern of equally-spaced fringes, but the spacings will be different: the first will give bright fringes at $\theta_{1,m} = \sin^{-1}(m\lambda_1/d)$, the second at $\theta_{2,m} = \sin^{-1}(m\lambda_2/d)$. The central bright fringe ($m = 0$) coincides for both (and indeed all) wavelengths; for this reason it is often called the *white light fringe*. The higher-order fringes of the two wavelengths will be separated, however, and the separation will increase with m.

We can thus use a two-slit apparatus as a simple **spectrometer**: a tool for measuring wavelengths. Since different wavelengths of light place fringes at different places, by measuring the fringe spacings we can estimate the wavelength of the incident light. Unfortunately, two slits make for a very poor spectrometer: the cosine-squared pattern in Equation 1 yields wide, blurry fringes. The fringes of different wavelengths overlap and are difficult to distinguish.

3 Multi-slit interference

Suppose we now add a third slit to our apparatus, each separated by a distance d from its neighbors and all illuminated by the same wavelength λ. Let's think qualitatively about what the diffraction pattern from this assembly will look like (see the left panel of Figure 1):

- The **maxima (bright fringes)** will be in the **same locations**. To see this, note that if the light from slits 1 and 2 are in-phase as seen from some angle, then the common spacing means that light from slit 3 must also be in phase with that from slit 2. Fully-constructive interference fringes thus occur at $d \sin\theta = m\lambda$, as before.

- There are now **more minima (dark fringes)**, because there are more ways to get fully-destructive interference. It's easiest to see this with phasors: if the relative phase difference between adjacent slits is $2\pi/3$, then the three phasors will form an equilateral triangle and sum to zero. The same will happen at a phase difference of $4\pi/3$, so there are now two equally-spaced dark fringes between each bright fringe.

- Note that the additional minima serve to make each bright fringe **narrower** than in the two-slit case, with two dark minima and one dimmer fringe in between each pair of bright fringes.

We can extend this logic to the general case of N slits. A full phasor analysis leads to the N-slit formula given in Lab 1:

$$I(\theta) = I_1 \left(\frac{\sin(N\phi/2)}{\sin(\phi/2)} \right)^2 . \tag{2}$$

By plotting this equation we can observe that as N increases at fixed d the bright fringes become brighter and narrower (because of the additional minima).

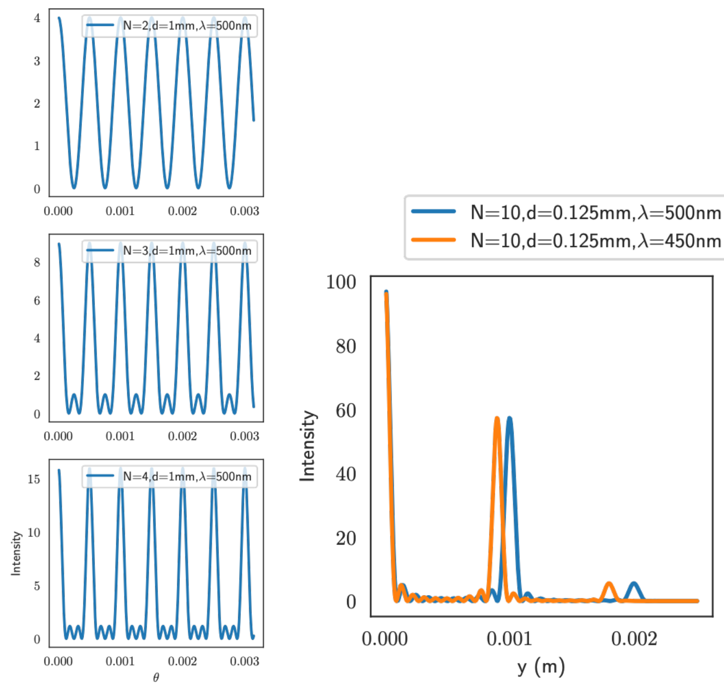

Figure 1: *Left:* Representative 2-, 3-, and 4-slit interference patterns with the same spacing and wavelength, illustrating how the bright fringes remain in the same places but become narrower with increasing N. *Right:* 10-slit interference patterns for two wavelengths that just satisfy the distinguishability criterion at first order: the peak of one fringe lines up with the first minimum of the next. Note that the 2nd-order fringes are more easily distinguishable, though dimmer. Typical gratings have far higher N, and can thus distinguish more closely-separated wavelengths. Figures by L. Wagner.

4 The diffraction grating: a better spectrometer

A diffraction grating is a sheet of material inscribed with a periodic pattern (*e.g.* grooves) that affects the transmission or reflection of light. Such a grating acts like an N-slit interference system, where N is the number of grooves illuminated by the light. Note that it's the number of illuminated grooves that matters - it's no help to have a million slits if the light only shines on a hundred of them!

Like our two-slit spectrometer above, a grating directs different wavelengths of light to different angles: the m-th order fringe of wavelength λ appears at angle $\theta_m = \sin^{-1}(m\lambda/d)$. But now each fringe is much, much *narrower*, allowing us to measure wavelength precisely and distinguish wavelengths of light that are very similar to one another. Moreover, the angular spacing between fringes of different wavelengths is greater for higher-order fringes, making it easier to distinguish nearby spectral features.

In Labs 1 and 2 you will use gratings to measure the wavelengths of light from various sources: a laser and several LED lamps, respectively. In Lab 4 you will combine a grating with a high-precision tool for angle measurement in order to measure the energies of quantum transitions in unknown gas samples.

5 Resolution limit of the diffraction grating

Suppose that we have a light source (*e.g.* a gas discharge lamp) that emits at two nearby wavelengths, $\lambda_1 = \lambda$ and $\lambda_2 = \lambda + \Delta\lambda$, with $\Delta\lambda/\lambda \ll 1$. This light illuminates N lines of a diffraction grating with line spacing d, and we examine the m-th order fringes projected onto a distant screen. Assume for simplicity that the small-angle approximation is valid ($\sin\theta \approx \theta$).

1. The m-th order principle maxima (bright fringes) from each wavelength appear at angles $d\sin\theta_{m,i} = m\lambda_i$. With the small-angle approximation, $\theta_{m,i} \approx m\lambda_i/d$ and the angular separation between these maxima is given by $\Delta\theta_m = \theta_{m,2} - \theta_{m,1} \approx m\Delta\lambda/d$.

2. With N slits there are $(N-1)$ minima equally spaced between the order-m and order-$(m+1)$ fringes. The half-width of each bright fringe is thus the angular distance between the fringe center and the nearest minimum, which is $\delta\theta_m \approx \frac{\lambda}{Nd}$.

3. The fringes from λ and $\lambda+\Delta\lambda$ will be barely distinguishable if the peak of one fringe lines up with the first minimum of the next, *i.e.* if $\Delta\theta_m = \delta\theta_m$. So finally, the two fringes will be distinguishable if

$$\boxed{\frac{\Delta\lambda}{\lambda} \gtrsim \frac{1}{Nm}} \tag{3}$$

This last is our final expression for the approximate resolution of a spectrometer built from a grating with N illuminated slits observed at order m. An example is shown in the right panel of Figure 1.