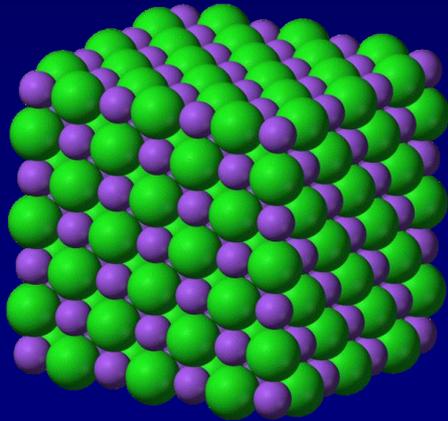


Lecture 6: Waves Review, Crystallography, and Examples



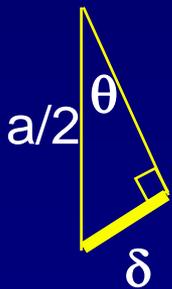
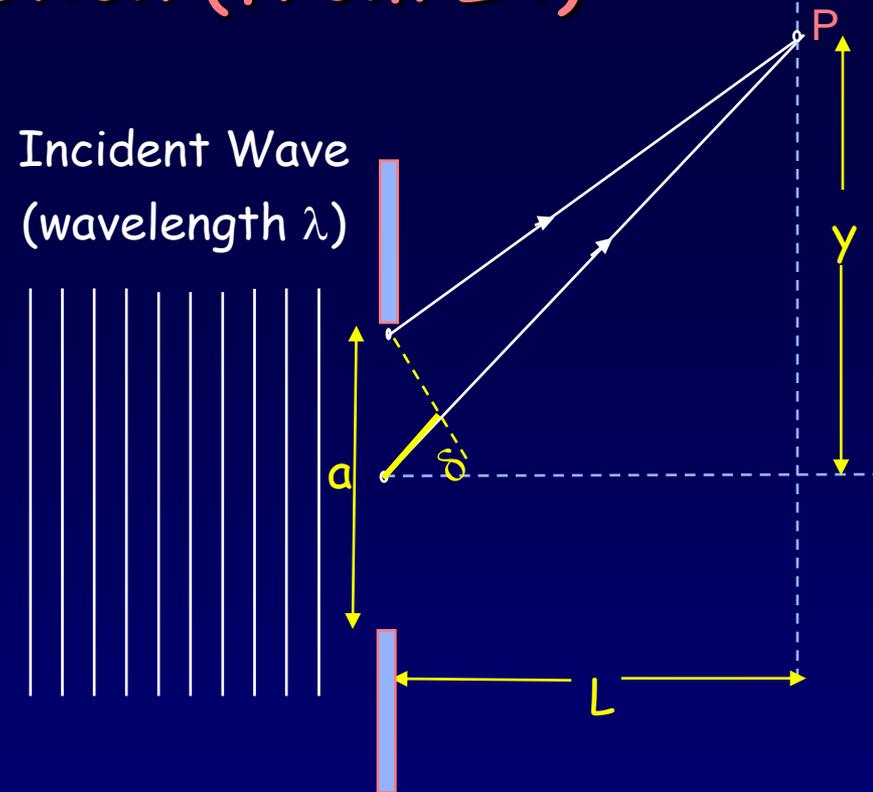
Single-Slit Diffraction (from L4)

Slit of width a . Where are the minima?

Use Huygens' principle: treat each point across the opening of the slit as a wave source.

The first minimum is at an angle such that the light from the top and the middle of the slit destructively interfere.

This works, because for every point in the top half, there is a corresponding point in the bottom half that cancels it.



$$\delta = \frac{a}{2} \sin \theta_{\min} = \frac{\lambda}{2}$$

$$\Rightarrow \sin \theta_{\min} = \frac{\lambda}{a}$$

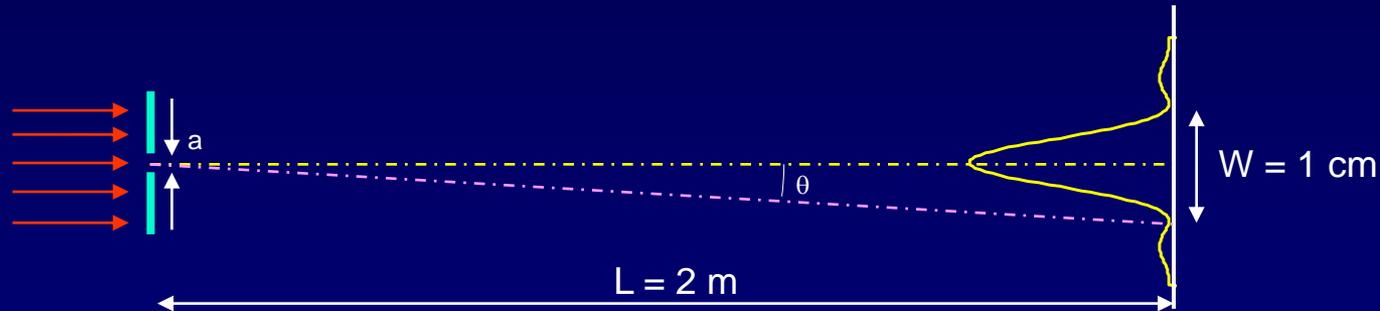
The second minimum is at an angle such that the light from the top and a point at $a/4$ destructively interfere:

$$\delta = \frac{a}{4} \sin \theta_{\min,2} = \frac{\lambda}{2} \Rightarrow \sin \theta_{\min,2} = \frac{2\lambda}{a}$$

$$\text{Location of } n\text{th-minimum: } \sin \theta_{\min,n} = \frac{n\lambda}{a}$$

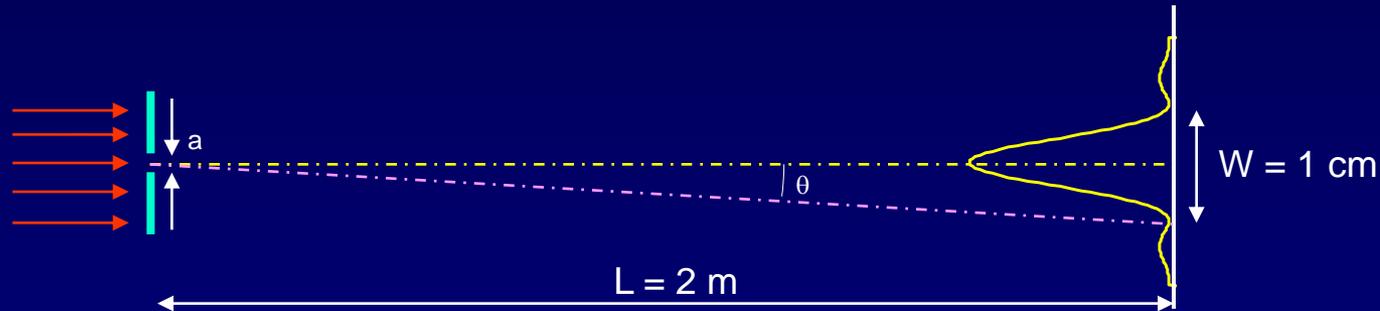
Single-Slit Diffraction Example

Suppose that when we pass red light ($\lambda = 600 \text{ nm}$) through a slit of unknown width a , the width of the spot (the distance between the first zeros on each side of the bright peak) is $W = 1 \text{ cm}$ on a screen that is $L = 2 \text{ m}$ behind the slit. How wide is the slit?



Solution

Suppose that when we pass red light ($\lambda = 600 \text{ nm}$) through a slit of unknown width a , the width of the spot (the distance between the first zeros on each side of the bright peak) is $W = 1 \text{ cm}$ on a screen that is $L = 2 \text{ m}$ behind the slit. How wide is the slit?

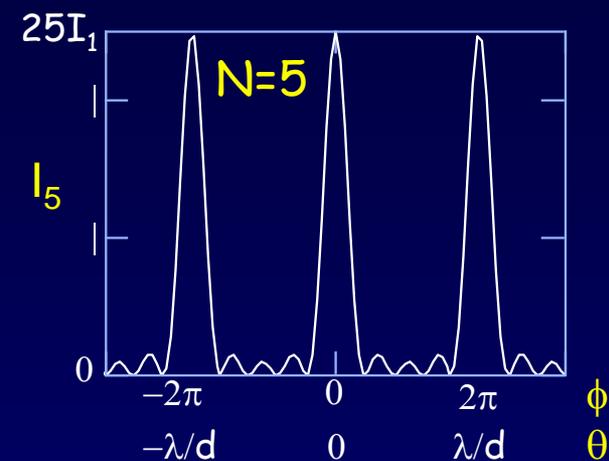
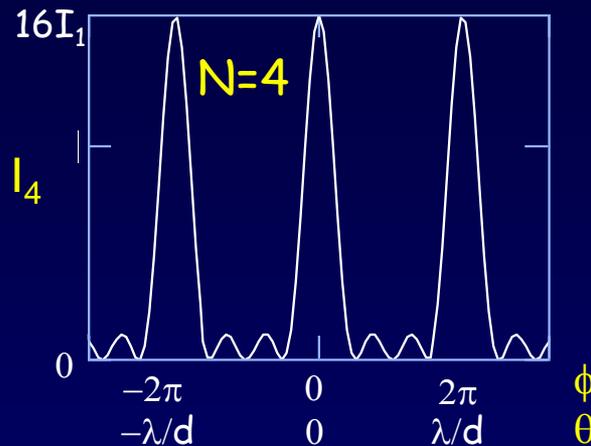
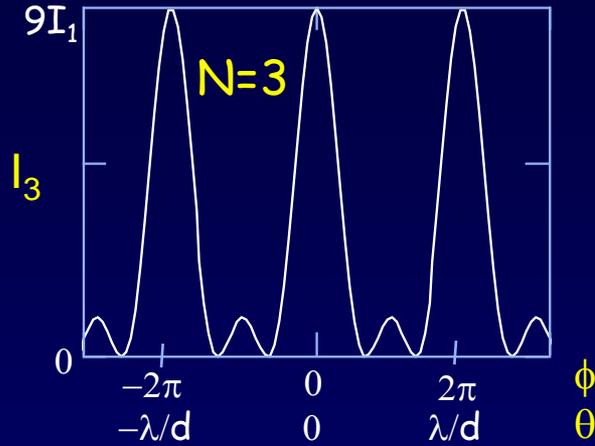


The angle to the first zero is: $\theta = \pm\lambda/a$

$$W = 2L \tan\theta \cong 2L\theta = 2L\lambda/a$$

$$a = 2L\lambda/W = (4\text{m})(6 \times 10^{-7} \text{ m}) / (10^{-2} \text{ m}) = 2.4 \times 10^{-4} \text{ m} = 0.24 \text{ mm}$$

Multiple Slit Interference (from L4)



The positions of the principal maxima occur at $\phi = 0, \pm 2\pi, \pm 4\pi, \dots$
 where ϕ is the phase between adjacent slits. $\theta = 0, \pm \lambda/d, \pm 2\lambda/d, \dots$

The intensity at the peak of a principal maximum goes as N^2 .

3 slits: $A_{\text{tot}} = 3A_1 \Rightarrow I_{\text{tot}} = 9I_1$. N slits: $I_N = N^2I_1$.

Between two principal maxima there are $N-1$ zeros and $N-2$ secondary maxima \Rightarrow The peak width $\propto 1/N$.

The total power in a principal maximum is proportional to $N^2(1/N) = N$.

Act 1

Light interfering from 10 equally spaced slits initially illuminates a screen. Now we double the number of slits, keeping the spacing constant.

1. What happens to the intensity I at the principal maxima?
a. stays same (I) b. doubles ($2I$) c. quadruples ($4I$)
2. What happens to the net power on the screen?
a. stays same b. doubles c. quadruples

Solution

Light interfering from 10 equally spaced slits initially illuminates a screen. Now we double the number of slits, keeping the spacing constant.

1. What happens to the intensity I at the principal maxima?

- a. stays same (I) b. doubles ($2I$) c. quadruples ($4I$)

$$I_N = N^2 I_1. \quad 10 \rightarrow 20 \text{ means } 100 \rightarrow 400.$$

2. What happens to the net power on the screen?

- a. stays same b. doubles c. quadruples

Solution

Light interfering from 10 equally spaced slits initially illuminates a screen. Now we double the number of slits, keeping the spacing constant.

1. What happens to the intensity I at the principal maxima?

- a. stays same (I) b. doubles ($2I$) **c. quadruples ($4I$)**

$$I_N = N^2 I_1. \quad 10 \rightarrow 20 \text{ means } 100 \rightarrow 400.$$

2. What happens to the net power on the screen?

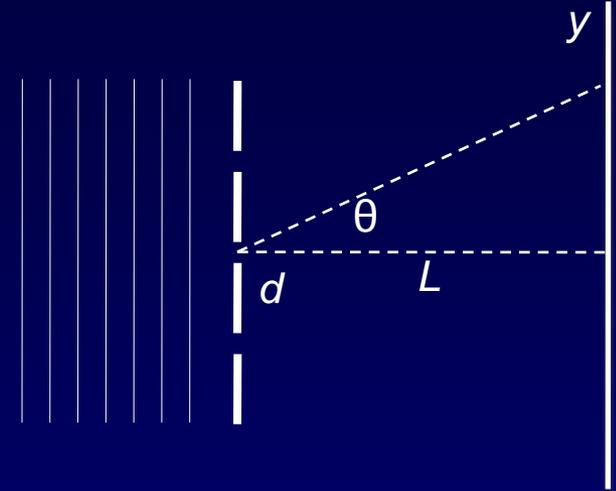
- a. stays same **b. doubles** c. quadruples

If we double the number of slits, we expect the power on the screen to double. How does this work?

- The number of principal maxima (which have most of the power) does *not* change.
- The principal maxima become 4x brighter.
- But they also become only half as wide.
- Therefore, the net power (integrating over all the peaks) increases two-fold, as we would expect.

Multiple-slit Example

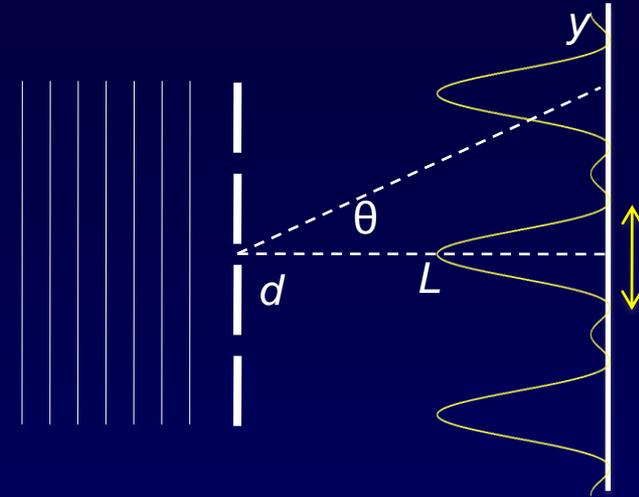
Three narrow slits with equal spacing d are at a distance $L = 1.4 \text{ m}$ away from a screen. The slits are illuminated at normal incidence with light of wavelength $\lambda = 570 \text{ nm}$. The first principal maximum on the screen is at $y = 2.0 \text{ mm}$.



1. What is the slit spacing, d ?
2. If the wavelength, λ , is increased, what happens to the width of the principal maxima?
3. If the intensity of each slit alone is I_1 , what is the intensity of the secondary maximum?

Solution

Three narrow slits with equal spacing d are at a distance $L = 1.4 \text{ m}$ away from a screen. The slits are illuminated at normal incidence with light of wavelength $\lambda = 570 \text{ nm}$. The first principal maximum on the screen is at $y = 2.0 \text{ mm}$.



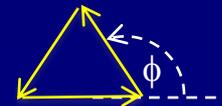
1. What is the slit spacing, d ?

The first maximum occurs when the path difference between adjacent slits is λ . This happens at $\sin\theta = \lambda/d$. We are told that $\tan\theta = y/L = 1.43 \times 10^{-3}$, so the small angle approximation is OK. Therefore, $d \approx \lambda/\theta = 0.40 \text{ mm}$.

2. If the wavelength, λ , is increased, what happens to the width of the principal maxima?

The relation between θ and ϕ is $\phi/2\pi = \delta/\lambda = d \sin\theta / \lambda$.

Therefore, for every feature that is described by ϕ (peaks, minima, etc.) $\sin\theta$ is proportional to λ . The width increases.



3. If the intensity of each slit alone is I_1 , what is the intensity of the secondary maximum?

Phasor diagram: = \longrightarrow

Two phasors cancel, leaving only one $\rightarrow I_1$

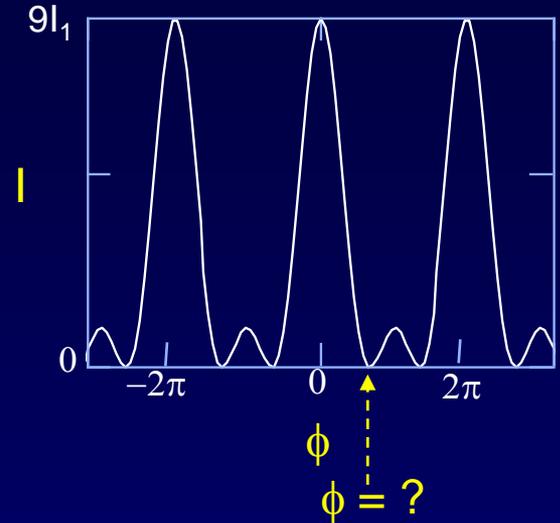
ACT 2: Multiple Slits

1. What value of ϕ corresponds to the first zero of the **3-slit** interference pattern?

a) $\phi = \pi/2$

b) $\phi = 2\pi/3$

c) $\phi = 3\pi/4$



2. What value of ϕ corresponds to the first zero of the **4-slit** interference pattern?

a) $\phi = \pi/2$

b) $\phi = 2\pi/3$

c) $\phi = 3\pi/4$

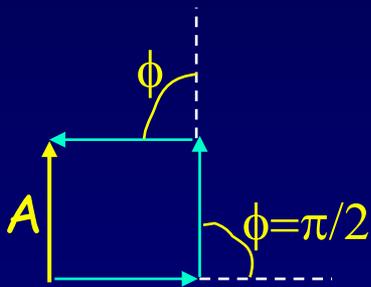
Solution

1. What value of ϕ corresponds to the first zero of the 3-slit interference pattern?

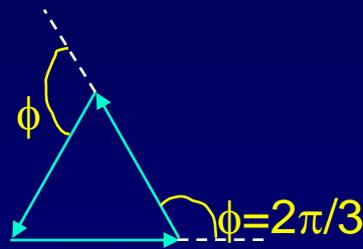
a) $\phi = \pi/2$

b) $\phi = 2\pi/3$

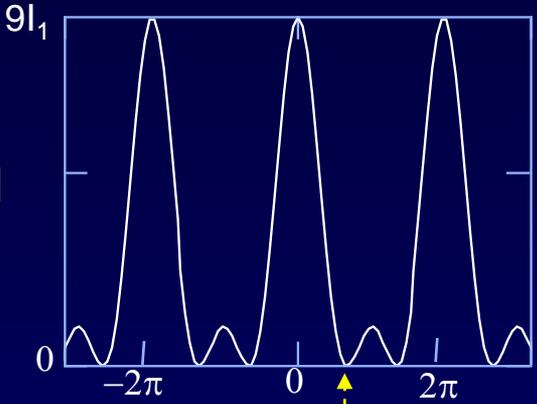
c) $\phi = 3\pi/4$



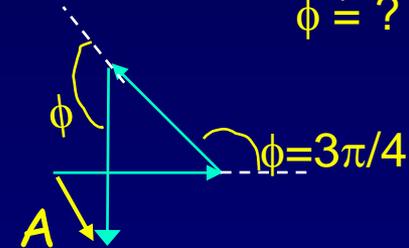
No.
A is not zero.



Yes!
Equilateral triangle
gives $A = 0$.



ϕ
 $\phi = ?$



No.
Triangle does not close.

2. What value of ϕ corresponds to the first zero of the 4-slit interference pattern?

a) $\phi = \pi/2$

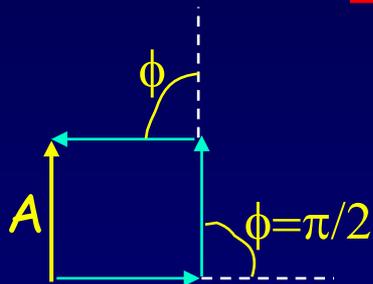
b) $\phi = 2\pi/3$

c) $\phi = 3\pi/4$

Solution

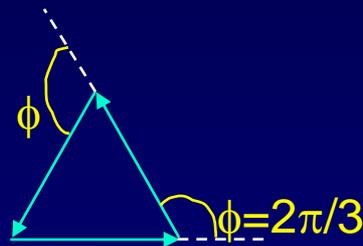
1. What value of ϕ corresponds to the first zero of the 3-slit interference pattern?

a) $\phi = \pi/2$



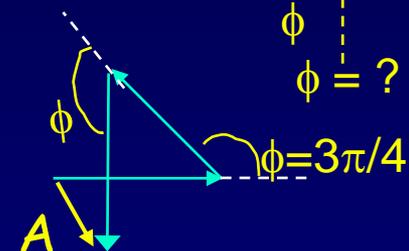
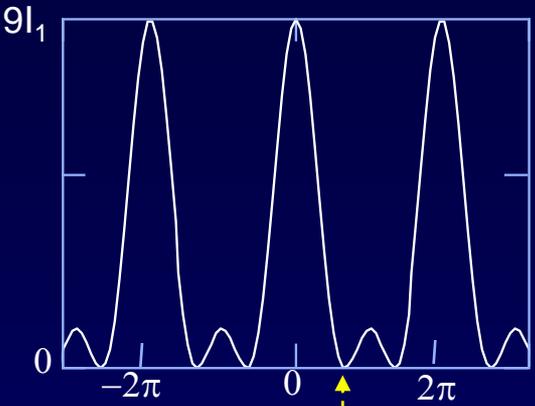
No.
A is not zero.

b) $\phi = 2\pi/3$



Yes!
Equilateral triangle gives $A = 0$.

c) $\phi = 3\pi/4$



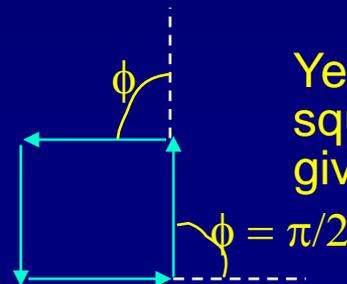
No.
Triangle does not close.

2. What value of ϕ corresponds to the first zero of the 4-slit interference pattern?

a) $\phi = \pi/2$

b) $\phi = 2\pi/3$

c) $\phi = 3\pi/4$



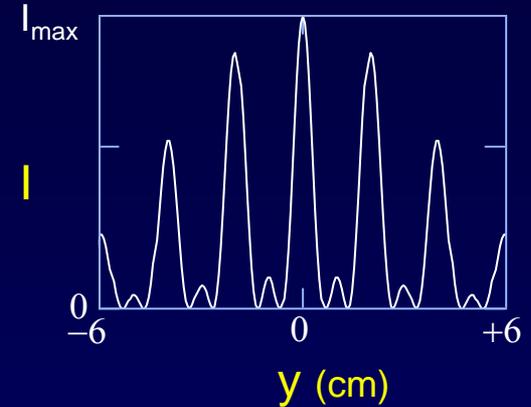
Yes. The square gives $A = 0$.

To get a zero, we need a closed figure. $N\phi$ must be a multiple of 2π , so the first zero is at $2\pi/N$.

Interference & Diffraction Exercise

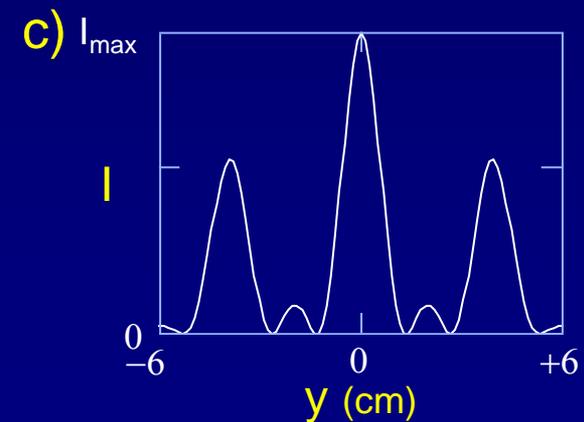
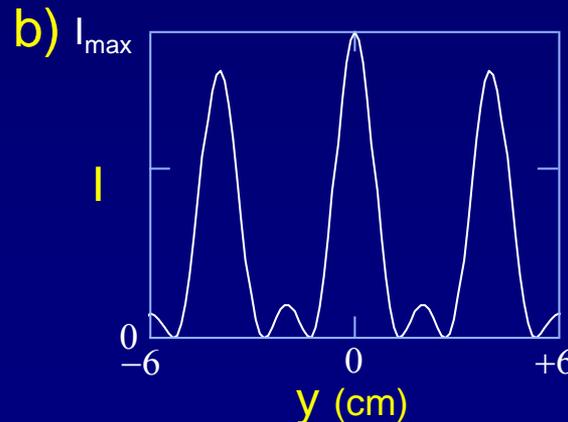
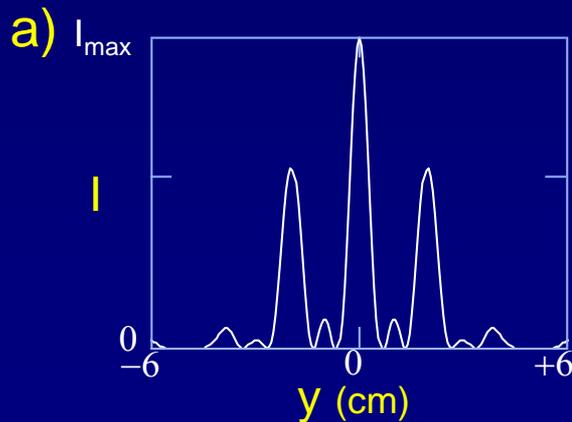
Light of wavelength λ is incident on an N -slit system with slit width a and slit spacing d .

1. The intensity I as a function of y at a viewing screen located a distance L from the slits is shown to the right. $L \gg d, y, a$. What is N ?



- a) $N = 2$ b) $N = 3$ c) $N = 4$

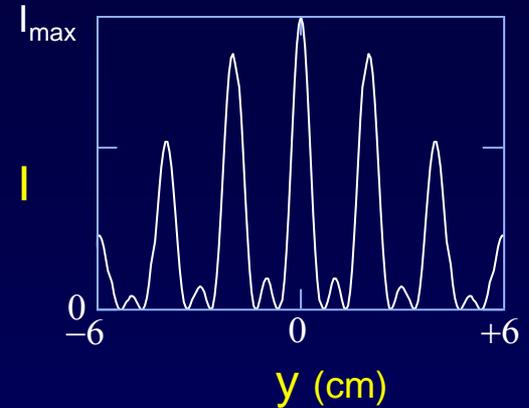
2. Now the slit spacing d is halved, but the slit width a is kept constant. Which of the graphs best represents the new intensity distribution?



Solution

Light of wavelength λ is incident on an N -slit system with slit width a and slit spacing d .

1. The intensity I as a function of y at a viewing screen located a distance L from the slits is shown to the right. $L \gg d, y, a$. What is N ?



a) $N = 2$

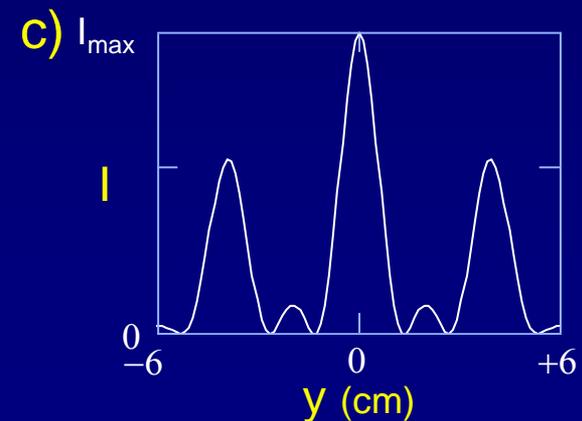
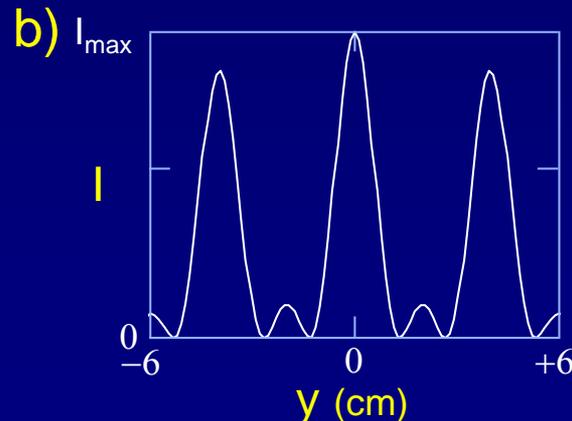
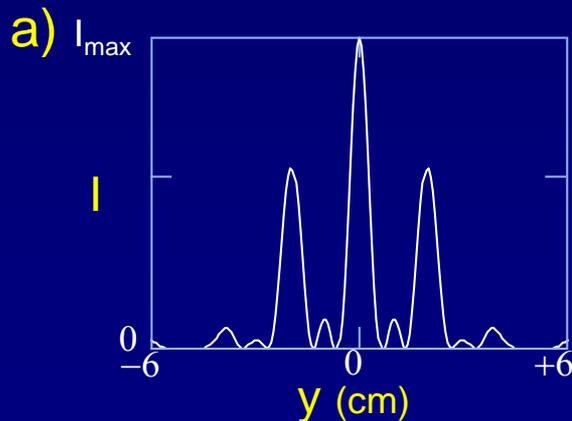
b) $N = 3$

c) $N = 4$

N is determined from the number of minima between two principal maxima.

$N = \#_{\text{minima}} + 1$ Therefore, $N = 3$.

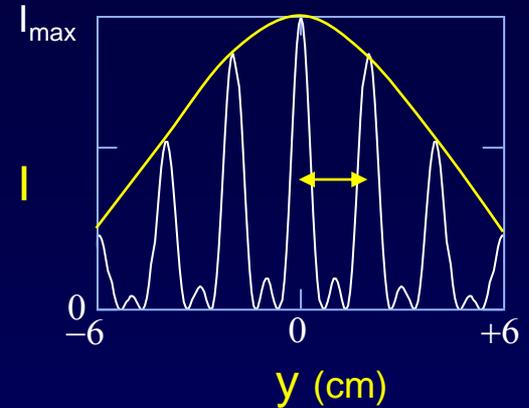
2. Now the slit spacing d is halved, but the slit width a is kept constant. Which of the graphs best represents the new intensity distribution?



Solution

Light of wavelength λ is incident on an N -slit system with slit width a and slit spacing d .

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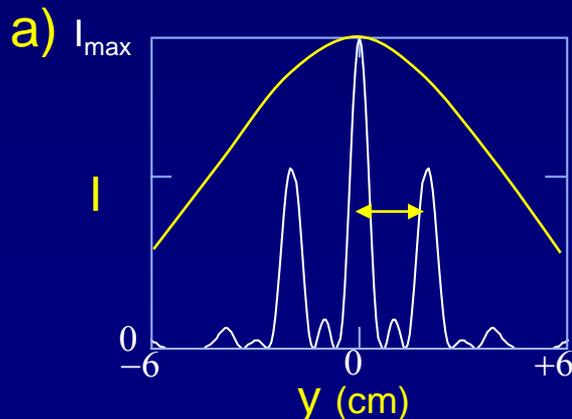
b) $N = 3$

c) $N = 4$

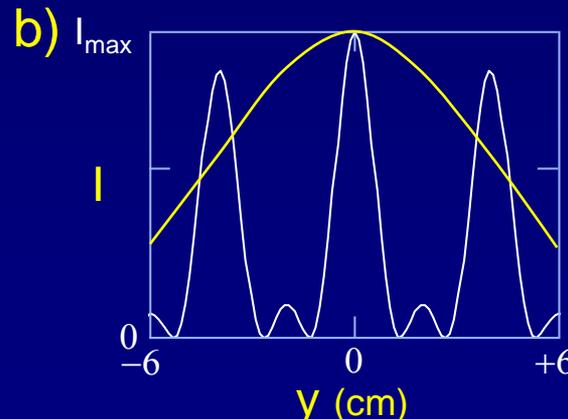
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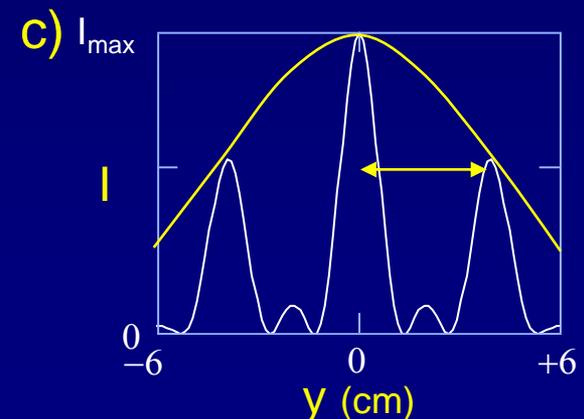
2. Now the slit spacing d is halved, but the slit width a is kept constant. Which of the graphs best represents the new intensity distribution?



Interference spacing should change.



Diffraction profile shouldn't change.



Interference spacing doubles. Diffraction profile is unchanged.

Diffraction from gratings

$$I_N = I_1 \left(\frac{\sin(N\phi/2)}{\sin(\phi/2)} \right)^2$$

The slit/line spacing determines the location of the peaks (and the angular dispersing power $\theta(\lambda)$ of the grating:

The positions of the principal interference maxima are the *same* for any number of slits!

$$d \sin\theta = m\lambda$$

The number of slits/beam size determines the *width* of the peaks (narrower peaks easier to resolve).

$$\delta\theta \approx \lambda/Nd$$

Resolving power of an N-slit grating: The Rayleigh criterion

Diffraction Grating Example

Angular splitting of the Sodium doublet:

Consider the two closely spaced spectral (yellow) lines of sodium (Na), $\lambda_1 = 589 \text{ nm}$ and $\lambda_2 = 589.6 \text{ nm}$. If light from a sodium lamp fully illuminates a diffraction grating with 4000 slits/cm , what is the angular separation of these two lines in the second-order ($m=2$) spectrum?

Hint: First find the slit spacing d from the number of slits per centimeter.

Solution

Angular splitting of the Sodium doublet:

Consider the two closely spaced spectral (yellow) lines of sodium (Na), $\lambda_1 = 589 \text{ nm}$ and $\lambda_2 = 589.6 \text{ nm}$. If light from a sodium lamp fully illuminates a diffraction grating with 4000 slits/cm, what is the angular separation of these two lines in the second-order ($m=2$) spectrum?

Hint: First find the slit spacing d from the number of slits per centimeter.

$$d = \frac{1 \text{ cm}}{4000} = 2.5 \times 10^{-4} \text{ cm} = 2.5 \mu\text{m}$$

$$\theta_1 = \sin^{-1}\left(m \frac{\lambda_1}{d}\right) = 28.112^\circ$$

Small angle approximation is not valid.

$$\Delta\theta = \sin^{-1}\left(m \frac{\lambda_2}{d}\right) - \sin^{-1}\left(m \frac{\lambda_1}{d}\right) = 0.031^\circ$$

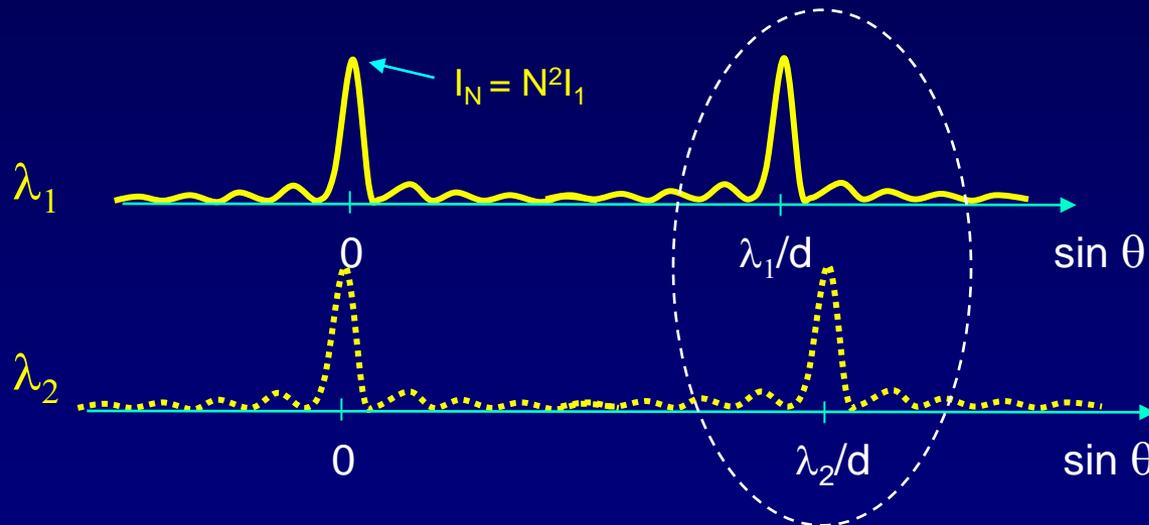
Diffraction Gratings (1)

Diffraction gratings rely on N-slit interference.

They consist of a large number of evenly spaced parallel slits.

An important question:

How effective are diffraction gratings at resolving light of different wavelengths (*i.e.* separating closely-spaced ‘spectral lines’)?



$\sin \theta$ depends on λ .

Example: Na has a spectrum with two yellow “lines” very close together: $\lambda_1 = 589.0 \text{ nm}$, $\lambda_2 = 589.6 \text{ nm}$ ($\Delta\lambda = 0.6 \text{ nm}$)

Are these two lines distinguishable using a particular grating?

We need a “resolution criterion”.

Diffraction Gratings (2)

We use Rayleigh's criterion:

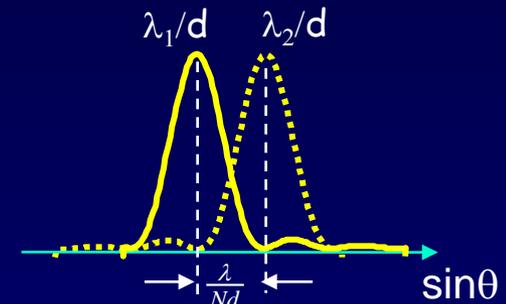
The minimum wavelength separation we can resolve occurs when the λ_2 peak coincides with the first zero of the λ_1 peak:

So, the Rayleigh criterion is $\Delta(\sin\theta)_{\min} = \lambda/Nd$.

However, the location of the peak is $\sin\theta = m\lambda/d$.

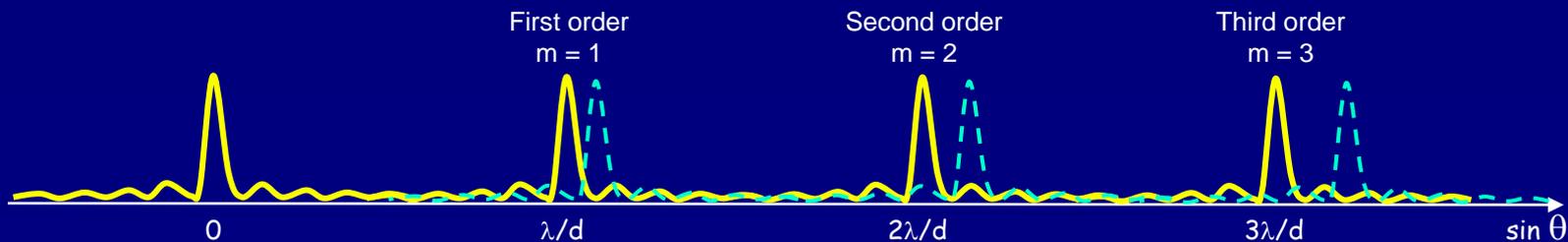
Thus, $(\Delta\lambda)_{\min} = (d/m)\Delta(\sin\theta)_{\min} = \lambda/mN$:

$$\frac{\Delta\lambda_{\min}}{\lambda} = \frac{1}{Nm}$$



Comments:

- It pays to use a grating that has a large number of lines, N . However, one must illuminate them all to get this benefit.
- It also pays to work at higher order (larger m): The widths of the peaks don't depend on m , but they are farther apart at large m .



ACT 2

1. Suppose we fully illuminate a grating for which $d = 2.5 \mu\text{m}$. How big must it be to resolve the Na lines (589 nm, 589.6 nm), if we are operating at second order ($m = 2$)?
 - a. 0.12 mm
 - b. 1.2 mm
 - c. 12 mm
2. How many interference orders can be seen with this grating?
 - a. 2
 - b. 3
 - c. 4
3. Which will reduce the maximum number of interference orders?
 - a. Increase λ
 - b. Increase d
 - c. Increase N

Solution

1. Suppose we fully illuminate a grating for which $d = 2.5 \mu\text{m}$. How big must it be to resolve the Na lines (589 nm, 589.6 nm), if we are operating at second order ($m = 2$)?

a. 0.12 mm

b. 1.2 mm

c. 12 mm

We need to make N large enough to satisfy Raleigh's criterion.

$$\frac{\Delta\lambda}{\lambda} = \frac{1}{Nm} \quad \text{So: } N \geq \frac{\lambda}{m \Delta\lambda} = \frac{589\text{nm}}{2(0.6\text{nm})} = 491$$

$$\text{Size} = Nd \geq 491 \times 2.5 \mu\text{m} \approx 1.2 \text{ mm}$$

2. How many interference orders can be seen with this grating?

a. 2

b. 3

c. 4

Solution

1. Suppose we fully illuminate a grating for which $d = 2.5 \mu\text{m}$. How big must it be to resolve the Na lines (589 nm, 589.6 nm), if we are operating at second order ($m = 2$)?

a. 0.12 mm

b. 1.2 mm

c. 12 mm

2. How many interference orders can be seen with this grating?

a. 2

b. 3

c. 4

The sin of the diffraction angle can never be larger than 1: $\sin\theta \leq 1$.

From $\sin\theta = m\lambda/d$, we obtain $m \leq d/\lambda = 2.5 \mu\text{m}/0.589 \mu\text{m} = 4.2$.

So, $m = 4$.

3. Which will reduce the maximum number of interference orders?

a. Increase λ

b. Increase d

c. Increase N

Solution

1. Assuming we fully illuminate the grating from the previous problem ($d = 2.5 \mu\text{m}$), how big must it be to resolve the Na lines (589 nm, 589.6 nm)?

a. 0.12 mm

b. 1.2 mm

c. 12 mm

2. How many interference orders can be seen with this grating?

a. 2

b. 3

c. 4

3. Which will reduce the maximum number of interference orders?

a. Increase λ

b. Increase d

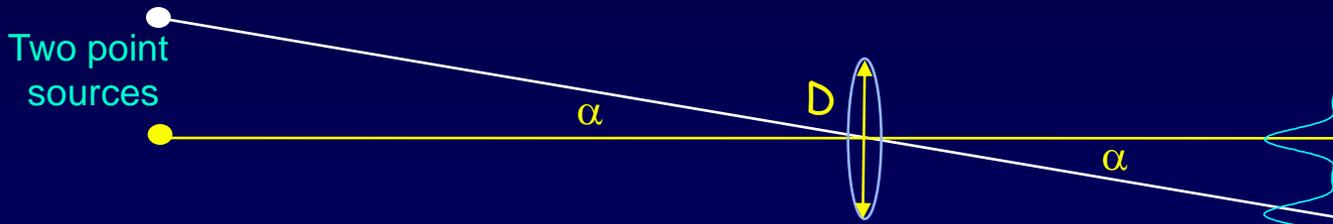
c. Increase N

$m \leq d/\lambda$, so increase λ , or decrease d .

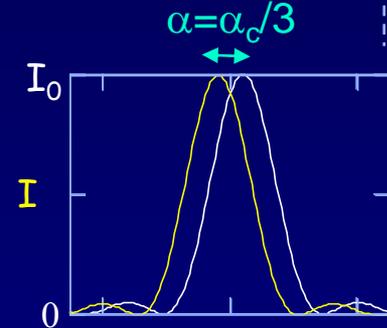
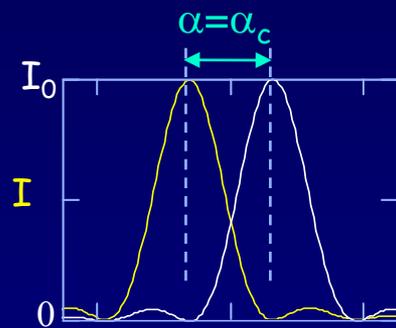
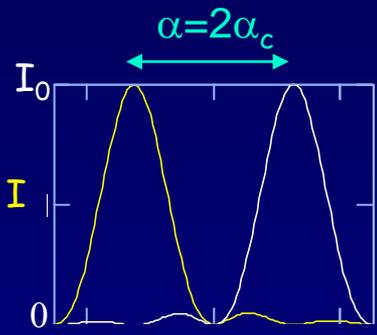
Changing N does not affect the number of orders.

Angular Resolution (from L5)

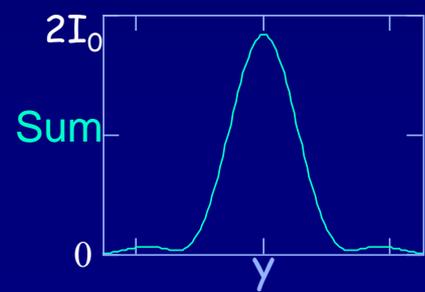
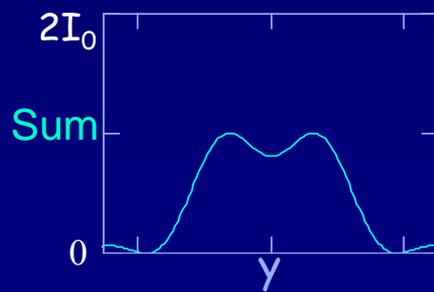
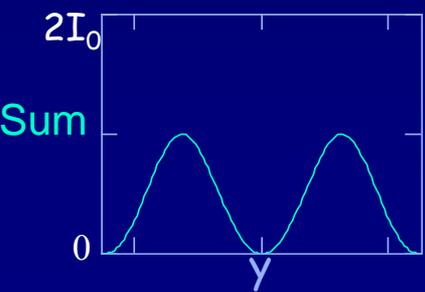
Diffraction also limits our ability to “resolve” (*i.e.*, distinguish) two point sources. Consider two point sources (e.g., stars) with angular separation α viewed through a circular aperture or lens of diameter D .



Just as before, Rayleigh's Criterion define the images to be resolved if the central maximum of one image falls on the first minimum of the second image.



$$\alpha_c = 1.22 \frac{\lambda}{D}$$



Two images resolvable

'Diffraction limit' of resolution

Two images not resolvable

NOTE:
No interference!!
Why not?

Exercise: Angular resolution

Car headlights in the distance:

What is the **maximum distance L** you can be from an oncoming car at night, and still distinguish its two headlights, which are separated by a distance **$d = 1.5 \text{ m}$** ?

Assume that your pupils have a diameter **$D = 2 \text{ mm}$** at night, and that the wavelength of light is **$\lambda = 550 \text{ nm}$** .



Solution

Car headlights in the distance:

What is the **maximum distance L** you can be from an oncoming car at night, and still distinguish its two headlights, which are separated by a distance **d = 1.5 m**?

Assume that your pupils have a diameter **D = 2 mm** at night, and that the wavelength of light is **$\lambda = 550$ nm**.



Use Rayleigh's criterion: $\alpha_c = 1.22 \frac{\lambda}{D} = 3.4 \times 10^{-4}$ (radians)

Then, $L \approx d/\alpha_c = 4500$ m = 2.8 miles (assuming perfect eyes).

The small angle approximation is valid.

Crystal diffraction

How do we know the atomic scale structure of matter around us?

A crystal is a very large number of atoms or molecules arranged in a periodic fashion. It acts like a grating with an extremely large number (\sim Avagadro's number) of units that diffract waves coherently.

Every crystal has its own “signature” of the various spacings between atoms.

By measuring the diffraction, we can determine the atomic scale structure.

Typical distances between atoms are of order 0.1-0.3 nm ($1-3 \times 10^{-10}$ m).

What characteristic wavelengths are needed to study crystals?

We want:

- $\lambda < \text{spacing}$ (so that we can get $\delta > \lambda$).
- λ not too small (so that θ isn't too small).

That is: $\lambda \sim 10^{-10}$ m. \Rightarrow **X-rays!**

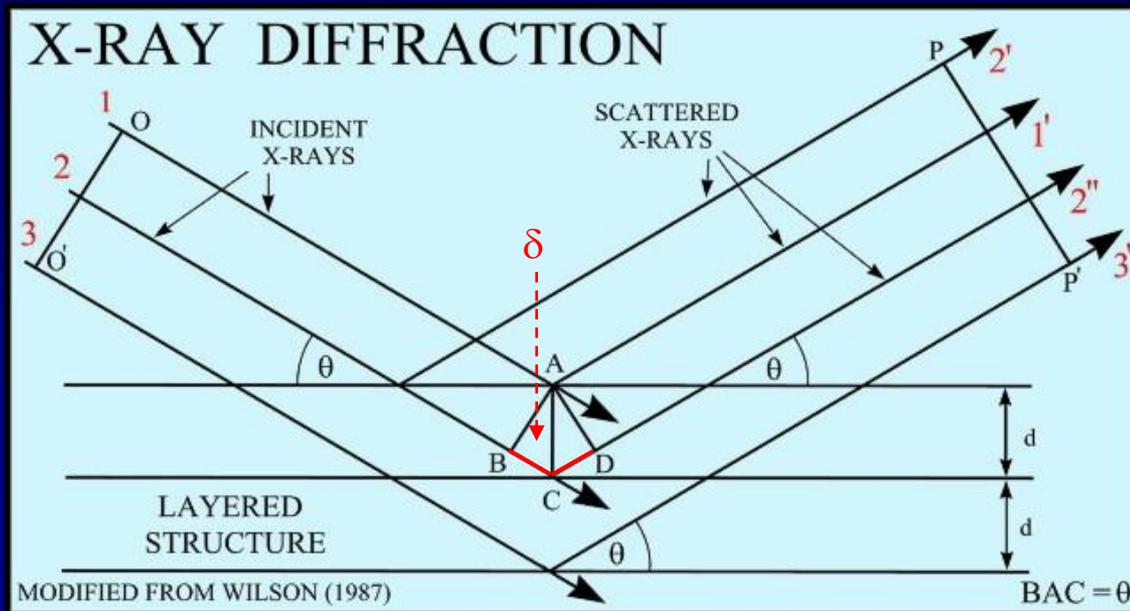
FYI: Crystal diffraction

The structure of the crystal can be found using almost the same law we have for optical gratings!

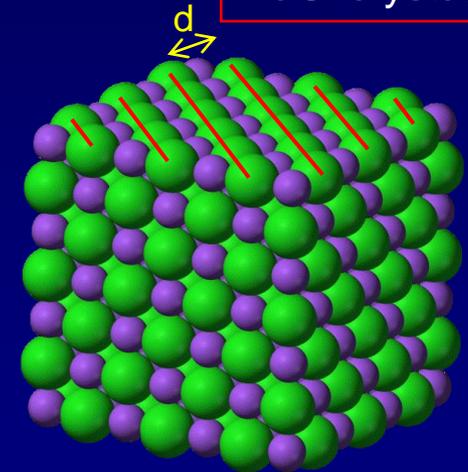
Bragg Law for constructive interference: $\delta = 2d \sin\theta = m\lambda$

d = lattice spacing, λ = x-ray wavelength

θ = x-ray angle (with respect to plane of crystal)



Example of planes in a NaCl crystal



Each crystal has many values of d - the distances between different planes. For a known wavelength λ the observed angles θ can be used to determine the crystal structure.

FYI: Application: Structure of DNA

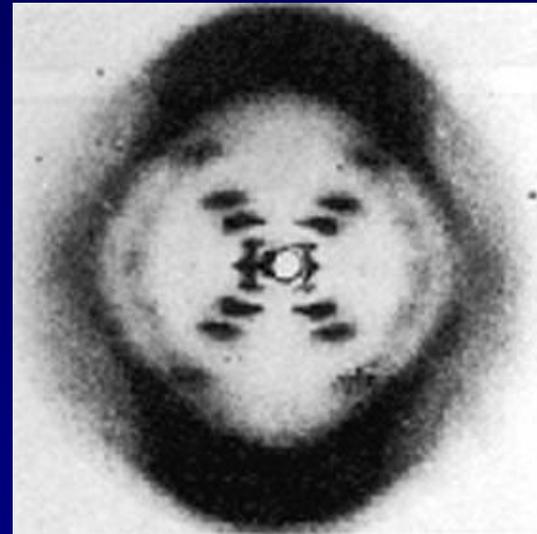
How do we know the structures of DNA, proteins and other biological molecules? X-ray diffraction!

The molecules are crystallized to create a crystal in which the molecules are arranged in a periodic lattice. By using the sharp Bragg diffraction from many molecules, the structure of each molecule is determined - the positions of thousands of atoms

The original diffraction image of DNA taken by Rosalind Franklin in 1953

Actually the DNA was not crystallized for the first DNA images, but the DNA was dehydrated in a fiber, and formed a “quasi-crystalline structure that showed the helical structure.

The dark bands arranged in a cross were the first evidence of the helical structure.



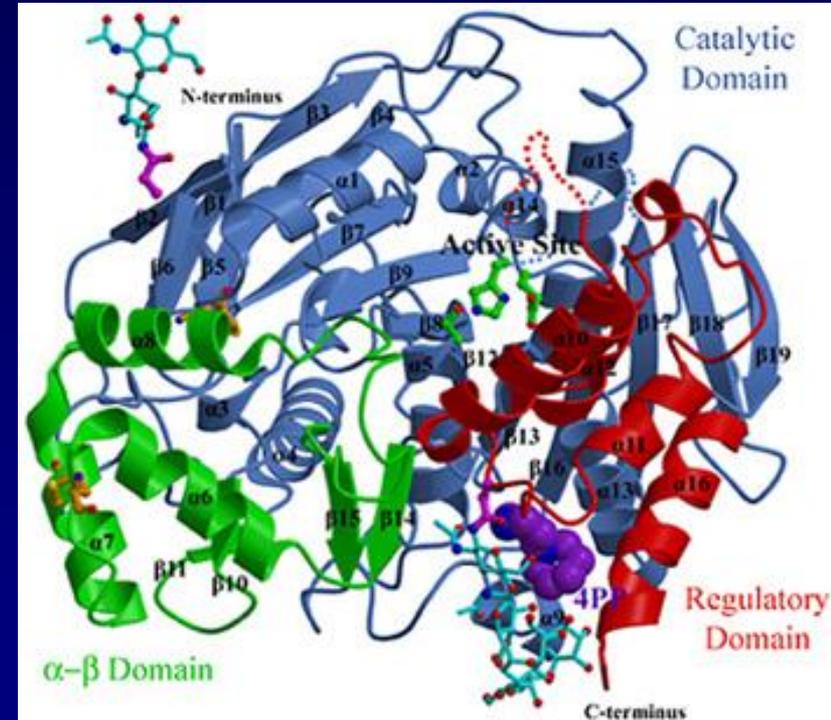
See figure and more details on this and other X-ray scattering in the text, sec. 36.6

FYI: Modern Applications in Biology

X-rays remain the primary methods for establishing the atomic scale structures of complex molecules.

Example: Rabbit liver carboxylesterase (one molecule showing atomic groups and attached large scale structures with atoms not shown).

“Alternative strategies to improve the antitumor efficacy have concentrated upon the design of novel camptothecin analogs. To effect this, we have determined the x-ray crystal structures of the rabbit liver carboxylesterase ...”



Source: St. Jude Children Research Hospital
http://www.stjude.org/faculty/0,2512,407_2030_4278,00.html