"But why must I treat the measuring device classically? What will happen to me if I *don't*??"

--Eugene Wigner

"There is obviously no such limitation – I can measure the energy and look at my watch; then I know both energy and time!"

--L. D. Landau, on the time-energy uncertainty principle

Lab 3 Comments

Lab 3 meets this week if you are normally in 132 Loomis.
Lab 3 meets next week if you are normally in 164 Loomis.
So does Discussion, and there is a quiz, so don't skip...

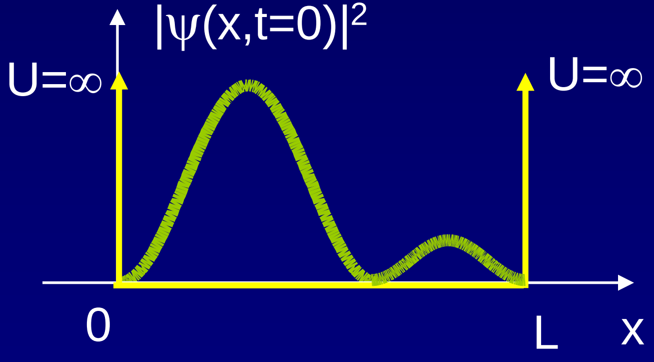
For the lab:

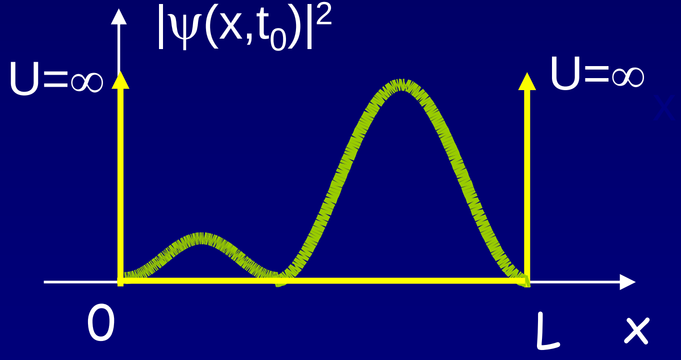
You will need your "Active Directory" Login

See: http://www.ad.uiuc.edu

You can save a lot of time by reading the lab ahead of time.
It's a tutorial on how to draw wave functions

Lecture 13: Superposition & Time-Dependent Quantum States





Last Week

Time-independent Schrodinger's Equation (SEQ):

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2}+U(x)\psi(x)=E\psi(x)$$

- It describes a particle that has a definite energy, *E*.
- The solutions, ψ(x), are time independent (stationary states).

We considered two potentials, U(x):

Finite-depth square well

- Boundary conditions.
- Particle can "leak" into forbidden region.
- Comparison with infinite-depth well.

Harmonic oscillator

- Energy levels are equally spaced.
- A good approximation in many problems.

Today

Time dependent SEQ:

Superposition of states and particle motion

Measurement in quantum physics

Time-energy uncertainty principle

Time-Dependent SEQ

To explore how particle wave functions evolve with time, which is useful for a number of applications as we shall see, we need to consider the time-dependent SEQ:

$$\boxed{-\frac{\hbar^2}{2m}\frac{d^2\Psi(x,t)}{dx^2}+U(x)\Psi(x,t)=}\boxed{i\hbar\frac{d\Psi(x,t)}{dt}}$$

Changes from the time independent version:

• $E\psi \rightarrow i\hbar\, d\Psi/dt$	We no longer assume a definite E.
• $\psi(x) \rightarrow \Psi(x,t)$	The solutions will have time dependence.
• $i = \sqrt{(-1)}$ appears	The solutions will be complex.

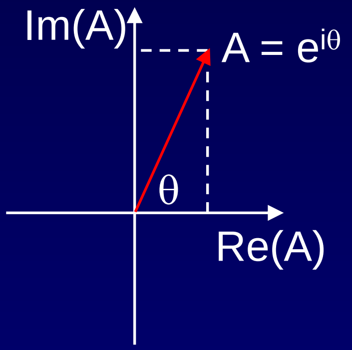
This equation describes the complete time and space dependence of a quantum particle in a potential U(x).
It replaces the classical particle dynamics law, F=ma.

The SEQ is linear in Ψ, and so the Superposition Principle applies:
If Ψ_1 and Ψ_2 are solutions to the time-dependent SEQ, then so is any linear combination of Ψ_1 and Ψ_2 (example: $\Psi = 0.6\Psi_1 + 0.8i\Psi_2$)

Review of Complex Numbers

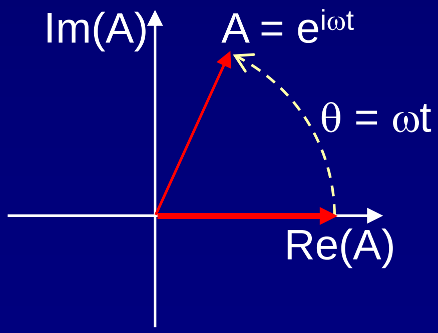
The equation, $e^{i\theta} = \cos\theta + i\sin\theta$, might be new to you. It is a convenient way to represent complex numbers. It also (once you are used to it) makes trigonometry simpler.

a. Draw an Argand diagram of $e^{i\theta}$.



The Argand diagram of a complex number, A, puts Re(A) on the x-axis and Im(A) on the y-axis. Notice the trig relation between the x and y components. θ is the angle of A from the real axis. In an Argand diagram, $e^{i\theta}$ looks like a vector of length 1, and components ($\cos\theta$, $\sin\theta$).

b. Suppose that θ varies with time, $\theta = \omega t$. How does the Argand diagram behave?



At $t = 0$, $\theta = 0$, so A = 1 (no imaginary component).
As time progresses, A rotates counterclockwise with angular frequency ω.
This is the math that underlies phasors.

The quantity, $ce^{i\theta}$ (c and θ both real), is a complex number of magnitude $|c|$. The magnitude of a complex number, A, is $|A| = \sqrt{(A^*A)}$, where A^* is the complex conjugate of A.

Act i

$$i \equiv \sqrt{-1}$$

1. What is (-i)i ?

   a. $-i$ b. -1 c. $+1$

2. What is 1/i?

   a. -1 b. $-i$ c. $+i$

3. What is $|e^{i\phi}|^2$?

   a. 0 b. $e^{2i\phi}$ c. 1

Solution

$$i \equiv \sqrt{-1}$$

1. What is (-i)i ?

a. –i b. -1 [✓ c. +1]

$$(-i)i = -i^2 = -(-1) = +1$$

2. What is 1/i?

a. –1 [✓ b. -i] c. +i

$$\frac{1}{i} = \left(\frac{1}{i}\right)\left(\frac{i}{i}\right) = \frac{i}{-1} = -i$$

3. What is $|e^{i\phi}|^2$?

a. 0 b. $e^{2i\phi}$ [✓ c. 1]

$$\underbrace{\left|e^{i\phi}\right|^2 \equiv \left(e^{i\phi}\right)^* \left(e^{i\phi}\right)} = \left(e^{-i\phi}\right)\left(e^{i\phi}\right)$$

This is the definition of the magnitude of a complex number.

$$= e^{-i\phi+i\phi} = e^0 = 1$$

Time-dependence of Energy Eigenstates

The time-independent SEQ is just a special case of the time-dependent SEQ.
So, if Ψ(x,t) is a state with definite energy, it is a solution to both equations.

Both equations have the same left-hand side, so the right sides must be equal:

RHS of time ind. SEQ

$$\boxed{E\Psi(x,t) = i\hbar\frac{d\Psi(x,t)}{dt}}$$

RHS of time dep. SEQ

This equation has the solution:

$$\boxed{\Psi(x,t) = \psi(x)e^{-i\omega t}} \quad \text{with } \omega = \frac{E}{\hbar}$$

Notes:

- ψ(x) is not determined. We need the LHS for that.
- E = ħω = hf, as expected.

Ψ is complex. However, we are interested in |Ψ|², because that's what we measure.

$$\left|\Psi(x,t)\right|^2 = \left(\psi^*(x)e^{+i\omega t}\right)\left(\psi(x)e^{-i\omega t}\right) = \left|\psi(x)\right|^2$$

This is *always* a real number.

So, for an energy eigenstate:
the probability density, |Ψ|², has no time dependence! (i.e., it's a "stationary state".)

We don't actually need the time-independent SEQ, but if we know we're dealing with energy eigenstates, the math is simpler.

Example: Motion of a Free Particle

A free particle moves without applied forces; so set U(x) = 0.

The SEQ is now reasonably simple:

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi(x,t)}{dx^2} = i\hbar\frac{d\Psi(x,t)}{dt}$$

The second x-derivative is proportional to the first t-derivative.
Here’s one solution:

$$\boxed{\Psi(x,t) = Ae^{i(kx-\omega t)}}$$

This is a traveling wave. The particle has

Momentum: p = ħk = h/λ
Energy: E = ħω = hf

Not a surprise, I hope.

Check it. Take the derivatives:

$$\frac{\partial\Psi}{\partial x} = ik\ Ae^{i(kx-\omega t)}$$

$$\frac{\partial^2\Psi}{\partial x^2} = (ik)^2 Ae^{i(kx-\omega t)} = -k^2 Ae^{i(kx-\omega t)}$$

$$\frac{\partial\Psi}{\partial t} = (-i\omega)Ae^{i(kx-\omega t)}$$

So, it works if: $\boxed{\frac{\hbar^2 k^2}{2m} = \hbar\omega}$

That’s the same as: $\boxed{\frac{p^2}{2m} = E}$

Time-dependence of Superpositions

A particle can be in a superposition of states that have different energies.

This superposition is still a solution of the time-dependent SEQ, but
not of the time-independent SEQ, because two different E's are involved.

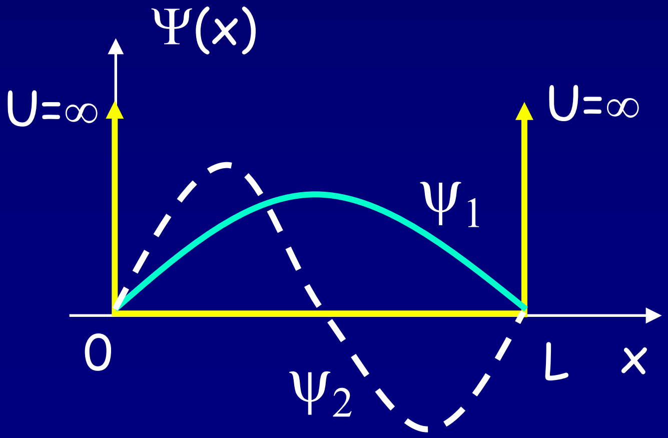
Two questions:

- How does this superposition evolve with time?
  For example, is the probability density still stationary?
- What happens if we measure the particle's energy?

First, look at the time dependence. Consider the first two energy states in an infinite well. Here's the superposition:

$$\Psi(x,t) = \psi_1(x)e^{-i\omega_1 t} + \psi_2(x)e^{-i\omega_2 t}$$

$$\boxed{\omega_1 = \frac{E_1}{\hbar},\ E_1 = \frac{h^2}{8mL^2}} \qquad \boxed{\omega_2 = \frac{E_2}{\hbar},\ E_2 = 4E_1}$$



The two terms have different frequencies,
so they oscillate in and out of phase.

falstad.com/qm1d/

Particle Motion in a Well

The probability density is given by: $|\Psi(x,t)|^2$:

$$|\Psi(x,t)|^2 = \psi_1^2 + \psi_2^2 + 2\psi_1\psi_2\cos((\omega_2 - \omega_1)t)$$

Interference term

We used the identity:

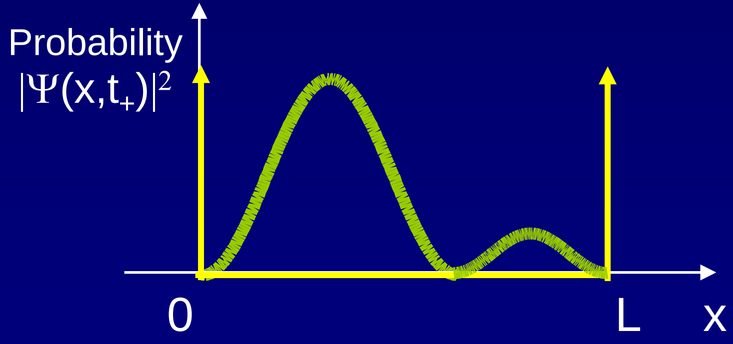
$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

So, $|\Psi(x,t)|^2$ oscillates between:

In phase: (cos = +1)

$$|\Psi(x,t)|^2 = (\psi_1 + \psi_2)^2$$

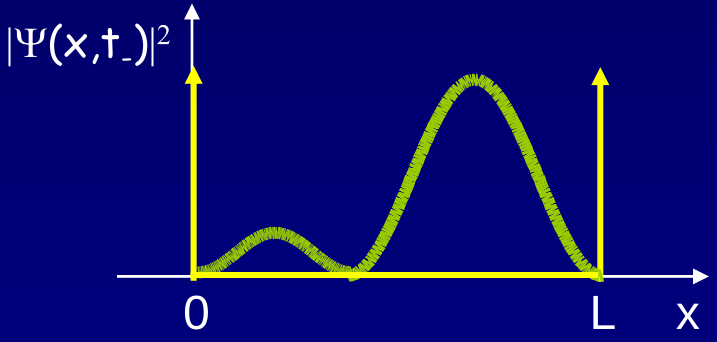
Particle localized on left side of well:



Out of phase: (cos = -1)

$$|\Psi(x,t)|^2 = (\psi_1 - \psi_2)^2$$

Particle localized on right side of well:



The frequency of oscillation is $\omega = \omega_2 - \omega_1 = (E_2 - E_1)/\hbar$, or $f = (E_2 - E_1)/h$.
This is precisely the frequency of a photon that would make a transition between the two states.

Interference Beats

The motion of the probability density comes from the changing interference between terms in Ψ that have different energy.

The beat frequency between two terms is the frequency difference, $f_2 - f_1$.

So energy differences are important: $E_1 - E_2 = hf_1 - hf_2$, etc.

Just like in Newtonian physics: absolute energies aren't important (you can pick where to call U=0 for convenience), only energy differences.

Normalizing Superpositions

We want the total probability to equal 1,
even when the particle is in a superposition of states:

$$P_{tot} = \int |\psi|^2 \, dx = \int |a\psi_1 + b\psi_2|^2 \, dx = 1$$

This looks like a mess. However, we're in luck. Multiply it out:

$$\int |a\psi_1 + b\psi_2|^2 \, dx = \int |a\psi_1|^2 \, dx + \int |b\psi_2|^2 \, dx + \int (a\psi_1)^* (b\psi_2) dx + \int (b\psi_2)^* (a\psi_1) dx$$

$$= |a|^2 \int |\psi_1|^2 \, dx + |b|^2 \int |\psi_2|^2 \, dx + a^* b \int (\psi_1)^* (\psi_2) dx + b^* a \int (\psi_2)^* (\psi_1) dx$$

$$= |a|^2 + |b|^2 + 0 + 0$$

If ψ_1 is normalized

If ψ_2 is normalized

It is a mathematical theorem that these integrals always = 0 if the energies are different.

Math 415

A normalized superposition must have $|a|^2 + |b|^2 = 1$.

$\psi = 0.8\psi_1 + 0.6\psi_2$ is normalized.
$\psi = 0.5\psi_1 + 0.5\psi_2$ is not normalized.

Act 2

Consider a particle in an infinite square well. At t = 0 it is in the state:

$$\Psi(x,t) = 0.5\psi_2(x) + A_2\psi_4(x)$$

with $\psi_2(x)$ and $\psi_4(x)$ both normalized.

1. What is A_2? a. 0.5 b. 0.707 c. 0.866

2. At some later time *t*,
what is the probability density at the center of the well?

a. 0 b. 1 c. It depends on the time *t.*

Solution

Consider a particle in an infinite square well. At t = 0 it is in the state:

$$\Psi(x,t) = 0.5\psi_2(x) + A_2\psi_4(x)$$

with $\psi_2(x)$ and $\psi_4(x)$ both normalized.

1. What is A_2? a. 0.5 b. 0.707 c. 0.866

As stated, the question is ambiguous. A_2 could be complex. However, let's assume that A_2 is real.

We are told that $\psi_2(x)$ and $\psi_4(x)$ are both normalized. Therefore: $0.5^2 + |A_2|^2 = 1 \Rightarrow |A_2| = \mathrm{sqrt}(1 - 0.25) = 0.866$

$A_2 = 0.866\, e^{i\phi}$ also works, for all ϕ.

2. At some later time *t*, what is the probability density at the center of the well?

a. 0 b. 1 c. It depends on the time *t.*

Solution

Consider a particle in an infinite square well. At t = 0 it is in the state:

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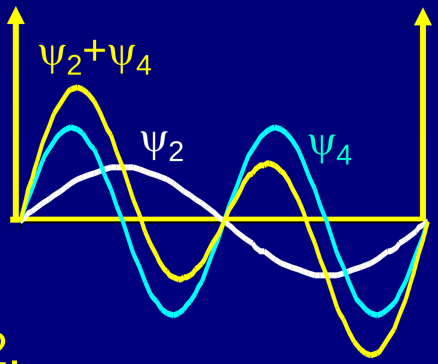
[✓ a. 0] b. 1 c. It depends on the time *t.*



In general, the probability distribution of a superposition of energy eigenstates depends on time.
However, ψ_2 and ψ_4 each have a node at L/2.
Therefore, every superposition of them also has a node at L/2.

Measurements of Energy

We are now ready to deal with the second question from earlier in the lecture:

> What happens when we measure the energy of a particle whose wave function is a superposition of more than one energy state?

If the wave function is in an energy eigenstate (E_1, say), then we know with certainty that we will obtain E_1 (unless the apparatus is broken).

If the wave function is a superposition ($\psi = a\psi_1+b\psi_2$) of energies E_1 and E_2, then we aren't certain what the result will be. However:

We know with certainty that we will only obtain E_1 or E_2 !!

We can't prove this statement. It is one of the fundamental postulates of quantum theory. Treat it as an empirical fact.

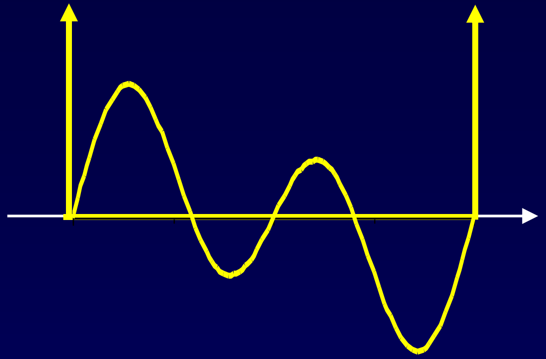
To be specific, we will never obtain $(E_1+E_2)/2$, or any other value.

What about a and b?

$|a|^2$ and $|b|^2$ are the probabilities of obtaining E_1 and E_2, respectively.

That's why we normalize the wave function to make $|a|^2 + |b|^2 =1$.

Act 3



Consider a particle in an infinite well.
It is in the state:

$$\Psi(x,t) = 0.5\Psi_2(x,t) + 0.866\Psi_4(x,t)$$

with $\psi_2(x)$ and $\psi_4(x)$ both normalized.

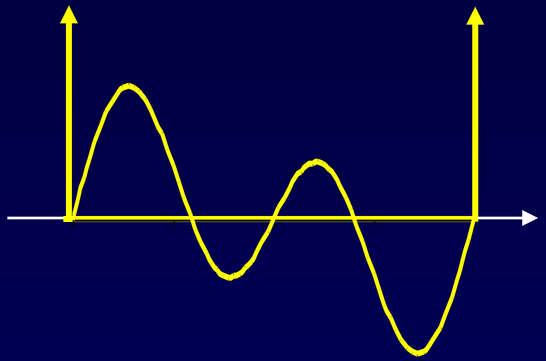
We now measure the energy of the particle. What value is obtained?

a. E_2 b. E_4 c. $0.25\ E_2 + 0.75\ E_4$ d. It depends on when we measure the energy.

Not part of this act, but an important question, nevertheless:

If E_2 is observed, what is the state of the particle after the measurement?

Solution



Consider a particle in an infinite well.
It is in the state:

$$\Psi(x,t) = 0.5\Psi_2(x,t) + 0.866\Psi_4(x,t)$$

with ψ_2 and ψ_4 both normalized.

We now measure the energy of the particle. What value is obtained?

$\boxed{\text{a. } E_2 \quad \text{b. } E_4}$ c. 0.25 E_2 + 0.75 E_4 d. It depends on when we measure the energy.

We can only get one of the eigenvalues, E_2 or E_4. (not answer c)
The probability of measuring E_2 is 25%.
The probability of measuring E_4 is 75%.
Note: Ψ depends on time, but 0.5 and 0.866 don't. So, d is not correct.

The <u>average</u> energy (if we were to measure a large number of similar particles) is the weighted sum of the energies: 0.25 E_2 + 0.75 E_4.

Not part of this act, but an important question, nevertheless:

If E_2 is observed, what is the state of the particle after the measurement?

The "Collapse" of the Wave Function

If E_2 is observed, what is the state of the particle after the measurement?

We start out with this wave function:

$$\Psi(x,t) = 0.5\Psi_2(x,t) + 0.866\Psi_4(x,t)$$

Before we make the measurement, we can't predict the result of an energy measurement with certainty.

However, after the measurement, we know with certainty that $E = E_2$.
To be specific: We know that a second measurement will yield E_2. (Why?)

Therefore, after obtaining E_2, the wave function must now be:

$$\Psi(x,t) = \Psi_2(x,t)$$

That is, the wave function has "collapsed" to the state that corresponds with the result we obtained.

This is one of the weirder features of QM, and is the principal reason that Einstein never accepted QM as a complete theory.

("God does not place dice!")

Supplement: Time-Energy Uncertainty Principle

Now that we are considering time-dependent problems, it is a good time to introduce another application of the Heisenberg Uncertainty Principle, based on measurements of energy and time. We start from our previous result:

$$\Delta p\,\Delta x \geq \hbar \quad \Rightarrow \quad (c\Delta p)\left(\frac{\Delta x}{c}\right) \geq \hbar \quad \Rightarrow \quad \Delta E\,\Delta t \geq \hbar$$

Sometimes this is further transformed as follows:

$$\Delta E\,\Delta t \geq \hbar \quad \Rightarrow \quad (\hbar\Delta\omega)\Delta t \geq \hbar$$

$$\Delta\omega\,\Delta t \geq 1 \quad \Rightarrow \quad \Delta f\,\Delta t \geq 1/2\pi$$

The last line is a standard result from Fourier wave analysis; this should not surprise us – the Uncertainty Principle arises simply because particles behave as waves that are oscillating in time as well as in space.

ΔE Δt Uncertainty Principle Example*

A particular optical fiber transmits light over the range 1300-1600 nm (corresponding to a frequency range of $2.3\text{x}10^{14}$ Hz to $1.9\text{x}10^{14}$ Hz). How long (approximately) is the shortest pulse that can propagate down this fiber?

$$\Delta\omega\,\Delta t \geq 1 \quad \Rightarrow \quad 2\pi\,\Delta f\,\Delta t \geq 1$$

$$\Delta t \geq 1/2\pi\,\Delta f$$

$$\geq 1/(2\pi\, 0.4 \times 10^{14}\,\text{Hz})$$

$$= 4 \times 10^{-15}\,\text{s} = \boxed{4\ \text{fs}}$$

Note: This means the upper limit to data transmission is ~1/(4fs) = $2.5\text{x}10^{14}$ bits/second = 250 Tb/s

*This problem obviously does not require “quantum mechanics” *per se*. However, due to the Correspondence Principle, QM had better give a consistent result.

Next Lectures

Tunneling of quantum particles

- Scanning Tunneling Microscope (STM)
- Nuclear Decay
- Solar Fusion
- The Ammonia Maser

Supplement:
Quantum Information References

- Quantum computing
  - Employs superpositions of quantum states - astoundingly good for certain parallel computation, may use entangled states for error checking
    - www.newscientist.com/nsplus/insight/quantum/48.html
    - http://www.cs.caltech.edu/~westside/quantum-intro.html

- Quantum cryptography
  - Employs single photons or entangled pairs of photons to generate a secret key. Can determine if there has been eavesdropping in information transfer
    - cam.qubit.org/articles/crypto/quantum.php
    - library.lanl.gov/cgi-bin/getfile?00783355.pdf

- Quantum teleportation
  - Employs an entangled state to produce an exact replica of a third quantum state at a different point in space
    - www.research.ibm.com/quantuminfo/teleportation
    - www.quantum.univie.ac.at/research/photonentangle/teleport/index.html
    - Scientific American, April 2000

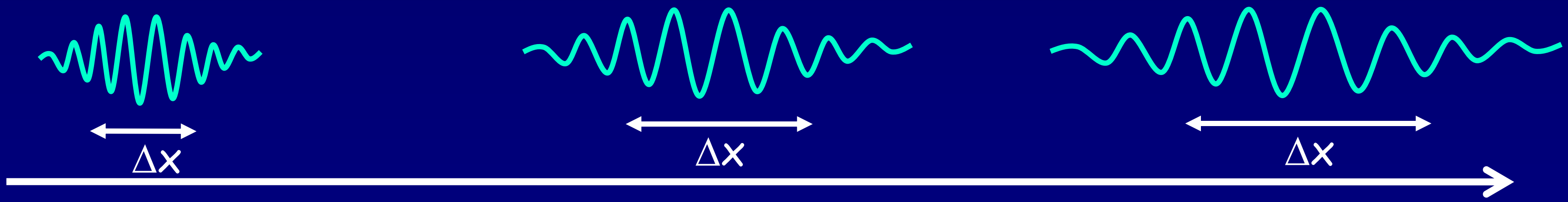
Supplement: Free particle motion

- It turns out (next slide) that the constructive interference region for a matter wavepacket moves at the "group velocity"

$$v = h/\lambda m = p/m$$

- So there's a simple correspondence between the quantum picture and our classical picture of particles moving around with momentum $p = mv$.

- But the quantum packet will spread out in the long run, since it has a range of p, so the correspondence is never perfect.



Position of maximum probability moves
and the width of probability distribution spreads out

Supplement: Group velocity

- Say a wave-packet starts out at x=0 at t=0.
  - meaning each harmonic component has the same phase there.
- After time t
  - the harmonic component at w_1 will have changed phase by $t\omega_1$
  - the harmonic component at w_2 will have changed phase by $t\omega_2$
  - The phase difference between these components at x=0 will now be $t(\omega_2 - \omega_1)$ To find the point x where they're in phase, we need to find where the phase difference from moving downstream by x cancels that:
    - $t(\omega_2 - \omega_1)=x(k_2-k_1)$ Or for small differences in ω,k: $td\omega=xdk$

Result

$$v_g=x/t=d\omega/dk$$

In this case $v_g=p/m$