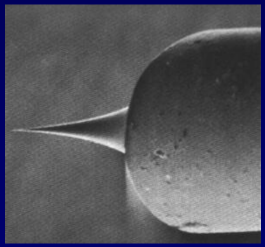
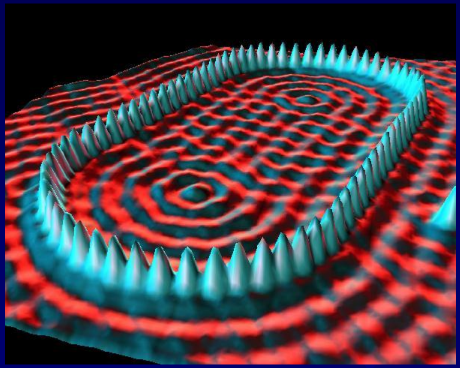
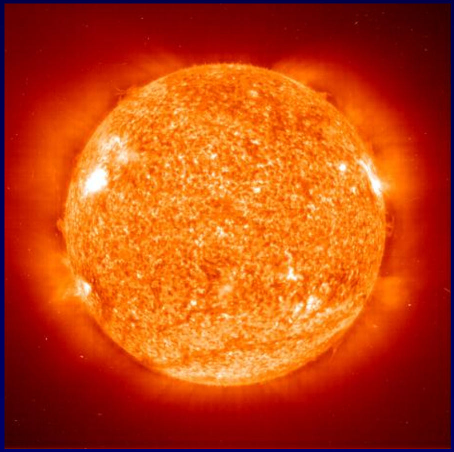
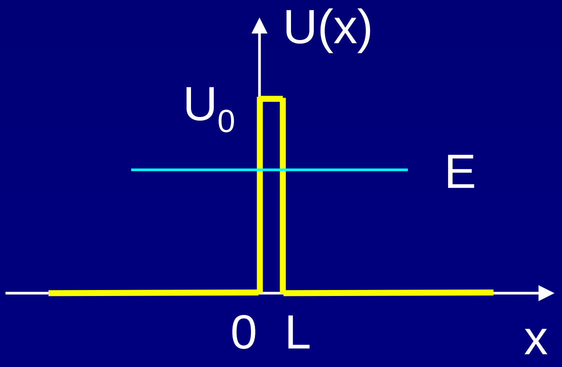
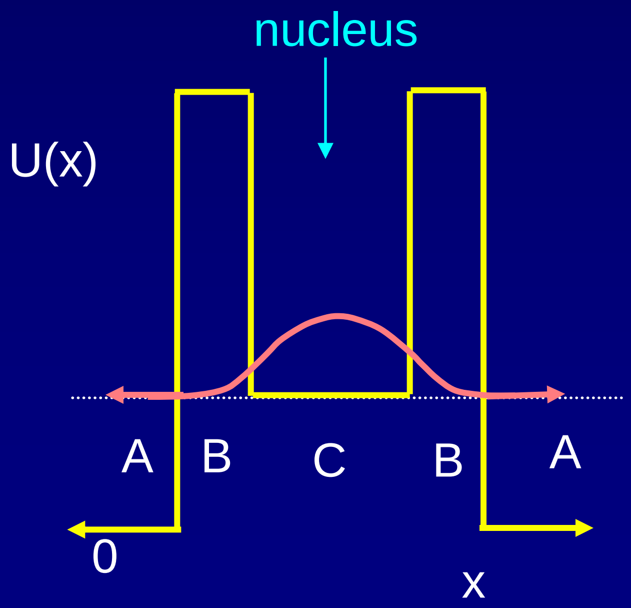
Lecture 13:
Barrier Penetration and Tunneling











Today

Tunneling of quantum particles

- Scanning Tunneling Microscope (STM)
- Nuclear Decay
- Solar Fusion

Next time: Time-*dependent* quantum mechanics

- Oscillations
- Measurements in QM
- Time-Energy Uncertainty Principle

The rest of the course:

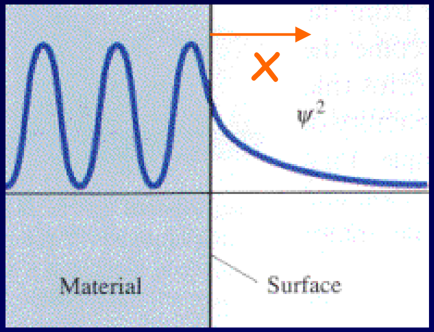
Next week: 3 dimensions - orbital and spin angular momentum
H atom, exclusion principle, periodic table

Last week: Molecules and solids.
Metals, insulators, semiconductors, superconductors, lasers, . .

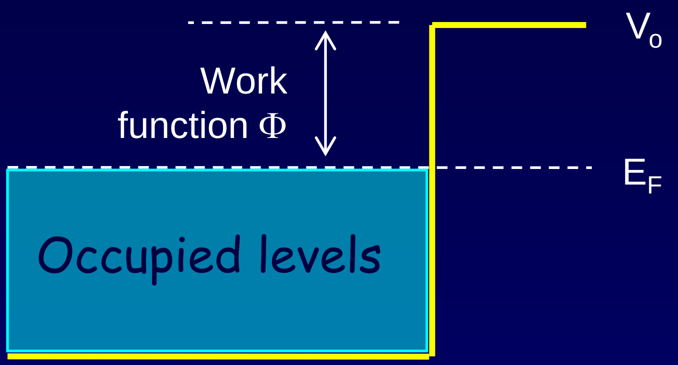
Good web site for animations http://www.falstad.com/qm1d/

"Leaky" Particles

Due to "barrier penetration", the electron density of a metal actually extends outside the surface of the metal!



x = 0



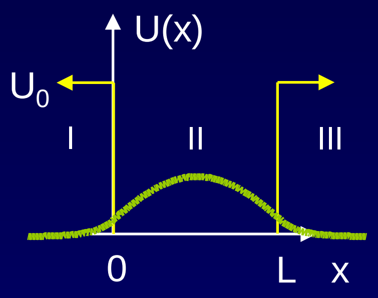
Assume that the work function (i.e., the energy difference between the most energetic conduction electrons and the potential barrier at the surface) of a certain metal is $\Phi = 5$ eV. Estimate the distance x outside the surface of the metal at which the electron probability density drops to 1/1000 of that just inside the metal.

Tunneling

In quantum mechanics a particle can penetrate into a barrier where it would be classically forbidden.

The finite square well:

In region III, $E < U_0$, and $\psi(x)$ has the exponential form $D_1 e^{-Kx}$. We did not solve the equations – too hard! You will do this using the computer in Lab #3.



The probability of finding the particle in the barrier region decreases as e^{-2Kx}.

The finite-width barrier:

Today we consider a related problem – a particle approaching a finite-width barrier and "tunneling" through to the other side.

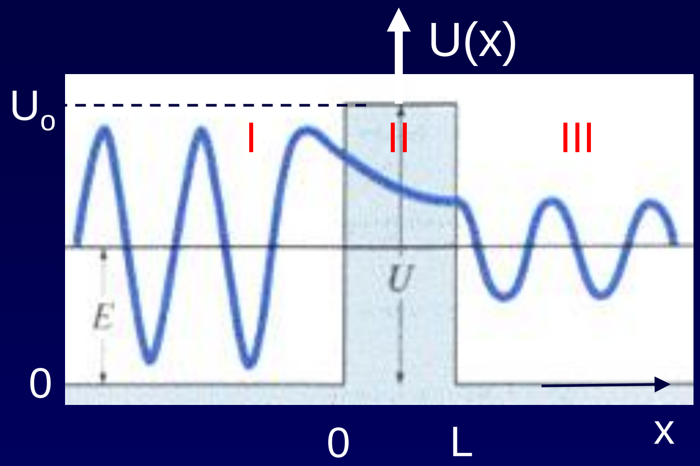
The result is very similar, and again the problem is too hard to solve exactly here:

The probability of the particle tunneling through a finite width barrier is approximately proportional to e^{-2KL} where L is the width of the barrier.

Tunneling Through a Barrier (1)

What is the the probability that an incident particle tunnels through the barrier?
It's called the "Transmission Coefficient, T".

Consider a barrier (II) of height U_0.
U = 0 everywhere else.



Getting an exact result requires applying the boundary conditions at x = 0 and x = L, then solving six transcendental equations for six unknowns:

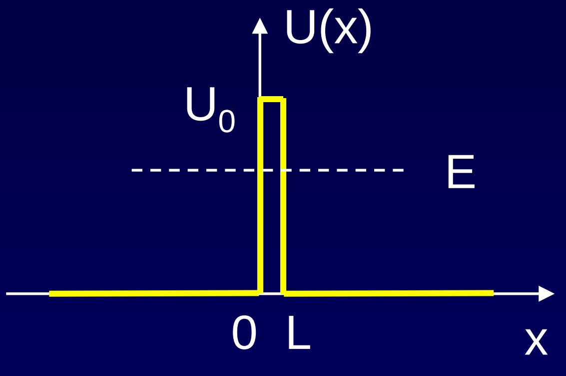
$$\psi_I(x) = A_1 \sin kx + A_2 \cos kx$$
$$\psi_{II}(x) = B_1 e^{Kx} + B_2 e^{-Kx}$$
$$\psi_{III}(x) = C_1 \sin kx + C_2 \cos kx$$

A_1, A_2, B_1, B_2, C_1, and C_2 are unknown. K and k are known functions of E.

This is more complicated than the infinitely wide barrier, because we can't require that $B_1 = 0$. (Why not?)

Tunneling Through a Barrier (2)

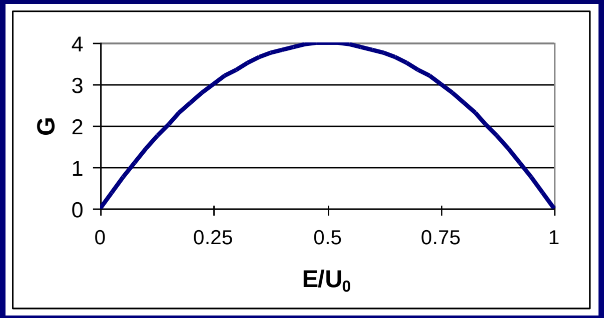
In many situations, the barrier width L is much larger than the 'decay length' 1/K of the penetrating wave (KL >> 1). In this case $B_1 \approx 0$ (why?), and the result resembles the infinite barrier. The tunneling coefficient simplifies:



$$T \approx G e^{-2KL} \text{ where } G = 16\frac{E}{U_0}\left(1 - \frac{E}{U_0}\right)$$

$$K = \sqrt{\frac{2m}{\hbar^2}(U_0 - E)}$$

This is nearly the same result as in the "leaky particle" example! Except for G:



We will often ignore G.
(We'll tell you when to do this.)

The important result is e^{-2KL}.

Act 1

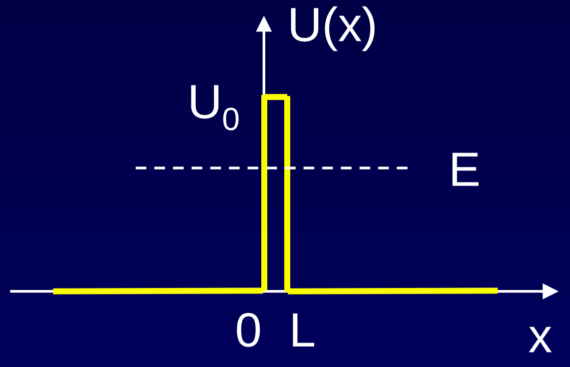
Consider a particle tunneling through a barrier.

1. Which of the following will increase the likelihood of tunneling?

   a. decrease the height of the barrier

   b. decrease the width of the barrier

   c. decrease the mass of the particle



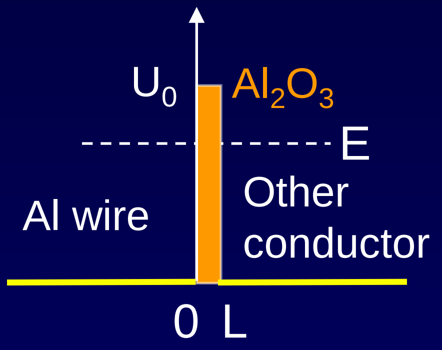
2. What is the energy of the emerging particles?

   a. < initial energy   b. = initial energy   c. > initial energy

Example: Aluminum wire

Why household electrical wire is not aluminum:

Aluminum is cheap and a good conductor. However, aluminum tends to form an oxide surface layer (Al_2O_3) which can be as much as several nanometers thick.



This layer could cause a problem in making electrical contacts, since it presents a barrier roughly 10 eV high to the flow of electrons in and out of the Al.

Your requirement is that your transmission coefficient across any contact must be $T > 10^{-10}$, or else the resistance will be too high for the high currents you're using, causing a fire risk. Should you use aluminum wiring or not? (You can neglect G here.)

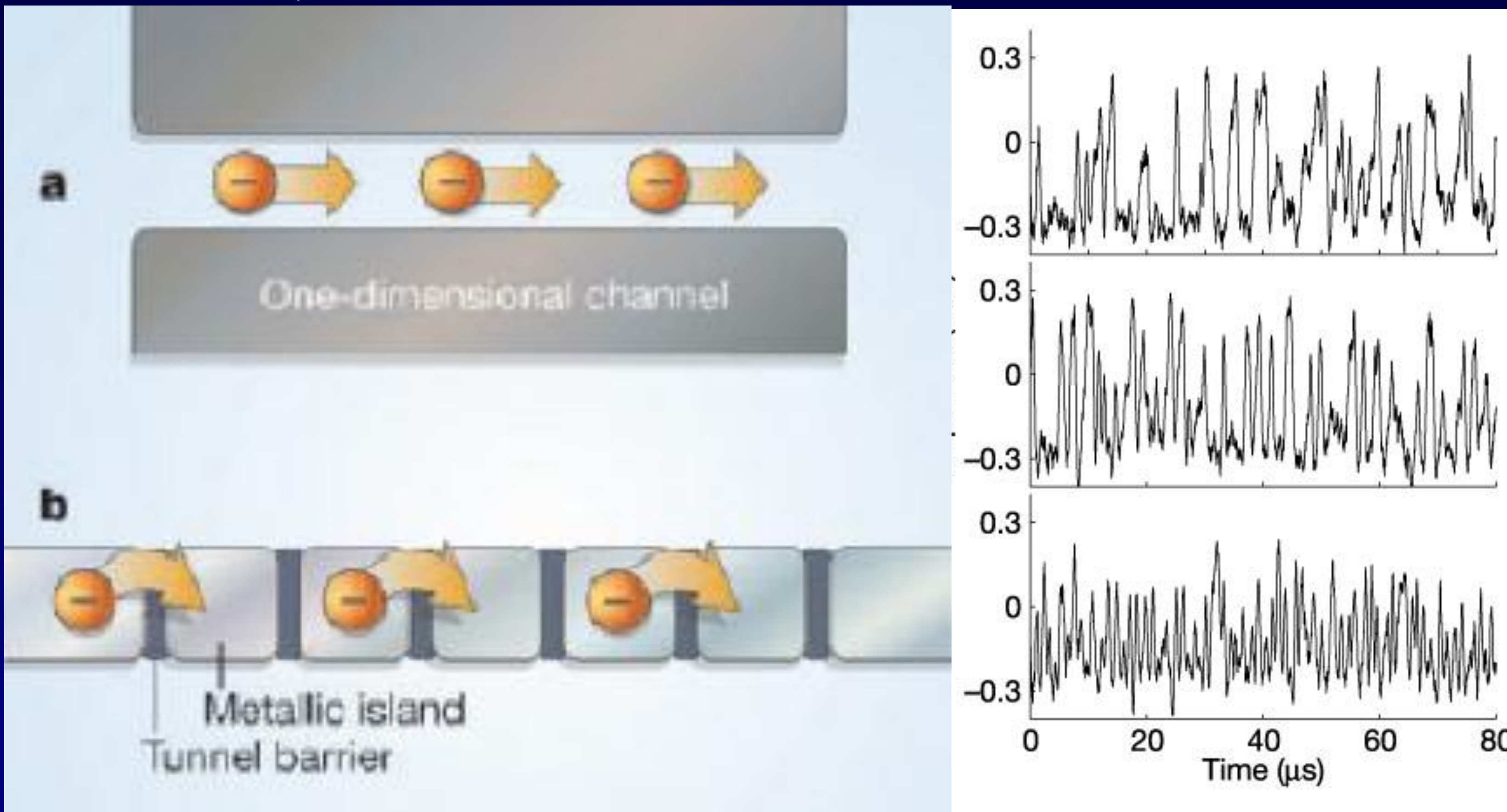
Example: Electrons in Nanoscale devices

letters to nature

Nature 434, 361 - 364 (17 March 2005)

Current measurement by real-time counting of single electrons

JONAS BYLANDER, TIM DUTY & PER DELSING



~40 fA

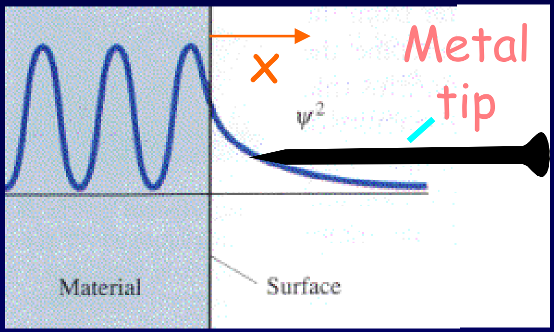
~80 fA

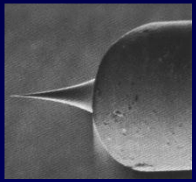
~120 fA

Electrons that successfully tunnel through the 50 junctions are detected using a fast single-electron transistor (SET).

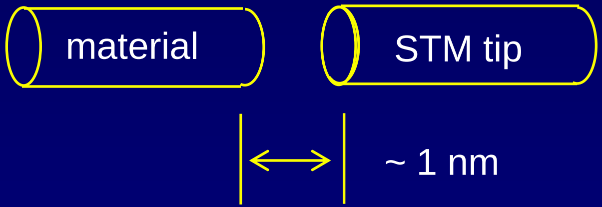
Application: Tunneling Microscopy

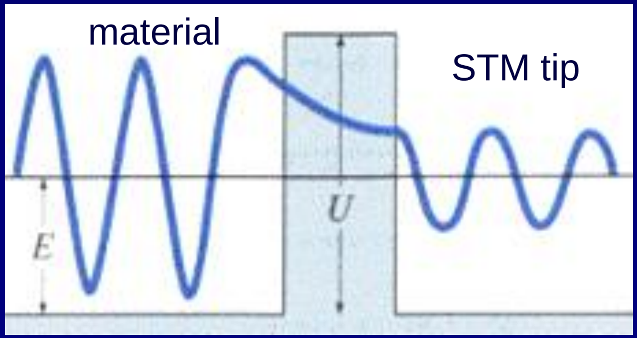
One can use barrier penetration to measure the electron density on a surface.





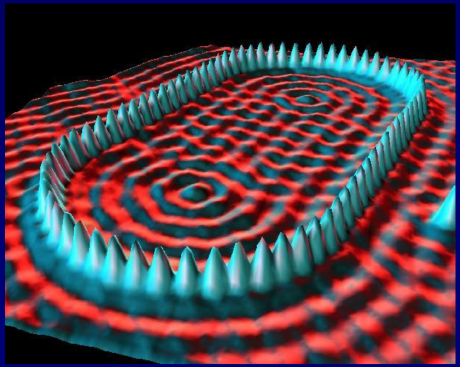
Real STM tip



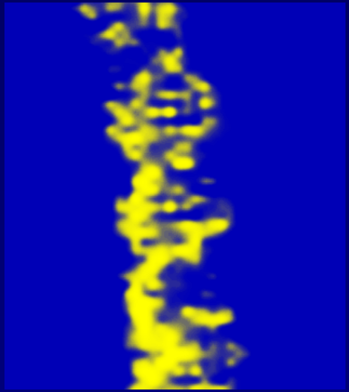


Scanning
Tunneling
Microscope
images

Na atoms on metal:



DNA Double Helix:



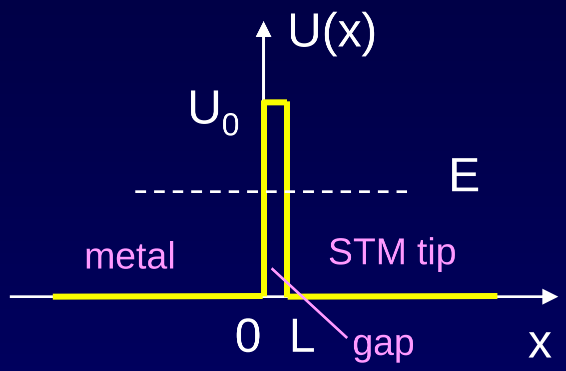
STM demo:
http://www.quantum-physics.polytechnique.fr/en/

Barrier penetration is a wave phenomenon, not only QM. It is used in optical microscopes also. See:
http://en.wikipedia.org/wiki/Total_internal_reflection_fluorescence_microscope

The STM

The STM (scanning tunneling microscope) tip is $L = 0.18$ nm from a metal surface.

An electron with energy of $E = 6$ eV in the metal approaches the surface. Assume the metal/tip gap is a potential barrier with a height of $U_o = 12$ eV. What is the probability that the electron will tunnel through the barrier?



Act 2

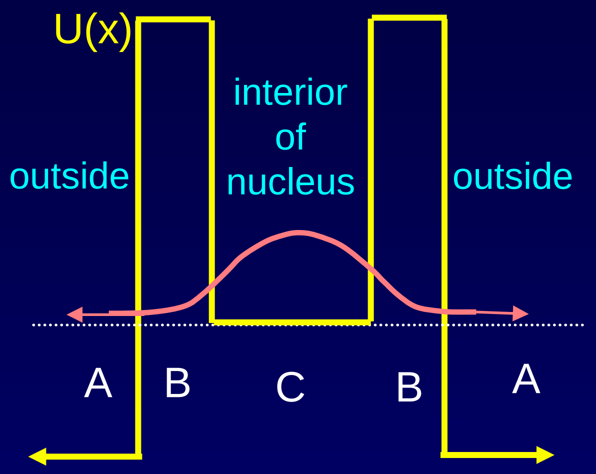
What effect does a barrier have on probability?

Suppose T = 0.05. What happens to the other 95% of the probability?

a. It's absorbed by the barrier.
b. It's reflected by the barrier.
c. The particle "bounces around" for a while, then escapes.

Tunneling and Radioactivity

In large atoms (e.g., Uranium), the nucleus can be unstable to the emission of an alpha particle (a He nucleus). This form of radioactivity is a tunneling process, involving transmission of the alpha particle from a low-energy valley through a barrier to a lower energy outside.



Why do we observe exponential decay?

- ψ leaks out from C through B to A – the particle "tunnels" out.
- The leakage is slow (T << 1), so ψ just outside the barrier stays negligible.
- The shape of ψ remaining in B-C shows almost no change: Its amplitude slowly decreases. That is, P_{inside} is no longer 1.
- The rate at which probability flows out is proportional to P_{inside} (by linearity) $\Rightarrow$ exponential decay in time.

$$\frac{dx}{dt} = -Ax \quad \Rightarrow \quad x = e^{-At} = e^{-t/\tau}$$

$t_{1/2} = (\tau \ln 2)$ is the "half life" of the substance

α-Radiation: Illustrations of the enormous range of decay rates in different nuclei

Consider a very simple model of a-radiation:

Assume the alpha particle ($m = 6.64 \times 10^{-27}$ kg) is trapped in a nucleus which presents a square barrier of width L and height U_o. To find the decay rate we consider:

(1) the “attempt rate” at which the alpha particle of energy E inside the nucleus hits the barrier

Rough estimate with E ~ 5 to 10 MeV: the alpha particle makes about 10^{21} “attempts” per second (~velocity/nuclear diameter)

(2) the tunneling probability for an alpha particle with energy E each time the particle hits the barrier. [For this order of magnitude calculation you may neglect G.] Here we use

$$T \approx e^{-2KL} \qquad K = \sqrt{\frac{2m}{\hbar^2}\left(U_0 - E\right)}$$

Because of the exponential this factor can vary enormously!

Act 3

Polonium has an effective barrier width of ~10 fermi, leading to a tunneling probability of ~10^{-15}. Now consider Uranium, which has a similar barrier height, but an effective width of about ~20 fermi.

Estimate the tunneling probability in Uranium:

a. 10^{-30}

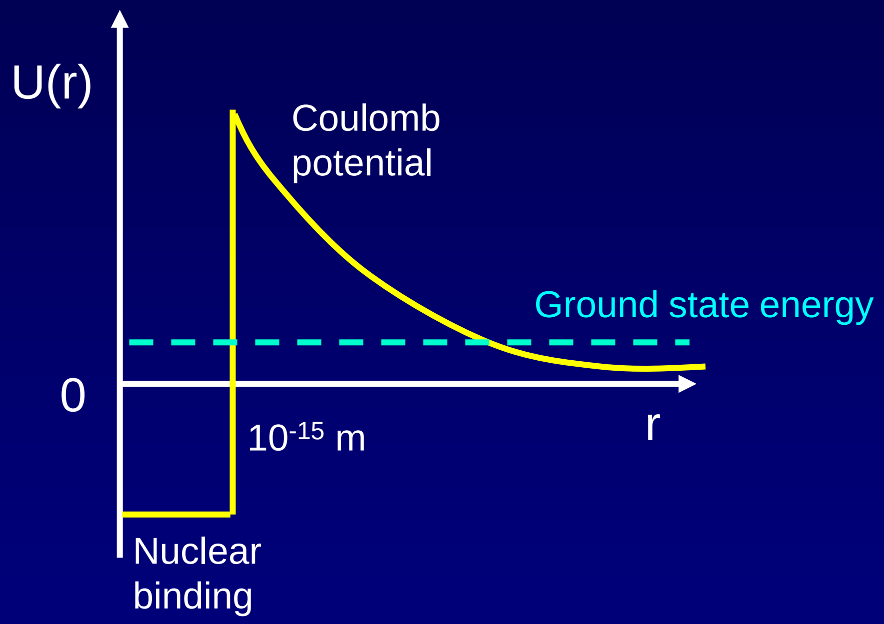
b. 10^{-14}

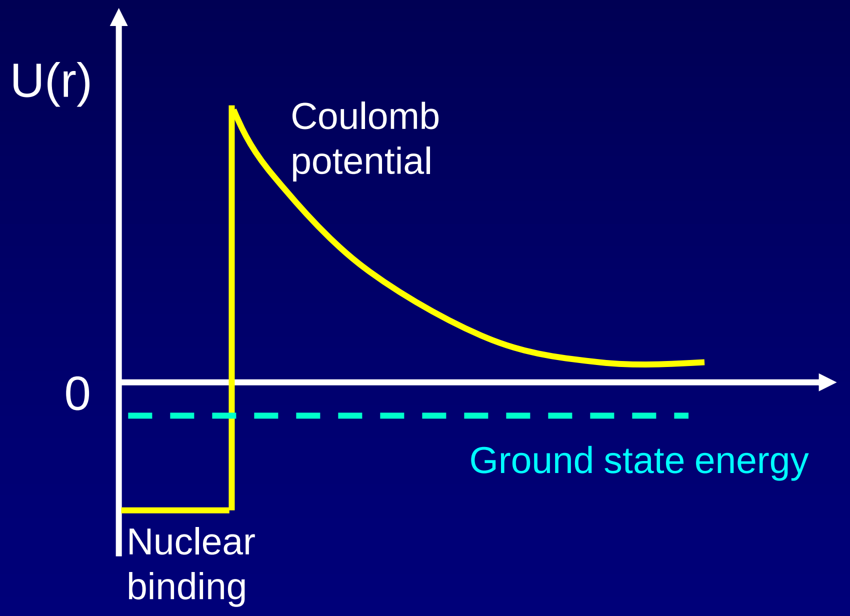
c. 10^{-7}

FYI: Why aren't *all* atoms radioactive?

Why do only some atoms undergo radioactive decay? It depends on whether they would have less energy in the decayed state than they do in the undecayed state.

Consider the following two possible potentials:



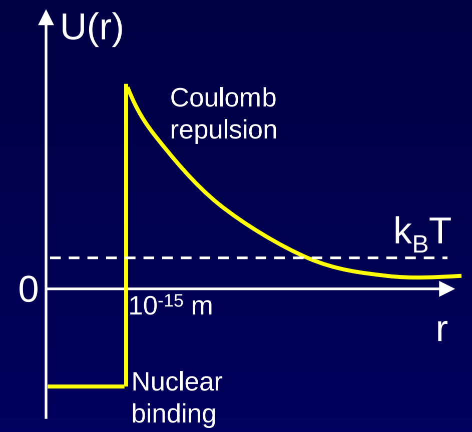


Here the α-particle can tunnel to an allowed region, and gain KE (used for power, cancer treatment, bombs, etc.)

Here the α-particle has NO allowed region to tunnel to, so the nucleus is stable against radioactive α-decay.

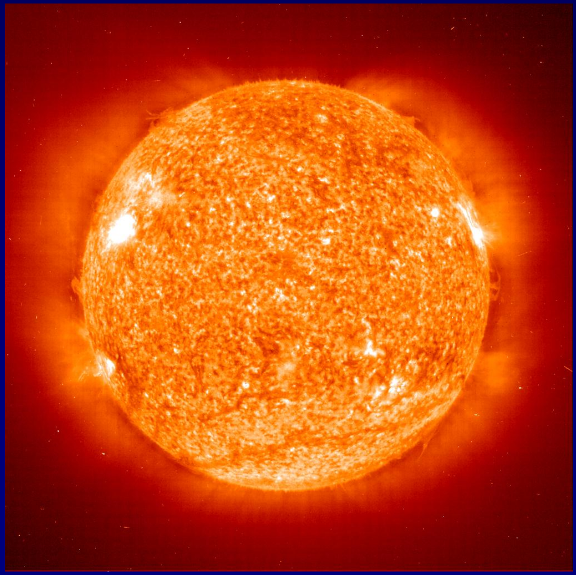
Tunneling Example: The Sun

The solar nuclear fusion process starts when two protons fuse together. In order for this reaction to proceed, the protons must “touch” (approach to within 10^{-15} m of each other).
The potential energy, U(r), looks something like this:



The temperature of the sun’s core is $T \sim 1.3 \times 10^7$ K.
This corresponds to an average kinetic energy:
$k_BT = 2 \times 10^{-16}$ J   (k_B = Boltzman’s constant)
At $r = 10^{-15}$ m the height of the Coulomb barrier is:

$$U(r) = (1/4\pi\varepsilon_0)e^2/r = (9\times10^9)\times(1.6\times10^{-19}\ \mathrm{C})^2/10^{-15}\ \mathrm{m}$$
$$= 2 \times 10^{-13}\ \mathrm{J}$$



Thus, the protons in the sun very rarely have enough thermal energy to go over the Coulomb barrier.

How do they fuse then? By tunneling through the barrier!

Next Lectures

Time-Dependent Schroedinger Equation

- Complex wave functions
- Superpositions of Energy Eigenstates
- Particle ‘motion’