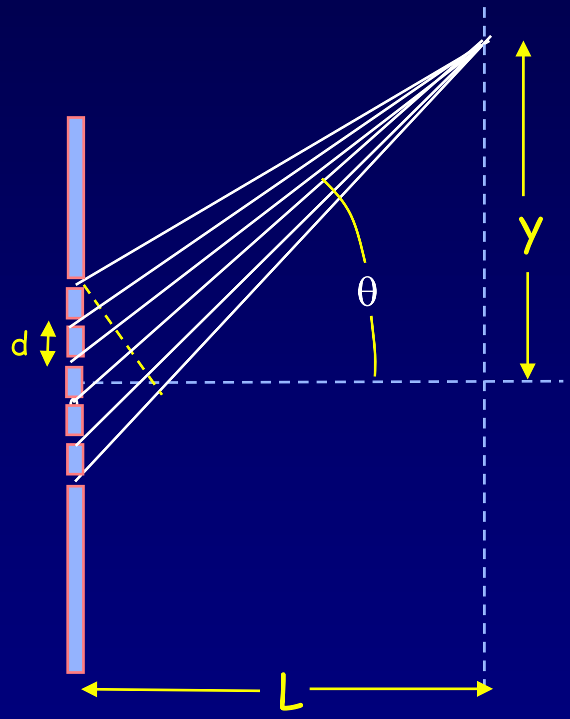
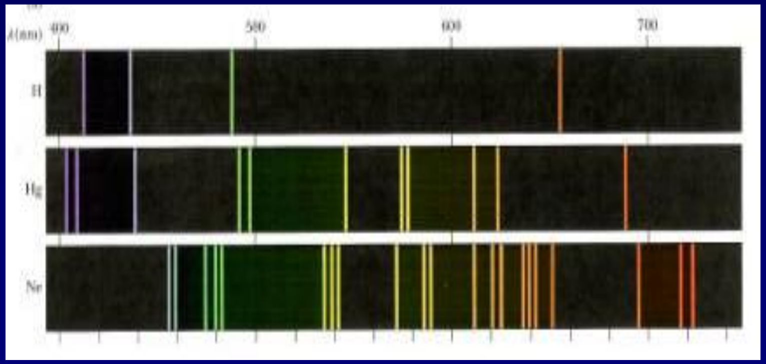
Lecture 4:
Diffraction & Spectroscopy





Spectra of atoms reveal the quantum nature of matter

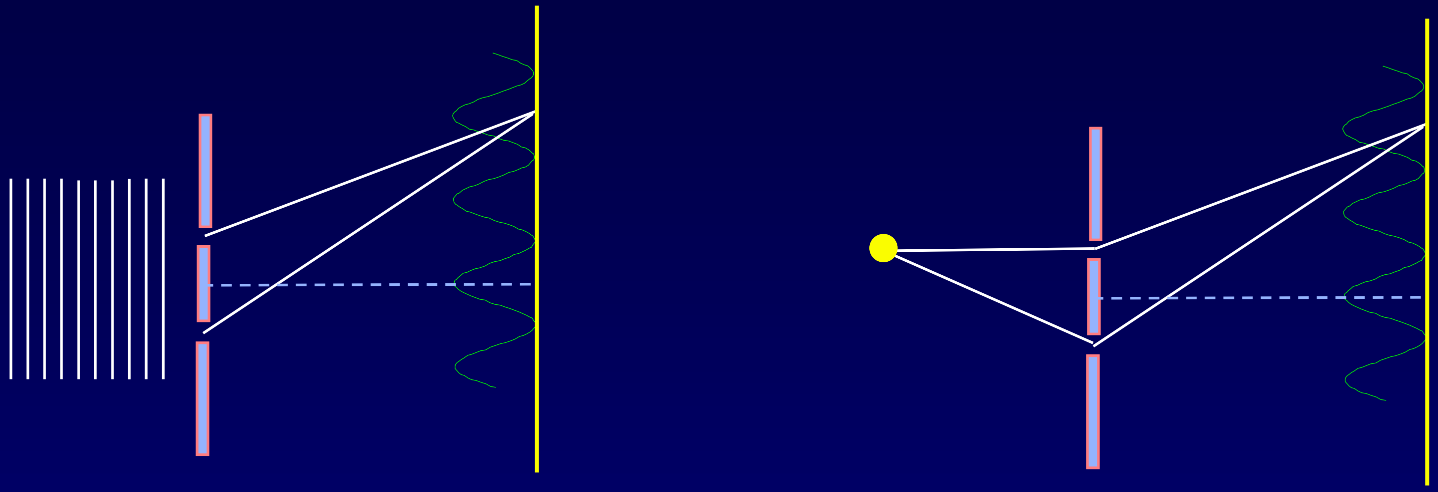
Take a plastic grating from the bin as you enter class.

Today's Topics

- Single-Slit Diffraction*
- Multiple-slit Interference*
- Diffraction Gratings
  - Spectral Resolution
- Optical Spectroscopy
- Interference + Diffraction

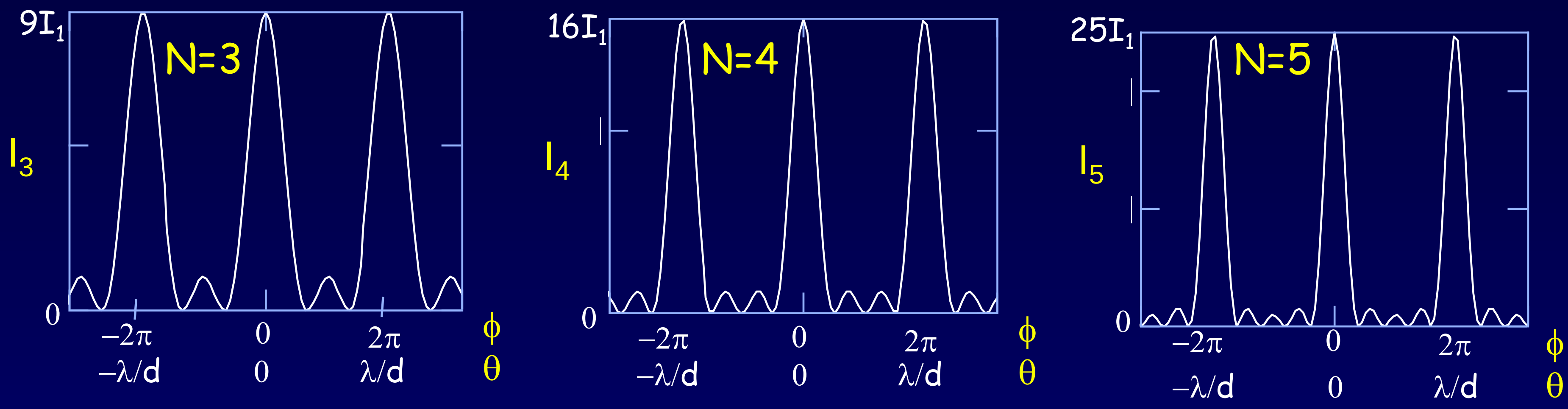
* Derivations in Appendix
(also in Young and Freeman, 36.2 and 36.4)

Review of 2-Slit Interference



- Only the phase difference matters.
  Phase difference is due to source phases and/or path difference.
- In a more complicated geometry (see figure on right),
  one must calculate the <u>total</u> path from source to screen.
- If the amplitudes are equal $\Rightarrow$ Use trig identity: $A = 2A_1\cos(\phi/2)$.
- Phasors: Phasors are amplitudes.
  Intensity is the square of the phasor length.

Multiple Slit Interference



The positions of the principal maxima occur at $\phi = 0, \pm 2\pi, \pm 4\pi, \ldots$
where ϕ is the phase between adjacent slits. $\theta = 0, \pm\lambda/d, \pm 2\lambda/d, \ldots$

The intensity at the peak of a principal maximum goes as N^2.
3 slits: $A_{tot} = 3A_1 \Rightarrow I_{tot} = 9I_1$. N slits: $I_N = N^2 I_1$.

Between two principal maxima there are N-1 zeros and
N-2 secondary maxima $\Rightarrow$ The peak width $\propto 1/N$.

The total power in a principal maximum is proportional to $N^2(1/N) = N$.

N-Slit Interference

The Intensity for N equally spaced slits is given by:

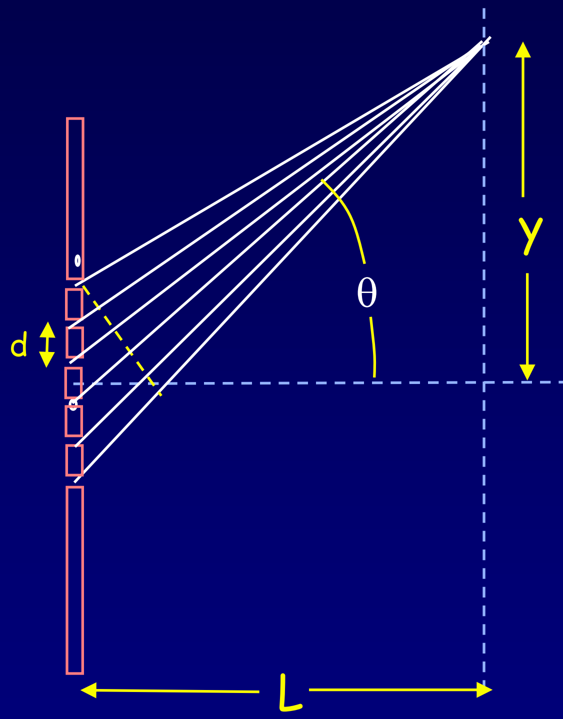
$$\boxed{I_N = I_1\left(\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right)^2} \; *$$

Derivation (using phasors) is in the supplementary slides.

As usual, to determine the pattern at the screen, we need to relate ϕ to θ or $y = L\tan\theta$:

$$\boxed{\frac{\phi}{2\pi} = \frac{\delta}{\lambda} = \frac{d\sin\theta}{\lambda} \approx \frac{d\theta}{\lambda} \quad \text{and} \quad \theta \approx \frac{y}{L}}$$

ϕ is the phase difference between adjacent slits.

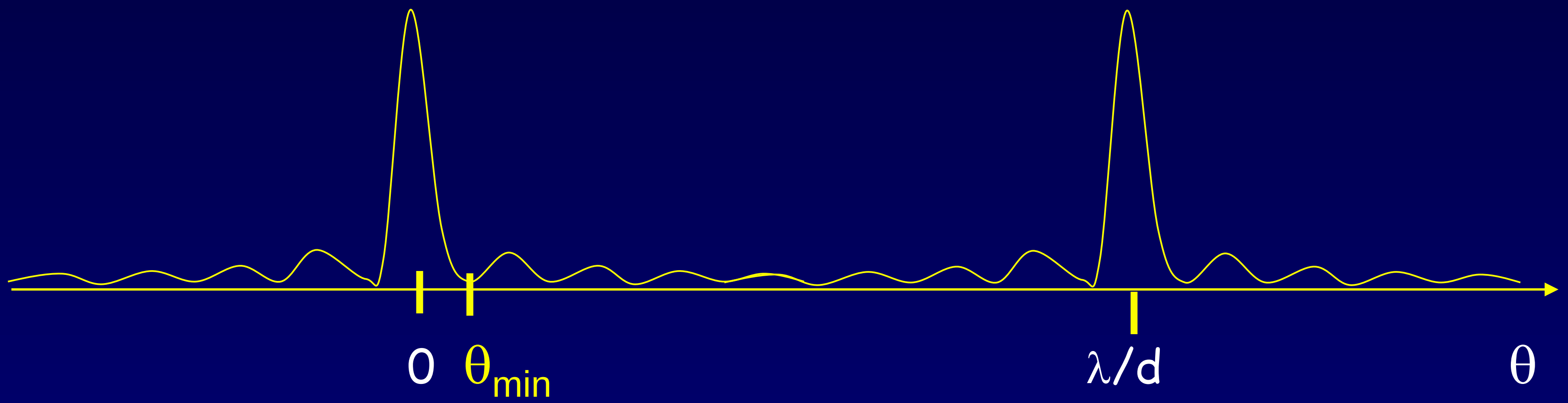


You will not be able to use the small angle approximations unless d >> λ.

* Your calculator can probably graph this. Give it a try.

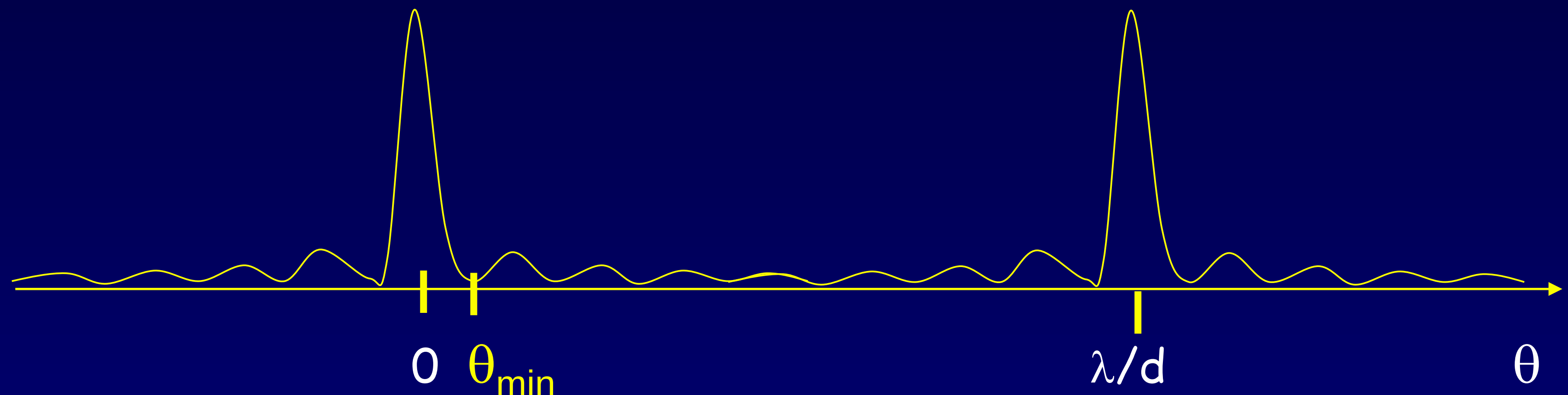
Example Problem

In an N-slit interference pattern, at what angle θ_{min} does the intensity first go to zero? (In terms of λ, d and N).



Solution

In an N-slit interference pattern, at what angle θ_{min} does the intensity first go to zero? (In terms of λ, d and N).



$I_N = I_1\left(\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right)^2$ has a zero when the numerator is zero. That is, $\phi_{min} = 2\pi/N$.

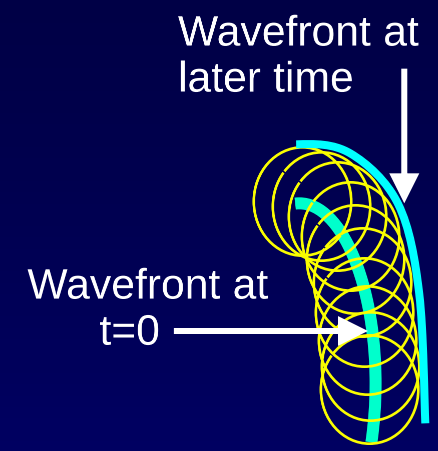
Exception: When the denominator is also zero.
That's why there are only N-1 zeros.

But $\phi_{min} = 2\pi(d \sin\theta_{min})/\lambda \approx 2\pi d\ \theta_{min}/\lambda = 2\pi/N$. Therefore, $\theta_{min} \approx \lambda/Nd$.

As the number of illuminated slits increases, the peak widths decrease!
General feature: Wider slit features → narrower patterns
Narrower slit features → wider patterns....

Huygens' principle

A Consequence of Superposition

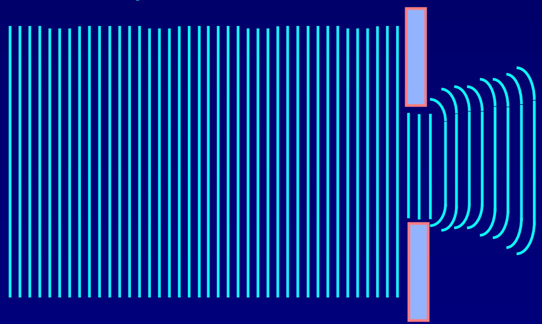


We will next study what happens when waves pass through one slit. We will use Huygens' principle (1678):
All points on a wave front (*e.g.*, crest or trough) can be treated as point sources of secondary waves with speed, frequency, and phase equal to the initial wave.

Q: What happens when a plane wave meets a small aperture?

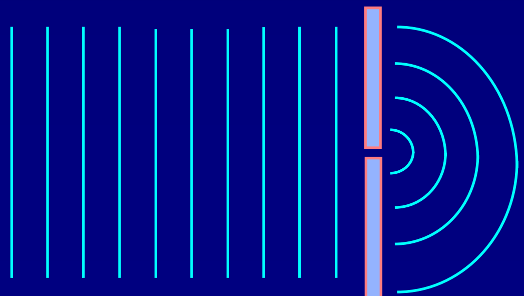
A: The result depends on the ratio of the wavelength λ to the size of the aperture, a:

$\lambda \ll a$



The transmitted wave is concentrated in the forward direction, and at near distances the wave fronts have the shape of the aperture. The wave eventually spreads out.

$\lambda \gg a$



Similar to a wave from a point source.

This effect is called *diffraction*.

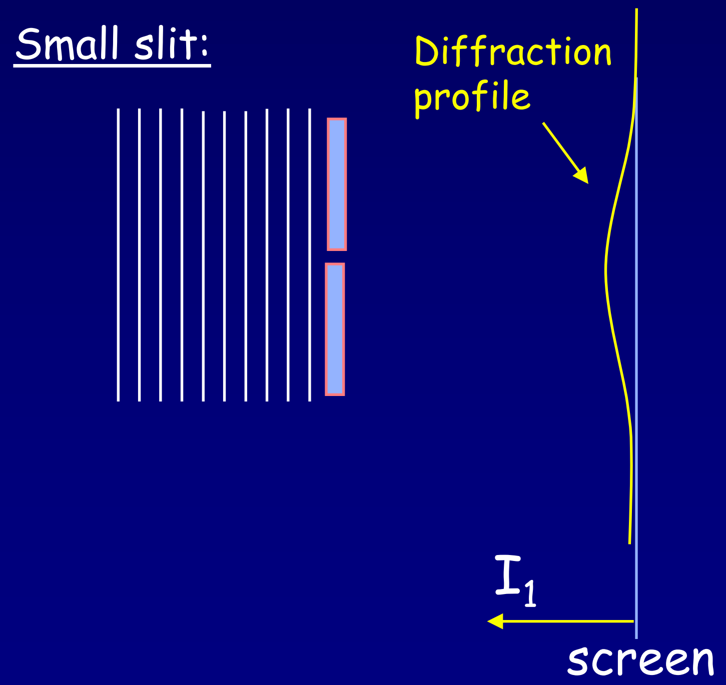
Single-slit Diffraction

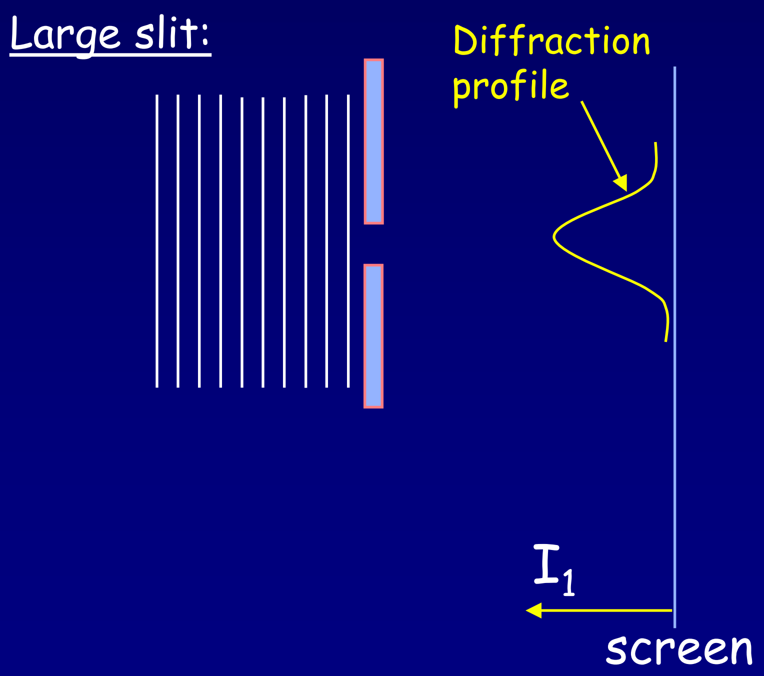
So far in the multiple-slit interference problems we have assumed that each slit is a point source.

Point sources radiate equally in all directions.

Real slits have a non-zero extent – - a “slit width” a.
The transmission pattern depends on the ratio of a to λ.

In general, the smaller the slit width, the more the wave will diffract.





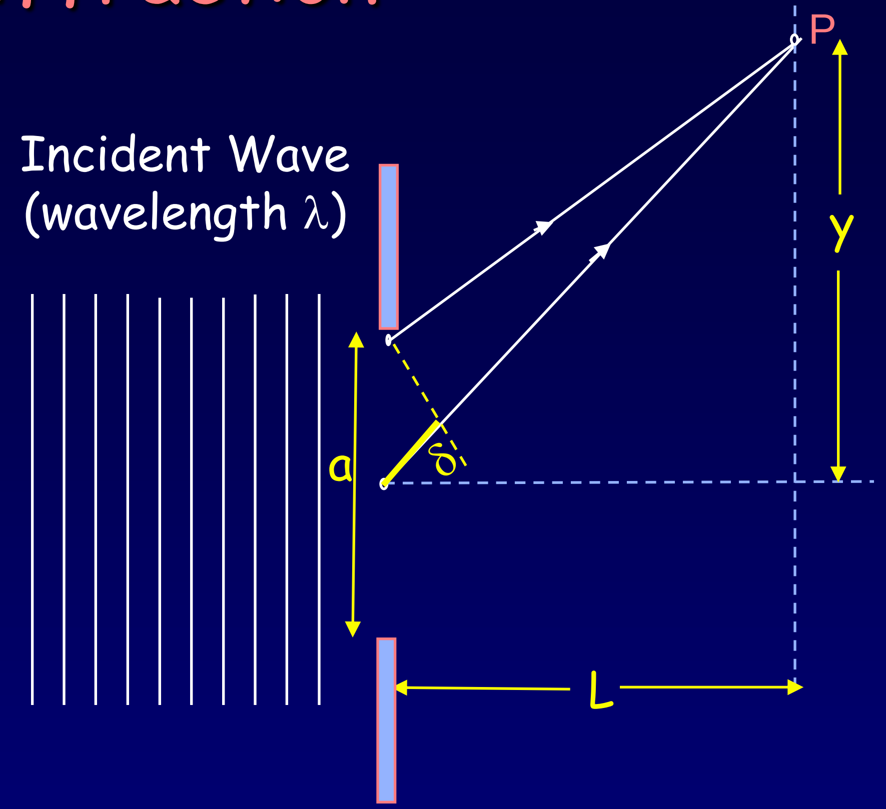
Single-Slit Diffraction

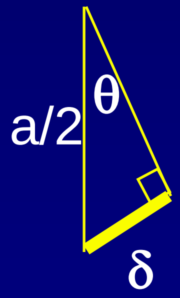
Slit of width a. Where are the *minima*?

Use Huygens' principle: treat each point across the opening of the slit as a wave source.

The first minimum is at an angle such that the light from the top and the middle of the slit destructively interfere.

This works, because for every point in the top half, there is a corresponding point in the bottom half that cancels it.





$$\delta = \frac{a}{2}\sin\theta_{\min} = \frac{\lambda}{2}$$

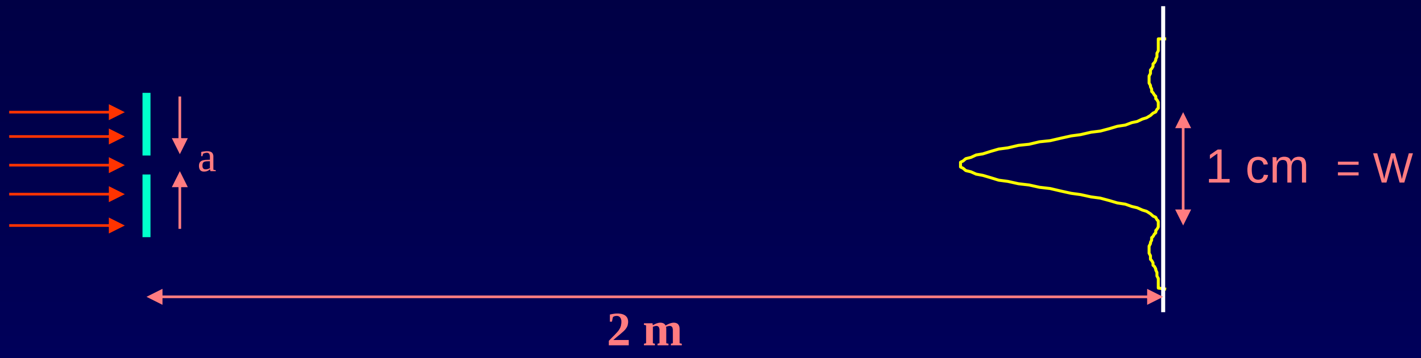
$$\Rightarrow \boxed{\sin\theta_{\min} = \frac{\lambda}{a}}$$

The second minimum is at an angle such that the light from the top and a point at a/4 destructively interfere:

$$\delta = \frac{a}{4}\sin\theta_{\min,2} = \frac{\lambda}{2} \qquad \Rightarrow \sin\theta_{\min,2} = \frac{2\lambda}{a}$$

Location of nth-<u>minimum</u>: $\sin\theta_{\min,n} = \dfrac{n\lambda}{a}$

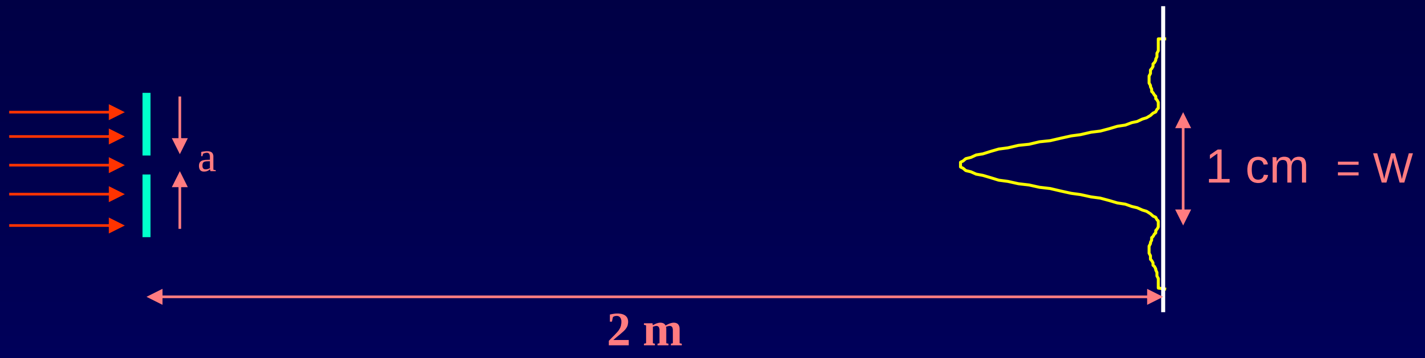
ACT 1



Which of the following would broaden the diffraction peak?

a. reduce the laser wavelength
b. reduce the slit width
c. move the screen further away
d. a. and b.
e. b. and c.

ACT 1 - Solution



Which of the following would broaden the diffraction peak?

a. reduce the laser wavelength

b. reduce the slit width

c. move the screen further away

d. a. and b.

$\boxed{\text{e. b. and c.}}$

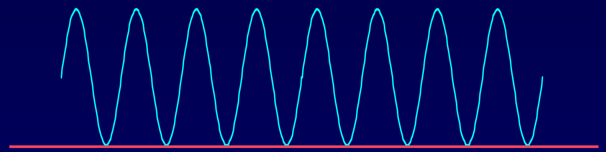
Laboratory 1: Interference

In this week's lab you will combine:

Two slits:

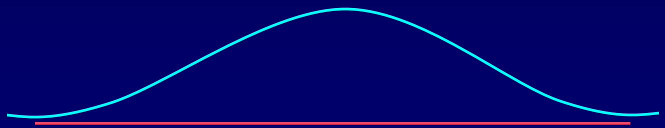
Multi-slit Interference $I = 4I_1\cos^2(\phi/2)$

For point sources, I_1 = constant.



and

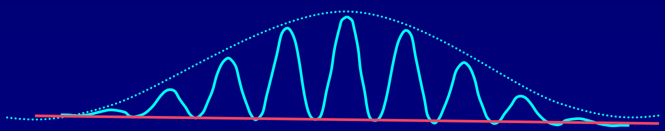
Single-slit Diffraction $I_1(\theta)$



For finite sources, $I_1 = I_1(\theta)$.

to obtain

The total pattern, $I = 4I_1(\theta)\cos^2(\phi/2)$



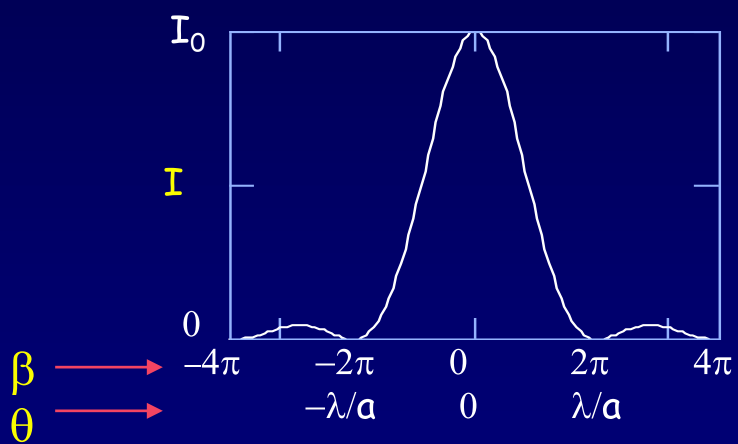
Don't forget … The prelab is due at the beginning of lab !!

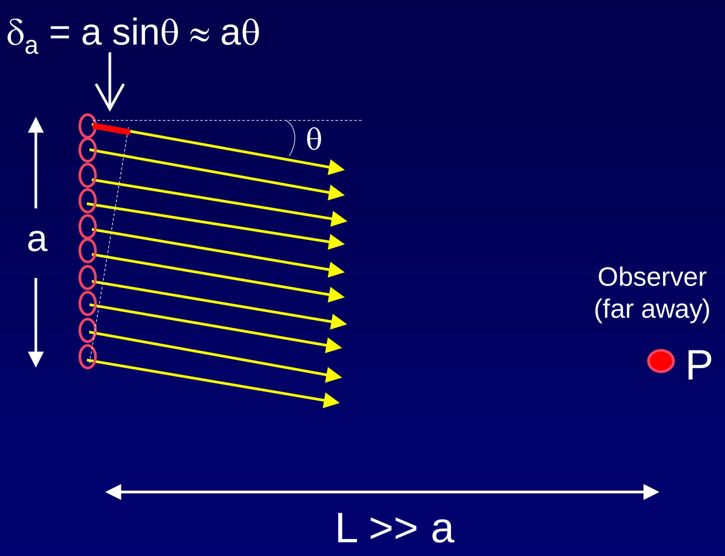
Single-slit Diffraction — Summary

The intensity of a single slit has the following form:

$$I_1 = I_0 \left(\frac{\sin(\beta/2)}{\beta/2} \right)^2$$

Derivation is in the supplement.





β = phase difference between waves coming from the top and bottom of the slit.

Single Slit Diffraction Features:

First zero: $\beta = 2\pi \;\Rightarrow\; \theta \approx \lambda/a$

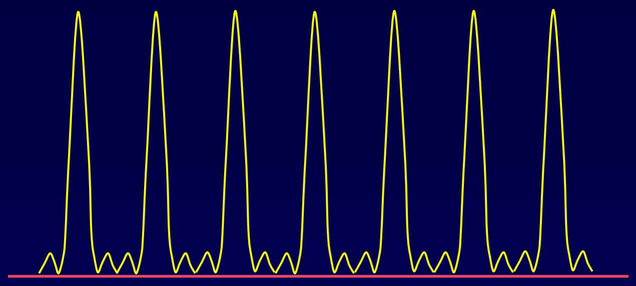
Secondary maxima are quite small.

Multi-Slit Interference + Diffraction

Combine:

Multi-slit Interference,

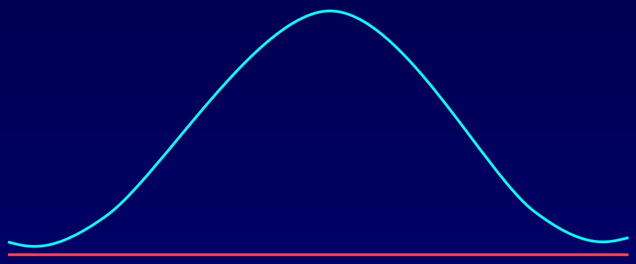
$$I_N = I_1\left(\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right)^2$$



and

Single-slit Diffraction,

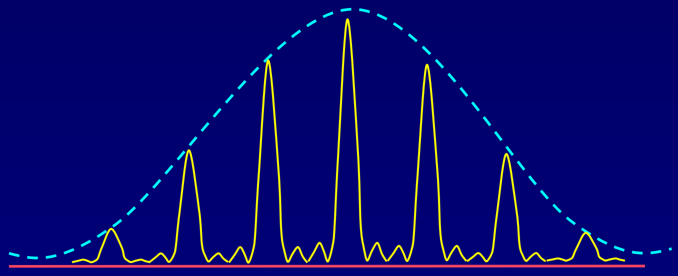
$$I_1 = I_0\left\{\frac{\sin(\beta/2)}{\beta/2}\right\}^2$$



to obtain

Total Interference Pattern,

$$I = I_0\left\{\frac{\sin(\beta/2)}{\beta/2}\right\}^2\left\{\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right\}^2$$



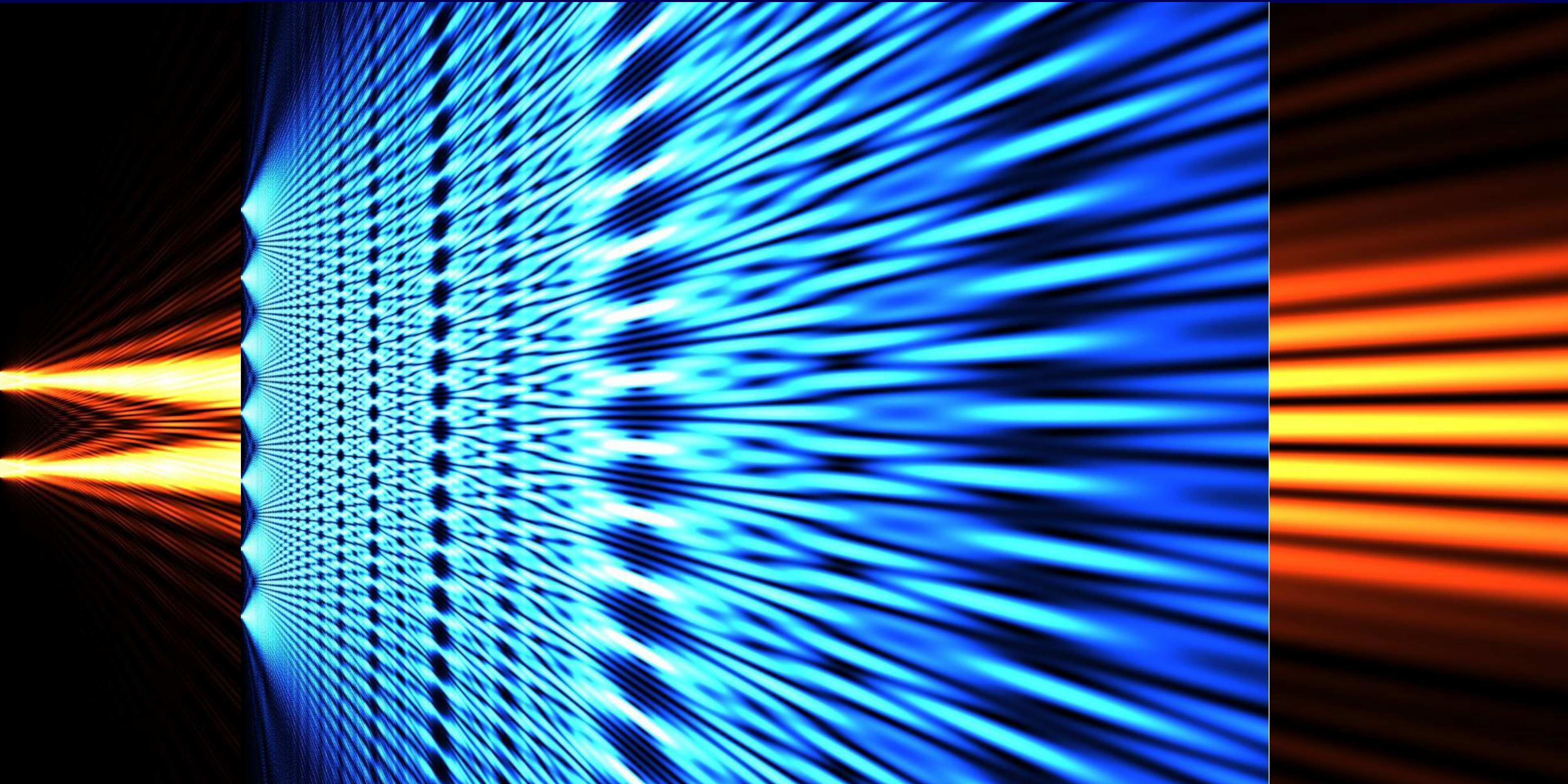
Remember: $\phi/2\pi = \delta/\lambda = (d \sin\theta)/\lambda \approx d\theta/\lambda$
$\beta/2\pi = \delta_a/\lambda = (a \sin\theta)/\lambda \approx a\,\theta/\lambda$

ϕ = phase between adjacent slits
β = phase across one slit

You will explore these concepts in lab this week.

Note: The simple calculations we have done only hold in the "far-field" (a.k.a. "Fraunhofer" limit), where $L \gg d^2/\lambda$.

Intermediate cases ("Fresnel diffraction") can be much more complex…



Interference Gratings

Examples around us.

CD disk – grooves spaced by ~wavelength of visible light.

The color of some butterfly wings!

The material in the wing in not pigmented!

The color comes from interference of the reflected light from the pattern of scales on the wing– a grating with spacing of order the wavelength of visible light!

(See the text Young & Freeman, 36.5)

Optical spectroscopy:
a major window on the world

Some foreshadowing:

Quantum mechanics → discrete energy levels,
e.g., of electrons in atoms or molecules.

When an atom transitions between energy levels it emits light of a very particular frequency.

Every substance has its own signature of what colors it can emit.

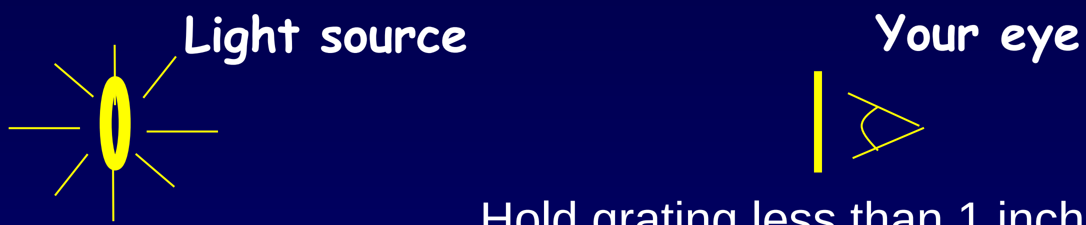
By measuring the colors, we can determine the substance, as well as things about its surroundings (*e.g.*, temperature, magnetic fields), whether its moving (via the Doppler effect), *etc*.

Optical spectroscopy is invaluable in materials research, engineering, chemistry, biology, medicine…

But how do we precisely measure wavelengths?

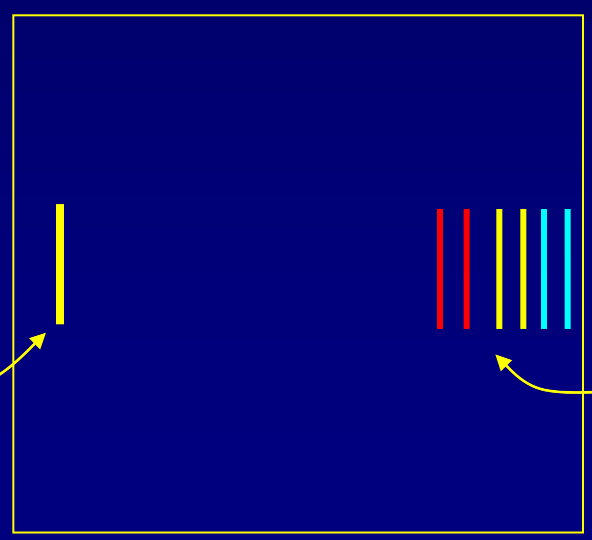
Spectroscopy Demonstration

We have set up some discharge tubes with various gases. Notice that the colors of the various discharges are quite different. In fact the light emitted from the highly excited gases is composed of many “discrete wavelengths.” You can see this with the plastic “grating” we supplied.



Hold grating less than 1 inch from your eye.

Your view through the grating:



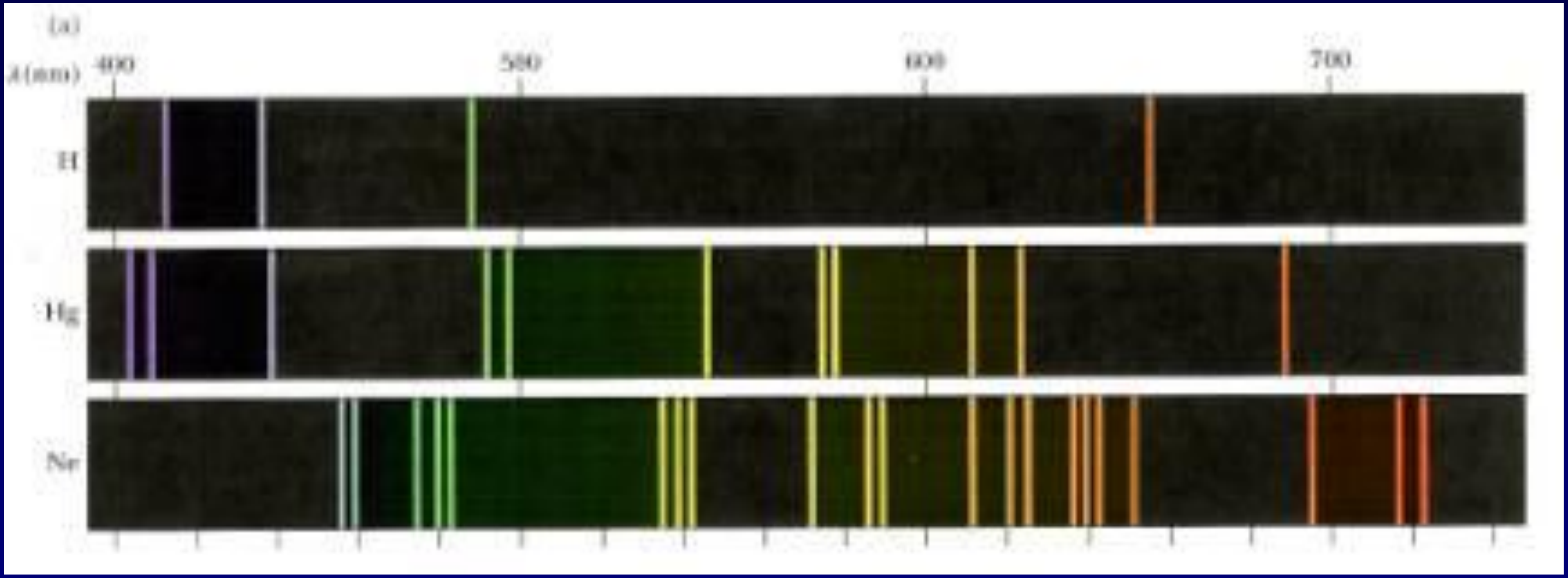
Put light source at left side of grating.

View spectral lines by looking at about 45^{o}. If you don’t see anything, rotate grating 90^{o}.

Atomic Spectroscopy

Figure showing examples of atomic spectra for H, Hg, and Ne.

Can you see these lines in the demonstration?



You can keep the grating. See if you can see the atomic lines in Hg or Na street lights or neon signs.

The lines are explained by Quantum Mechanics.

Source for figure:
http://www.physics.uc.edu/~sitko/CollegePhysicsIII/28-AtomicPhysics/AtomicPhysics.htm

Diffraction from gratings

$$I_N = I_1\left(\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right)^2$$

The slit/line spacing determines the location of the peaks (and the angular dispersing power $\theta(\lambda)$ of the grating:

The positions of the principal interference maxima are the *same* for any number of slits!

$$\boxed{d\sin\theta = m\lambda}$$

The number of slits/beam size determines the *width* of the peaks (narrower peaks easier to resolve).

$$\delta\theta \approx \lambda/Nd$$

Resolving power of an N-slit grating: The Rayleigh criterion

Diffraction Grating Example

Angular splitting of the Sodium doublet:

Consider the two closely spaced spectral (yellow) lines of sodium (Na), λ_1 = 589 nm and λ_2 = 589.6 nm. If light from a sodium lamp fully illuminates a diffraction grating with 4000 slits/cm, what is the angular separation of these two lines in the second-order ($m=2$) spectrum?

Hint: First find the slit spacing d from the number of slits per centimeter.

Solution

Angular splitting of the Sodium doublet:

Consider the two closely spaced spectral (yellow) lines of sodium (Na), λ_1 = 589 nm and λ_2 = 589.6 nm. If light from a sodium lamp fully illuminates a diffraction grating with 4000 slits/cm, what is the angular separation of these two lines in the second-order (m=2) spectrum?

Hint: First find the slit spacing d from the number of slits per centimeter.

$$d = \frac{1\,\text{cm}}{4000} = 2.5 \times 10^{-4}\,cm = 2.5\,\mu m$$

$$\theta_1 = \sin^{-1}\left(m\frac{\lambda_1}{d}\right) = 28.112°$$

Small angle approximation is not valid.

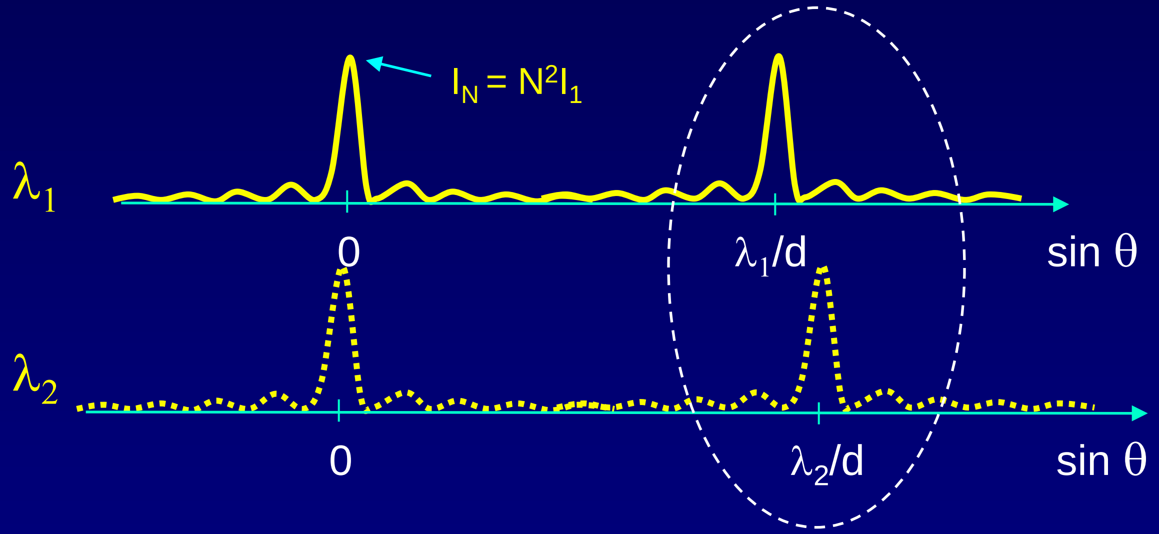
$$\Delta\theta = \sin^{-1}\left(m\frac{\lambda_2}{d}\right) - \sin^{-1}\left(m\frac{\lambda_1}{d}\right) = \boxed{0.031}°$$

Diffraction Gratings (1)

Diffraction gratings rely on N-slit interference.
They consist of a large number of evenly spaced parallel slits.

An important question:

How effective are diffraction gratings at resolving light of different wavelengths (*i.e.* separating closely-spaced 'spectral lines')?



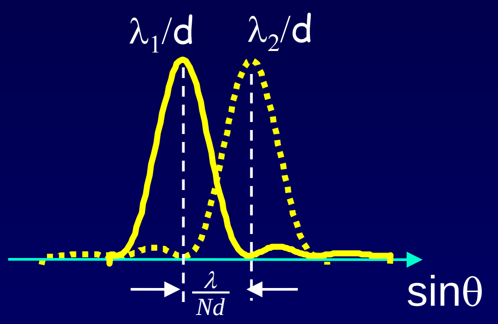
sinθ depends on λ.

Example: Na has a spectrum with two yellow "lines" very close together: λ_1 = 589.0 nm, λ_2 = 589.6 nm   ($\Delta\lambda$ = 0.6 nm)

Are these two lines distinguishable using a particular grating?
We need a "resolution criterion".

Diffraction Gratings (2)

We use Rayleigh's criterion:
The minimum wavelength separation we can resolve occurs when the λ_2 peak coincides with the first zero of the λ_1 peak:



So, the Raleigh criterion is $\Delta(\sin\theta)_{min} = \lambda/Nd$.

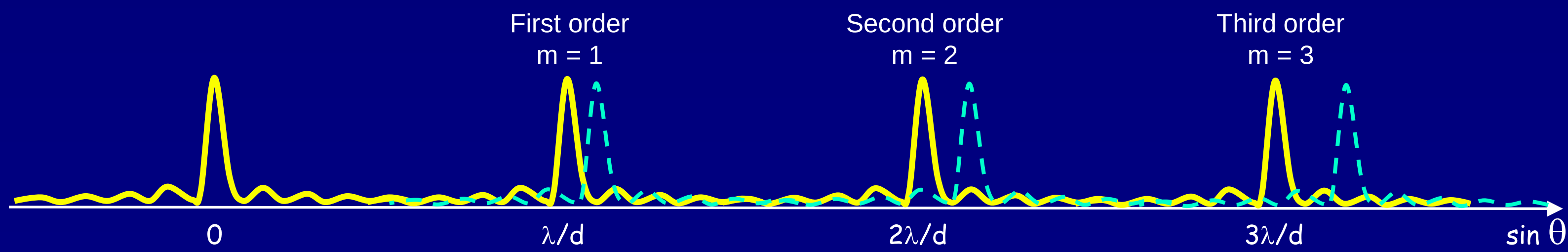
However, the location of the peak is $\sin\theta = m\lambda/d$.

Thus, $(\Delta\lambda)_{min} = (d/m)\Delta(\sin\theta)_{min} = \lambda/mN$:

$$\boxed{\frac{\Delta\lambda_{min}}{\lambda} = \frac{1}{Nm}}$$

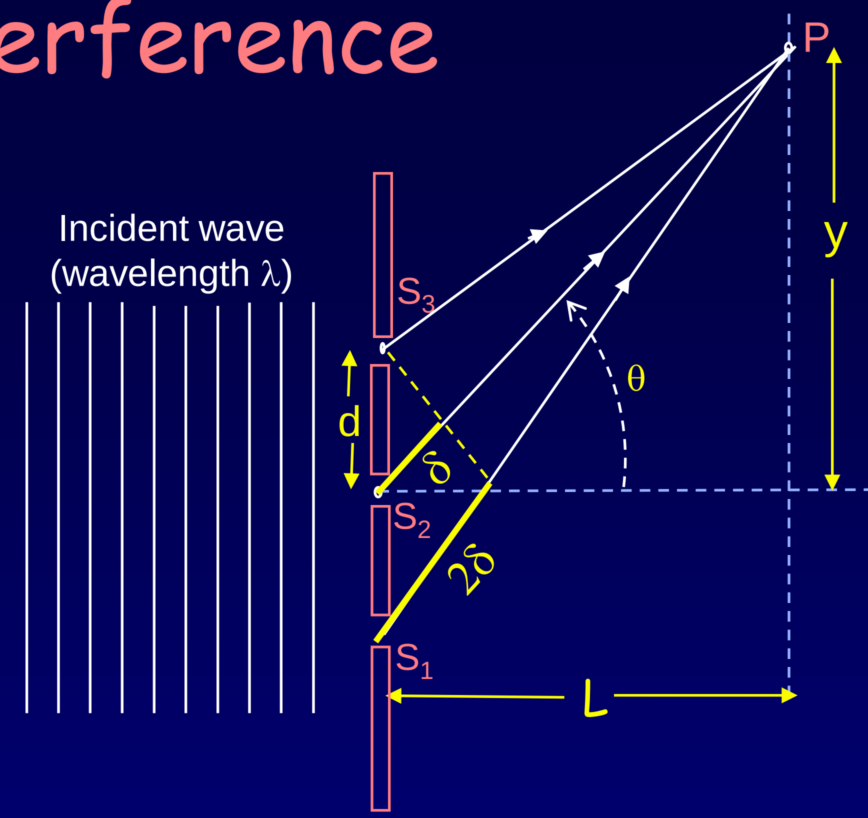
Comments:

- It pays to use a grating that has a large number of lines, N. However, one must illuminate them all to get this benefit.
- It also pays to work at higher order (larger m): The widths of the peaks don't depend on m, but they are farther apart at large m.



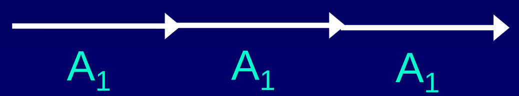
Supplementary Slides

Multi-Slit Interference



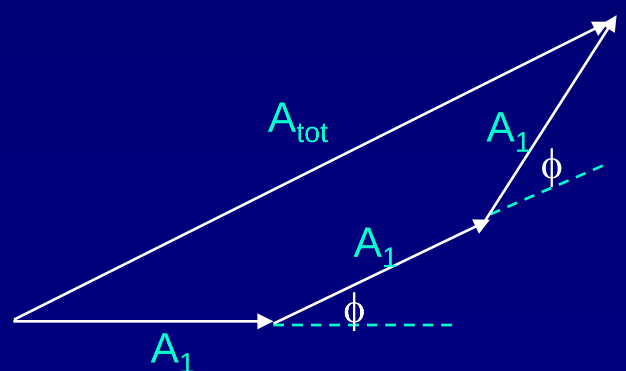
We already saw (slide 4) that the positions of the principal maxima are independent of the number of slits. Here, we will use phasors to determine the intensity as a function of θ.

At each principal maximum ($d \sin\theta = m\lambda$), the slits are all in phase, and the phasor diagram looks like this:



$$A_{tot} = N\,A_1 \rightarrow I_{tot} = N^2\,I_1$$

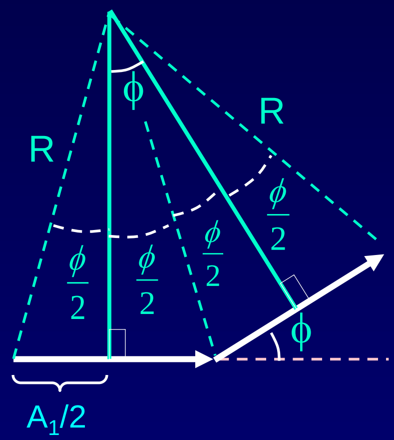
For other values of θ, the phasors are rotated, each by an angle ϕ with respect to its neighbors. Remember that $\phi/2\pi = \delta/\lambda = d/\lambda \sin\theta$.
We can calculate A_{tot} geometrically (next slide).



Multi-Slit Interference (2)

The intensity for N equally spaced slits is found from phasor analysis.

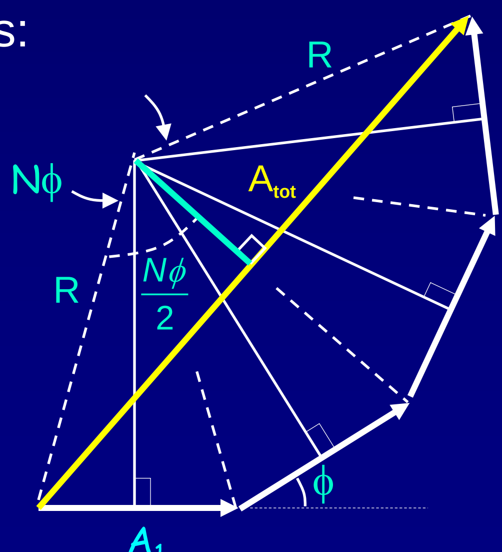
Draw normal lines bisecting the phasors. They intersect, defining R as shown:



$$\frac{A_1}{2} = R\sin\frac{\phi}{2} \Rightarrow R = \frac{A_1}{2\sin(\phi/2)}$$

Substitute

N slits:



$$\frac{A_{tot}}{2} = R\sin\frac{N\phi}{2} \Rightarrow$$

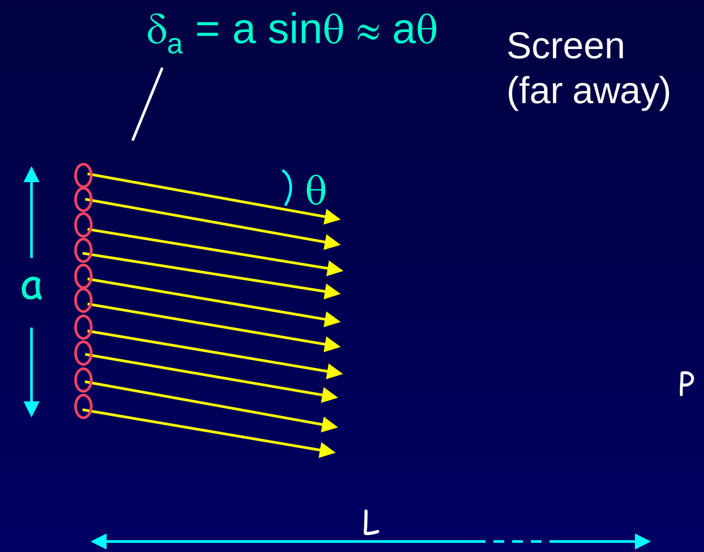
$$\boxed{\begin{aligned} A_{tot} &= A_1 \frac{\sin(N\phi/2)}{\sin(\phi/2)} \\ I_{tot} &= I_1 \left(\frac{\sin(N\phi/2)}{\sin(\phi/2)}\right)^2 \end{aligned}}$$

Single-slit Diffraction

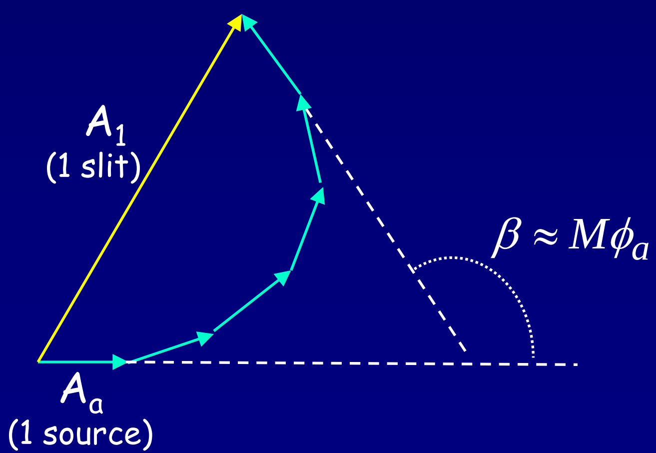
To analyze diffraction, we treat it as *interference* of light from many sources (*i.e.*, the Huygens wavelets that originate from each point in the slit opening).

Model the single slit as M point sources with spacing between the sources of a/M. We will let M go to infinity on the next slide.

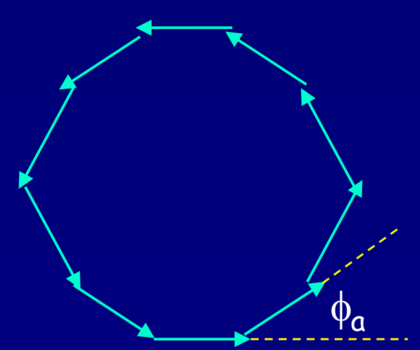
The phase difference β between first and last source is given by $\beta/2\pi = \delta_a/\lambda = a \sin\theta / \lambda \approx a\theta/\lambda$.



L >> a implies rays are parallel.



Destructive interference occurs when the polygon is closed ($\beta = 2\pi$):

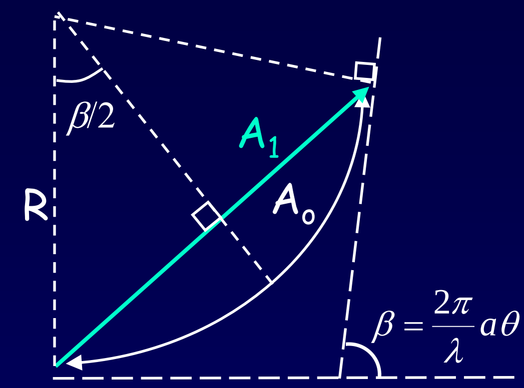


This means $\dfrac{a \sin\theta}{\lambda} = 1$

For small θ, $\boxed{\theta \approx \dfrac{\lambda}{a}}$

Destructive Interference

Single-slit Diffraction (2)



We have turned the single-slit problem into the M-slit problem that we already solved in this lecture.

However, as we let $M \to \infty$, the problem becomes much simpler because the polygon becomes the arc of a circle.

The radius of the circle is determined by the relation between angle and arc length: $\beta = A_o/R$.

Trigonometry: $A_1/2 = R\sin(\beta/2)$

With $R = A_o/\beta$: $$A_1 = A_0 \frac{\sin(\beta/2)}{\beta/2}$$

$I = A^2$: $$\boxed{I_1 = I_0 \left(\frac{\sin(\beta/2)}{\beta/2}\right)^2}$$

You can graph this function

Remember: $\beta/2\pi = \delta_a/\lambda = (a\sin\theta)/\lambda \approx a\,\theta/\lambda$

$\beta =$ angle between 1st and last phasor