Physics 225

Relativity and Math Applications

Fall 2011

Unit 2

The Lorentz Transformation

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Unit 2: Special Relativity Formalized: the Lorentz Transformation

The Axioms of Special Relativity

Today we will be formalizing our study of Special Relativity (SR). To kick things off, there's nothing more formal than a precise statement of the **Axioms of Special Relativity**:

1. The Principle of Relativity: All inertial frames are totally equivalent for the performance of all physical expts.
2. The Universality of the Speed of Light: The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

Every formula and physical consequence of special relativity is derivable from those two axioms. How do we know they're true? Experiment. No experiment to date has found any discrepancy with SR, and that's the only definition of "true" that we have for anything in science.

We worked with axiom #2 last week. We were tacitly using axiom #1 but we didn't really discuss it. Think about the Principle of Relativity for a moment. Here's another way of expressing it which might make its meaning more clear:

There is no experiment that can distinguish between
a stationary frame and one that is moving at constant velocity.

Thus, there is no such thing as an absolute "stop". If you say an object is "stopped" or "stationary" without any further information, your statement is meaningless. An object can be stopped *relative to* another object (hence "relativity"!), but not in any absolute sense.

Key concepts: space-time events and reference frames

The only quantitative features of special relativity we've seen so far are the rates at which moving clocks slow down and moving objects shrink. This week we'll incorporate those effects into the full-blown mathematical framework of special relativity, which is the **Lorentz transformation**. The Lorentz transformation ("LT" for short) is a set of formulae that tells us how events in space-time transform from one reference frame to another.

What do we mean by "**reference frame**"? It is a coordinate system in which all our clocks and rulers are at rest, and all our clocks are synchronized. It is simply the coordinate system in which a given **observer** measures positions and times.

What do we mean by "**space-time event**"? An event is an occurrence, like a rocket launch from Cape Canaveral, or you pouring your morning cup of coffee. The thing is: an event occurs somewhere in space *and* somewhere in time. The great lesson of SR is that *space and time must be treated together*. If I asked you where you poured your cup of coffee, you'd probably say "at my kitchen table". But relativity demands that we never ignore the time portion. A complete description of the event, in your reference frame, would be: "I poured my cup of coffee at my kitchen table at 8:30 am." If you poured another cup of coffee at 8:50 am, that would be a second event. And if your housemate poured a cup of coffee also at 8:50 am, but at the dining-room table, that would be a third event. So we always specify an event with *four* variables, not three: (x,y,z) denoting position plus t denoting time.

The reason this matters is that space and time *mix* under Lorentz transformations. Suppose an observer whizzes by your front window at close to the speed of light and observes you and your housemate pouring your cups of coffee at 8:50 am. In your reference frame, these two events occurred at the same time – i.e., they were simultaneous. *But the speeding observer will see the two events at **different** times*. The Lorentz transformation (LT) equations formalize what you discovered last week: that the time interval between two events in frame A depends on *both* the time *and* position intervals between the events in frame B. Only if two events are at precisely the same position *and* time in frame A will they be simultaneous in frame B – in that case their complete space-time coordinates are the same.

Exercise 2.1: Quantitative description of non-simultaneity

Using only our knowledge of time-dilation and length-contraction from last week, let's analyze two events from the persective of two different observers. As is customary, we will define one observer to be stationary, and the other to be moving at some relative velocity v. The observers will measure the position x and time t of each event. The LT equations are coming up in a few pages; those equations will only work if we follow two important **notational conventions**, so let's start using them now:

- We always define the **+x and +x' directions** of our coordinate systems so that they point in the direction of motion of the moving observer.
- We always use **primed variables** (t', x') to indicate the measurements of the moving observer; the static observer measures x and t.

We'll use the letters S and S' as shorthand for the two frames. It doesn't matter which frame you choose to be "stationary" (S) and which to be "moving" (S'). What matters is that you are *consistent*: once you pick a frame to be the moving one, S', that frame (a) determines the directions of the $+x$ and $+x'$ axes and (b) always measures primed quantities.

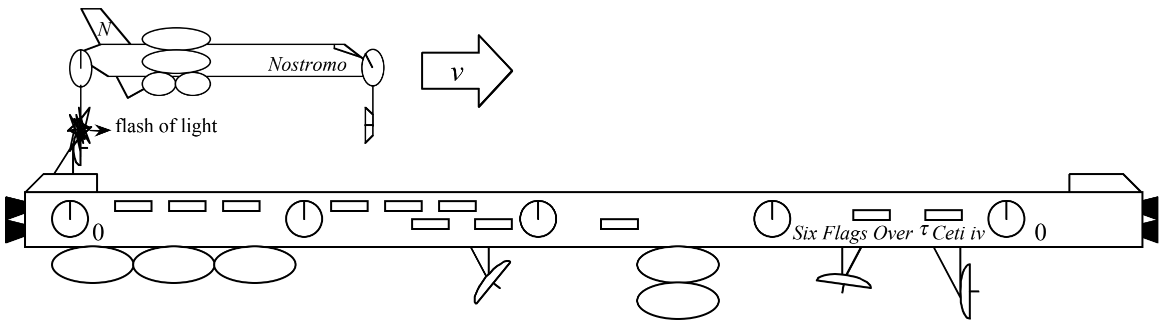
The starship *Nostromo* glides past the *Six Flags* recreational facility at speed $v = 0.8\ c$, traveling to the right. The rest length of the *Nostromo* is L_0. Aboard the *Nostromo*, the crew performs a routine test: they produce a flash of light from a signal laser at the tail of their ship, and then record the flash with a sensor at the ship's nose. (A figure is on the next page.)

The production and reception of the light form a pair of space-time events:

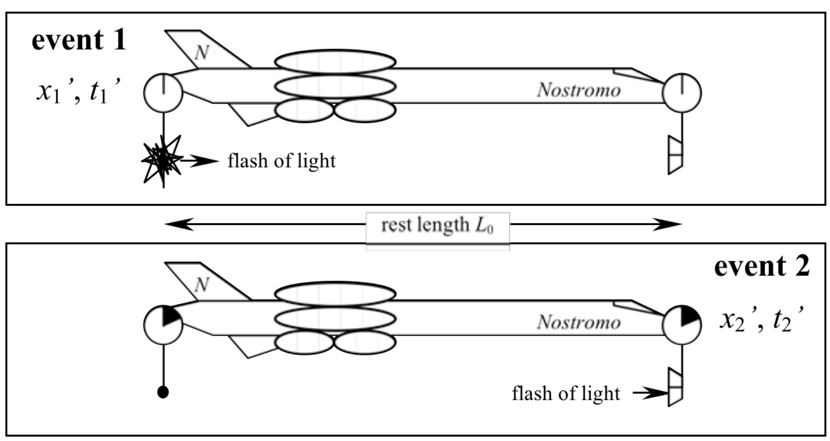
- event #1 is the production of the flash at the *Nostromo's* aft laser
- event #2 is the arrival of the light flash at the *Nostromo*'s forward sensor

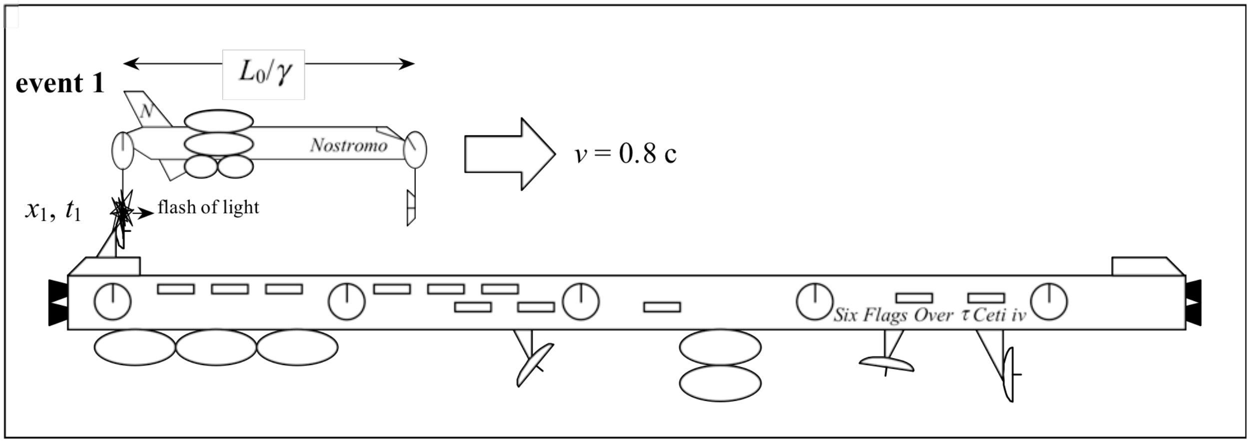
These events are observed by the *Nostromo* crew, of course, but also by the vacationers on the recreational facility who are staring out their windows. We have to pick one of these two as our "stationary" observer → let's make it the vacationers. So according to the conventions listed above, the vacationers will measure (t, x), while the *Nostromo* crew will measure (t', x').

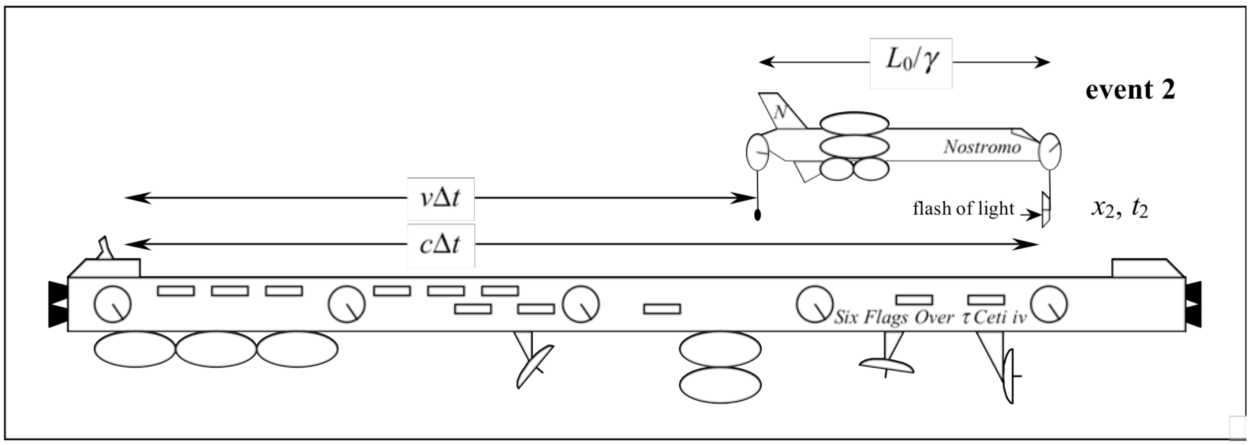
(a) To express the positions x and x' that our observers measure, we need to decide on our coordinate systems. Let's choose the *Six Flags* to be the S frame (the "stationary" one) and the *Nostromo* to be the S' frame (the "moving" one). According to our conventions, what directions do we choose for positive x and positive x' ? Go ahead and draw them on the picture.



(b) First let's work from the *Nostromo's* reference frame (S'). There is a well-defined spatial and temporal **interval** between the two events in this reference frame: $\Delta x' \equiv x_2' - x_1'$ is the difference between the positions at which the two events occur, and $\Delta t' \equiv t_2' - t_1'$ is the time difference. What intervals $\Delta x'$ and $\Delta t'$ do the crew measure? Express your answer in terms of the rest length L_0 of the ship.







(c) Now let's move to the recreational facility. From the perspective of the vacationers, the moving *Nostromo*'s length is Lorentz-contracted by a factor of $1/\gamma$. What space-time interval $(\Delta t, \Delta x) = (t_2, x_2) - (t_1, x_1)$ do the vacationers measure between the two events? To figure it out, look at the notations on the figure (and a hint: the light flash has to "catch up" with the *Nostromo* in this frame). Do the calculation with symbols – *as always!* – then *at the end* put in numbers for β and γ to get Δt and Δx in terms of L_0 and c. (Note: you do not need a calculator to get γ, see[1]).

(d) To define a space-time event completely, we need its time t and position (x,y,z) … yet we've been ignoring the y and z components of position and dealing only with x. Can you think why?

[1] You may have noticed that our examples always seem to use β=0.6 or 0.8. Here's why: 3,4,5 are **Pythagorean numbers** that satisfy the relation $3^2+4^2=5^2$ … and that makes γ very easy to calculate when $\beta = 3/5$ or $4/5$!

Exercise 2.2: The Lorentz transformation (“LT”)

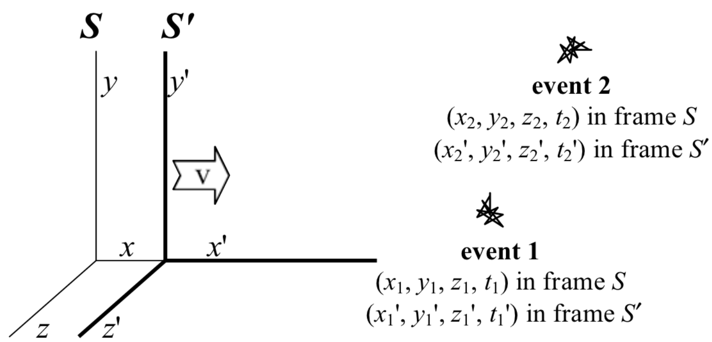
Here’s a summary of what we learned last week: Starting entirely from the principle that the speed of light is the same in all frames, we discovered that these three phenomena must occur:

1. **Time dilation**: A moving clock ticks slow (by a factor of γ) according to a stationary observer.
2. **Length contraction**: Objects in motion are shorter than their rest length (by a factor of γ) according to a stationary observer.
3. **Loss of simultaneity**: Events which occur simultaneously in one frame, but are separated by some distance, do *not* occur simultaneously in a second frame that is in motion relative to the first.

The Lorentz transformation absorbs all three of these effects into a coherent mathematical framework. It tells us how to transform measurements made in one frame to those made in another.

Imagine that a pair of events is seen by observers in the two different coordinate systems $\boldsymbol{S}$ and $\boldsymbol{S'}$. Observers in $\boldsymbol{S}$ (the “stationary” frame) measure the space-time intervals between the events to be Δx, Δy, Δz, Δt while observers in $\boldsymbol{S'}$ (the “moving” frame) measure the interval to be $\Delta x'$, $\Delta y'$, $\Delta z'$, $\Delta t'$. (Here $\Delta x \equiv x_2 - x_1$, $\Delta t \equiv t_2 - t_1$, and so forth.) If we set up our coordinate system so that $\boldsymbol{S'}$ is moving in the positive-x direction with speed $v = \beta c$ (relative to $\boldsymbol{S}$) then the Lorentz transformation is:

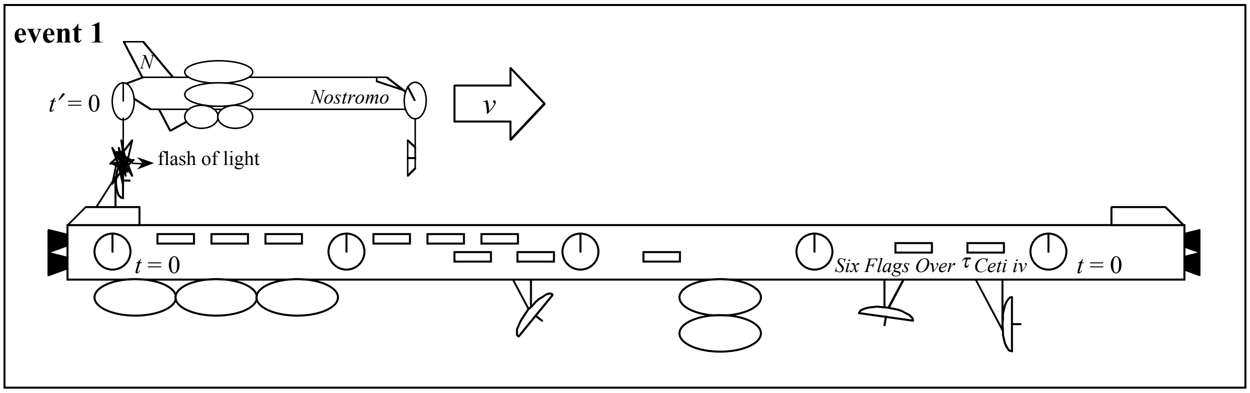
$$\Delta t' = \gamma(\Delta t - \beta \Delta x / c)$$
$$\Delta x' = \gamma(\Delta x - \beta \Delta t\, c)$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$

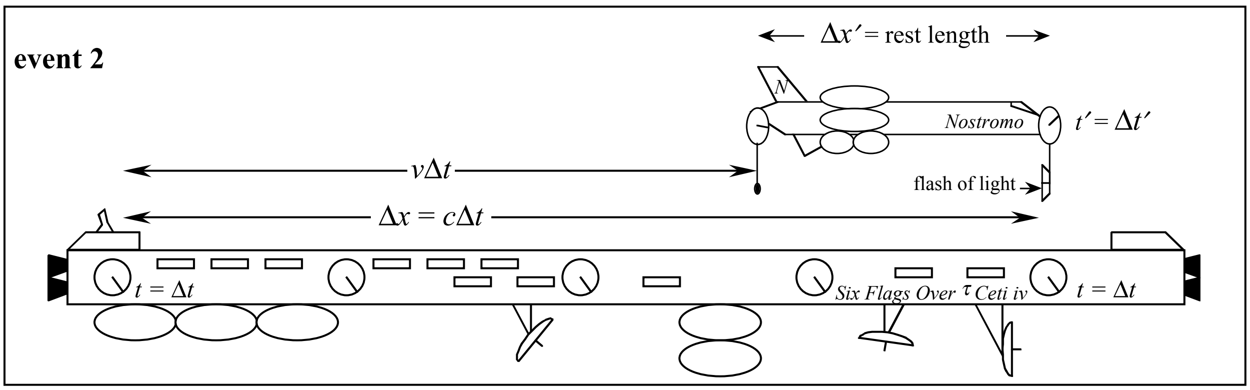


Here are some things to notice about the Lorentz transformation:

- Position (x) and time (t) appear on an equal footing in these expressions, and they mix: time in one frame depends on *both* time and position in the other frame, and vice versa.
- The Lorentz transformation has a beautifully symmetrical structure, which makes it easy to remember. The only “awkward-looking” elements are the factors of c, but they are also easy to remember: they are only there to keep the units correct.
- Note how the spatial coordinates y and z that are perpendicular to the relative motion between the frames are unaffected by the transformation, as we learned last week.

Now let’s apply the Lorentz transformation to the problem we solved in Exercise 2.1!





(a) Go all the way back to part (c) of the previous exercise. There, you calculated the space-time interval (Δt, Δx) between the emission and detection of the light flash by the *Nostromo* as observed by the vacationers on the rec facility. Start with that: $\Delta x = 3L_0$, $\Delta t = 3L_0/c$.
Now, apply the Lorentz transformation – your new equations! – to determine the interval ($\Delta t'$, $\Delta x'$) measured by the moving *Nostromo* crew. You should get back your answer to part (b)!

Was that a proof? It can be shown mathematically that the axioms of SR lead inescapably to the Lorentz transformation; we'll cover the proof in lecture so you can see for yourself! But what you've just done is actually pretty close to this proof already → In Exercise 2.1, your calculations were based purely on your findings from last week, and those came *directly* from the axioms of SR. What you've shown on this page is that the new Lorentz equations do indeed reproduce what you found, for a randomly selected pair of events, which is pretty convincing evidence that they're correct. The only major piece of the full proof that's missing is to convince yourself that the transformation equations must be *linear*.

(b) Suppose we wanted to do this calculation in reverse: start with the *Nostromo* measurement ($\Delta t'$, $\Delta x'$) and calculate the vacationers' measurement (Δt, Δx). To do that, you would need the **inverse Lorentz transform**. You can calculate the inverse algebraically if you like. Alternatively, you could try some physical thinking: what modification of the LT equations would turn an $S \to S'$ transform into an $S' \to S$ inverse transform?[2] Whichever method you choose, write down the inverse LT equations that give you (Δt, Δx) in terms of ($\Delta t'$, $\Delta x'$). Check with your instructor, then put these useful relations in a box!

Exercise 2.3: The Lorentz transformation does it all

Next, let's discover how the three basic phenomena of special relativity – time dilation, length contraction, and loss of simultaneity – can all be derived from the Lorentz transformation.

(a) **Time dilation** tells us that moving clocks are slowed down by a factor of γ. Let's be precise about this by considering a specific situation. Consider a single digital clock mounted on the outside of the *Nostromo*, which is speeding past the *Six Flags* with speed $\beta = 0.8$ as before. Around noon, the vacationers look out their windows and see the reading on the *Nostromo* clock change from 12:00 to 12:01 (*i.e.*, a change of 1 minute = 60 seconds). According to the vacationers' own watches, how long did it take for this change to occur?

Solution procedure:

- First, *make a sketch!* (Remember the scientist's toolkit from last week? No picture provided this time. ☺) It is essential to *visualize* what's going on, just sketch as you read the problem. Also remember to label your sketch with what you know and what you want to know. Assign well-chosen symbols to each of these quantities.
- Next, identify the two space-time events involved: what are they exactly?
- Finally, consider the space-time interval between those events. There are four variables describing this interval: Δt, Δx, $\Delta t'$, and $\Delta x'$. The situation fixes two of them; the LT allows you to solve for the other two. Which two do you know? → put in those values below. Which one do you *want* to know? → flag that one with a question mark. Figuring out what you know and what you want to know is 95% of the problem!

  $\Delta t =$
  $\Delta x =$
  $\Delta t' =$
  $\Delta x' =$

[2] Suggestion: try writing down the normal and inverse *Galilean* transformations; that should give you a major clue.

(b) **Length contraction** tells us that moving objects are shortened by a factor of $1/\gamma$. Again let's be precise. What does "length" really mean? Well, no matter how you accomplish it in practice (rulers, light flashes, GPS readings), you always find the length of an object as follows:

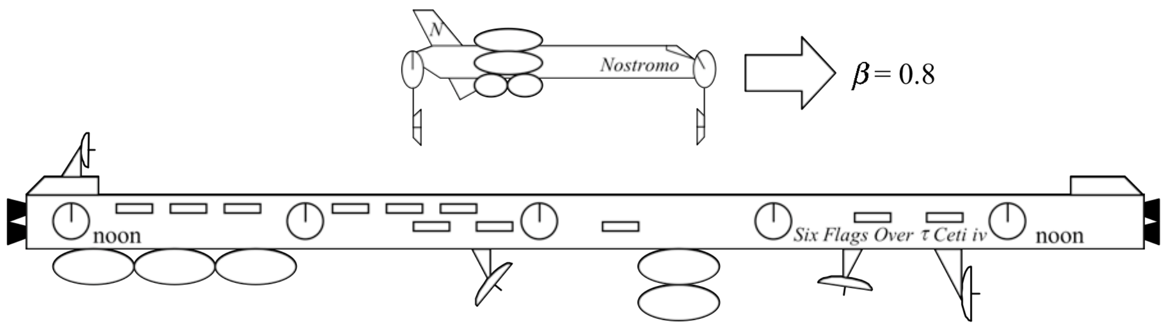
by determining the position of its two ends at the same time

and taking the difference. With this insight in hand, think about the measurements the vacationers must make to find the length of the *Nostromo*. The rest length of the *Nostromo* is 600′ … what length do the vacationers find? As before, identify the space-time events involved in their measurements, then use the Lorentz transformation to calculate the length they obtain.

(c) **Length contraction II**: You just showed that the stationary vacationers see a shortened version of the *Nostromo*, because it is moving. How does this work in reverse? "Common sense" suggests that the moving *Nostromo* crew should see a *lengthened* version of the *Six Flags* ... but that is not the case: to them, it is the *Six Flags* that is moving, and they also find that it is shorter than its rest length. It must be so, or the Principle of Relativity is blown.

The Lorentz transformation deals effortlessly with this seemingly paradoxical phenomenon. Consider the two spacetime events involved when the *Nostromo* crew measures the length of the rec facility (whose rest length is 2000′). As before, the situation fixes two of the four variables Δt, Δx, $\Delta t'$, $\Delta x'$ and the Lorentz transformation allows you to calculate the one you want. What length does the crew measure? Is it shorter or longer than the rest length of 2000′?

Loss of simultaneity. The *Nostromo* crew has a clock at the front end of the ship and another one on the back end. Naturally, they keep these clocks synchronized [using some experimental technique whose details are irrelevant]. Synchronized in the *Nostromo*'s frame, that is … When the vacationers on the *Six Flags* look at the *Nostromo* clocks, they see that one of them always reads *earlier* than the other. Which one, and by how much?



(d) Here's the question phrased precisely: According to the vacationers, when the aft *Nostromo* clock reads precisely 1:00 pm, what does the forward clock read? Think very carefully … *what spacetime events* does that question describe?

(e) So, which clock reads the *earlier* time according to the vacationers? The sign you just obtained for $\Delta t'$ (or Δt depending on the convention you picked) gives you the answer … as long as you can intepret it.

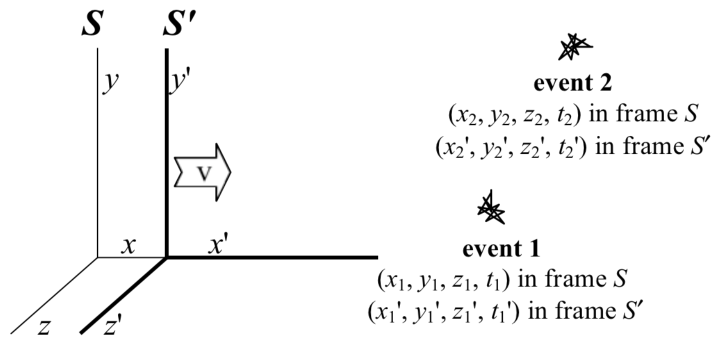
(f) Is your answer to (e) correct? It's very easy to make a sign error in questions like this, so check your result with physical thinking! First, invent a simple experimental technique that the *Nostromo* crew could use to synchronize their clocks.[3] Next, picture what that clock-setting exercise looks like to the vacationers. Which clock is started first? Did you correctly identify the one that reads earlier? If not, see if you can find your error in parts (d) or (e).

[3] e.g. recall Unit 1, where light flashes from a ship's midpoint were used to synchronize the clocks at the ends.

Summary

- The Lorentz transformations can be used to calculate the interval ($\Delta t'$, $\Delta x'$, $\Delta y'$, $\Delta z'$) between events observed from one reference frame, called S', using the interval ($\Delta t, \Delta x$, Δy, Δz) observed from another reference frame, called S. To apply the formulas correctly, your coordinate system must be set up so that the S' frame is moving with speed $v = \beta c$ in the positive x direction, with respect to the S frame. Then, the formulas are:

$$\Delta t' = \gamma(\Delta t - \beta \Delta x/c)$$
$$\Delta x' = \gamma(\Delta x - \beta \Delta t\, c)$$
$$\Delta y' = \Delta y$$
$$\Delta z' = \Delta z$$



- The Lorentz transformation mixes space and time → that's how simultaneity gets lost between frames.
- **Inverting** the Lorentz transformation is easy: just reverse the sign of the relative velocity β between the frames.

$$\Delta t = \gamma(\Delta t' + \beta \Delta x'/c)$$
$$\Delta x = \gamma(\Delta x' + \beta \Delta t'\, c)$$
$$\Delta y = \Delta y'$$
$$\Delta z = \Delta z'$$

- There's really no particular need to write the Lorentz transformation in terms of space-time *intervals* (Δt ,Δx, Δy, Δz). It also works for individual space-time events, as long as you **synchronize the origins** of your S and S' space-time coordinate systems. To be precise: at the moment when the spatial origins $(x,y,z) = (0,0,0)$ and $(x',y',z') = (0,0,0)$ of the frames are at the same spot, their clocks must also read the same: $t = 0$ and $t' = 0$. In that case, the forward and reverse Lorentz transformations apply directly to individual events:

$t' = \gamma(t - \beta x/c)$	$t = \gamma(t' + \beta x'/c)$
$x' = \gamma(x - \beta t\, c)$	$x = \gamma(x' + \beta t'\, c)$
$y' = y$	$y = y'$
$z' = z$	$z = z'$