Physics 225

Relativity and Math Applications

Fall 2012

Unit 3

The Interval, Causality, and Proper Time

N.C.R. Makins

University of Illinois at Urbana-Champaign

Unit 3: The Interval, Causality, and Proper Time

Last week, we introduced the **Lorentz transformation**: the master equations which allow us to transform **space-time events** from one intertial reference frame to another. The equations are simple and elegant … just don't forget the *conventions* behind them:

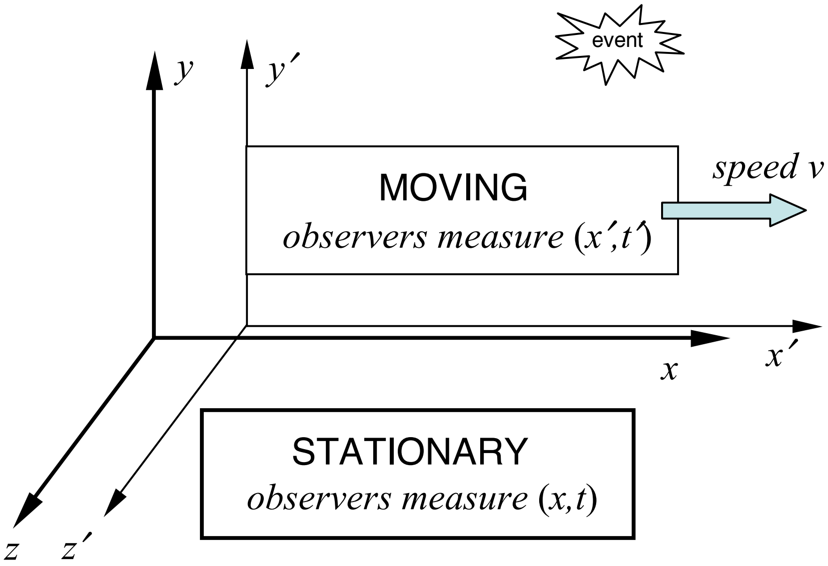
- the unprimed coordinates refer to our chosen "stationary" frame
- the primed coordinates refer to the "moving" frame
- the $+x$ and $+x'$ directions points in the direction-of-motion of the moving frame relative to the stationary one

All of this is summarized in the equations and figure below.

$$\boxed{\begin{aligned}\Delta t' &= \gamma(\Delta t - \beta\Delta x/c)\\ \Delta x' &= \gamma(\Delta x - \beta\Delta t\, c)\\ \Delta y' &= \Delta y\\ \Delta z' &= \Delta z\end{aligned}}$$

with $\beta = \frac{v}{c}$ and

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$



Also remember one of our findings from last week: the **inverse** Lorentz transformation, which translates from the primed (moving frame's) coordinates back to the unprimed (stationary frame's) coordinates, is easily obtained by simply *reversing the sign of β.*

Here's some new jargon for you: a Lorentz transformation is often called a **Lorentz boost**. ("Boost" as in "booster rockets", as in "jumping onto a moving ship". ☺)

Today, we'll derive a new measure of the "distance" between two events: it's called **the interval** and it is a **Lorentz invariant** quantity. Next, we'll consider what it means, and introduce the concept of **proper time**. Finally, we'll examine the fundamental concept of **causality** and see how it can be preserved in the strange world of relativity.

Exercise 3.1: The Invariant Interval

Length contraction is quite a shock when you first encounter it. In Newtonian physics, the distance $d=\sqrt{(\Delta x)^2+(\Delta y)^2+(\Delta z)^2}$ between any two physical locations (e.g. the ends of an object) is always the same: it is **invariant** under all changes of frame. But as we've discovered, when things are moving close to the speed of light we lose the concept of "length" as a frame-invariant quantity. We would very much like to replace it! Surely there is *something* about the "distance" between two events which all observers can agree on?

Yes there is, and it is called the **interval** between two space-time events. It is defined like this:

$$\boxed{I=(c\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2}$$

The entire reason we care about the interval is that it is **frame invariant**: a moving observer gets the same result as a stationary one.

To make sure the concept is clear, consider two observers: S is stationary and S′ is moving in the $+x$ direction. If they measure the space-time locations of two events A and B, they will get different values for the coordinate-distances between the events:

$$\Delta t_{A-B}\neq\Delta t'_{A-B}\qquad\text{and}\qquad\Delta x_{A-B}\neq\Delta x'_{A-B}\,.$$

However, if they both calculate I, they will get the *same answer*:

$$I=(c\Delta t)^2-(\Delta x)^2-(\Delta y)^2-(\Delta z)^2=(c\Delta t')^2-(\Delta x')^2-(\Delta y')^2-(\Delta z')^2=I'$$

The fact that this particular combination of space and time intervals is frame invariant follows immediately from the Lorentz transformation equations. Proving it is easy. You can do it yourself in 3 lines of algebra, or see page 84 of French.

At last we have *something* about event pairs that all observers agree on! ☺

(a) The formula for I gives the interval between two space-time events. Our observer S would measure event A at (ct_A, x_A, y_A, z_A) and event B at (ct_B, x_B, y_B, z_B), then go on to calculate $I_{(\text{B-A})}$. Do we have to pay attention to the *order* of these two events when determining the interval between them? In other words, is $I_{(\text{B-A})}$ different from $I_{(\text{A-B})}$? Compare this to the familiar spatial distance d between two points … does the order of the points matter in that case?

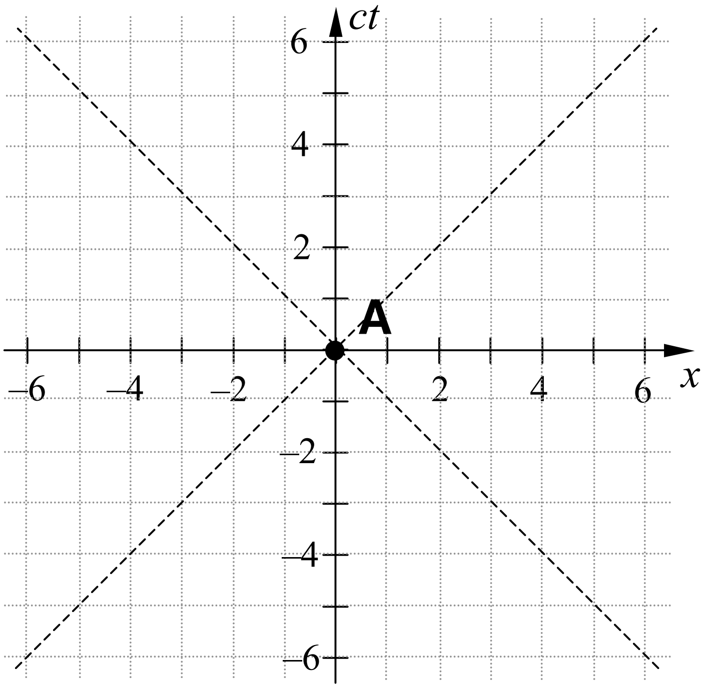
Let's take one event, event **A**, to supply the origin $(x,y,z,t) = (0,0,0,0)$ of our coordinate system. This event could be Astronaut Sandy sitting down at her desk in the morning. Every other space-time event **B** will be some interval I away from event **A**. More definitions: we classify event-pairs according to the *sign* of this interval:

- If $I_{(\mathrm{B\text{-}A})}$ is <u>positive</u>, the events A and B are separated by a **timelike** interval
- If $I_{(\mathrm{B\text{-}A})}$ is <u>negative</u>, the events A and B are separated by a **spacelike** interval
- If $I_{(\mathrm{B\text{-}A})}$ is <u>zero</u>, the events A and B are separated by a **light-like** interval

(b) To explore the significance of these terms, mark some events B on the graph below. First, use *filled* circles to indicate some space-time points **B** that are separated from Sandy-sitting-down (event **A**) by a *timelike* interval. In which parts of the space-time continuum do these events occur?

Second, use *open* circles to mark some space-time points that are a *spacelike* interval away from the origin.

Finally, uses X's to mark a couple of events with *lightlike* separation from **A**.



(c) As it happens, physical objects cannot travel faster than the speed of light. We haven't proved that yet (we will, patience …). Just assuming it for now, what class of interval must exist between two space-time points A and B for <u>*physical travel*</u> from A to B to be possible?

(d) I'm sure you got that one: timelike! So let's turn to spacelike event pairs. Any trajectory that connects two events of this type would have to exceed the speed of light at some point, which seems weird and non-physical. In fact, spacelike-separated events are perfectly ordinary. Think of two! Think of two simple events that are separated by a spacelike interval and that you can demonstrate live to your instructor. ☺

Exercise 3.2: Proper Time

The interval I between two space-time events is very useful: all observers agree on its value! Well that's just great … but what does the interval *mean*? We need to come up with a *physical interpretation* of this new measure of the space-time "distance" between two events.

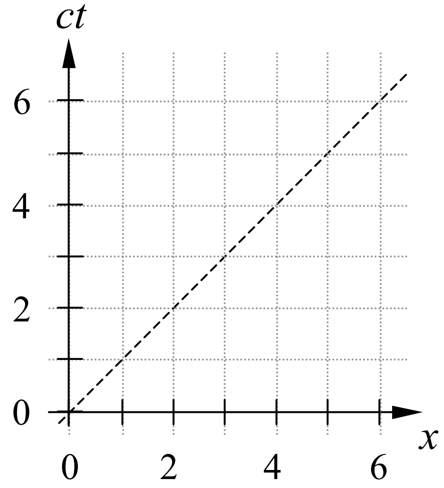
At time $t = 0$, two starships set out from their homebase on Earth, which they all agree is at $x = 0$. Both ships head directly for planet LV-426, which is infested with vicious aliens. According to Earth controllers, LV-426 is 3 lightyears away.[1] The starship *Flyby* travels at a constant speed 0.8 c; its mission is to take photos of the situation on LV-426 as it flies past the planet.
The second ship, the *Sulacco*, travels at a slower constant speed of 0.6 c. Its mission is to drop a nuclear bomb on LV-426 as soon as it arrives, thereby destroying the alien infestation.

Two events are pivotal in this story:

- Event A: the ships leave homebase
- Event B: the Sulacco destroys the aliens on LV-426

Also, there are three observers: the Earth controllers, the Sulacco's crew, and the Flyby's crew.

(a) Write down the distance Δx and time Δt between events A and B according to the Earth controllers. On the plot at right, sketch the story as it happens in the Earth controllers' frame: mark the space-time coordinates of events A and B, and draw the paths of the three observers.



More terminology: Spacetime graphs like this are often called **Minkowski diagrams** after Hermann Minkowski, one of the architects of special relativity. When the trajectories of objects are plotted on spacetime graphs, they are often called **worldlines** (e.g. the three paths you just drew). Why? Well, they're the paths the objects take through the world, I suppose … and "worldline" sounds cooler than "path". ☺

---

[1] A "**lightyear**" is a very convenient unit of distance for relativitistic problems: it is the distance light travels in 1 year. So just work the entire problem in lightyears (for distance) and years (for time), and a wonderful thing happens: in these units, **c = 1**. You can therefore *ignore all the c's*, both in the Lorentz-transformation equations and in the definition of the interval. ☺ Is there a down-side to such a clever choice of units? Yes, so beware: you lose some of your unit-checking ability.

(c) Using your description of the events in the Earth controllers' frame, calculate the interval I between events A and B. Take its square-root to get it in units of lightyears.

(d) Is this interval timelike or spacelike? Whichever you decide, does your answer make sense given that a *physical object* (the Sulacco) can travel between A and B at a physical speed $v < c$?

(e) Now calculate the distance $\Delta x'$ and time $\Delta t'$ between events A and B according to the Sulacco's crew. Have a look at your answers: who *aged more* between events A and B, the Earth controllers or the Sulacco's crew?

(f) Re-calculate the interval I and its square-root $\sqrt{I}$ between events A and B, this time using the distance and time separation you just found in the *Sulacco*'s frame. You'd better get the same answer as before!

(g) In (c) and (f) you calculated the interval $\sqrt{I}$ between A and B. Our goal is to figure out what $\sqrt{I}$ *means*. Well, have a look at (a) and (e) where you found the actual measurements made by Sulacco ($\Delta x'$, $\Delta t'$) and Earth ($\Delta x'$, Δt). Compare those 5 numbers … notice anything? If so, you're ready to complete our quest: *What is the physical significance of the interval* $\sqrt{I}$ *between two events with timelike separation?* I'll give you a clue in fill-in-the-blanks format:

> "The interval $\sqrt{I_{A-B}}$, which all inertial observers agree on, represents the ___________ between events A and B as seen by a *special observer*: one who ___________________
> ____"

The Answer: Proper Time

The interval $I = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$ between two space-time events is the same in all reference frames, but its *significance* is found by considering what it means in the frame of an inertial observer who *physically travels* between the two events. Since this observer is "at" both events, the events are at the same position in this special frame, and the expression for the interval collapses to a very simple form: $I = (c\Delta t)^2$. Thus:

> The interval $\sqrt{I}\,/\,c$ between two space-time events A and B represents "**watch-time**": the time that elapses on the wristwatch of an inertial observer who physically travels from A to B.

The interval $\sqrt{I}\,/\,c$ between two events with timelike separation — i.e. ones where a physical observer *can* actually travel between the two events — is known as **proper time**. It gets the symbol $\boldsymbol{\Delta\tau}$. We will find proper time very useful indeed, as it is a measure of space-time on which all observers agree.

Suppose an object moves from point A to point B in space at constant velocity. A stationary observer will find some time interval $\Delta t_{\text{A–B}}$ and spatial interval $\Delta x_{\text{A–B}}$ between these events. He can then calculate the interval of proper time, $\Delta\tau_{\text{A–B}}$. All other observers can do the same, and will get the same result. The significance of $\Delta\tau$ is this: **it is the time that elapses in the moving object's own frame on its trip from A to B**. Why is it called "proper"? It actually has nothing to do with the English word proper; the name comes from the French word "propre", which means "one's own". Proper time – *propre temps* – is the moving object's *own time.*

Interlude: Dropping the c's

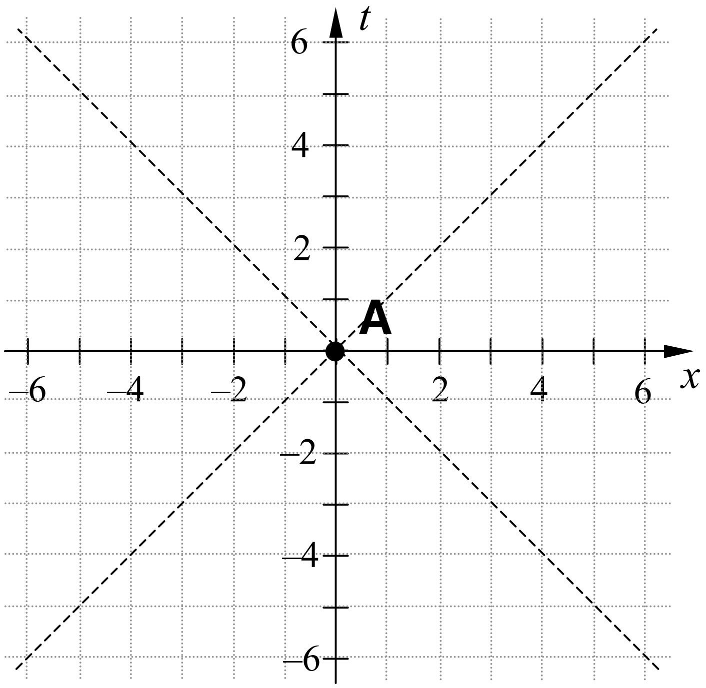
Working in years and light-years is convenient, isn't it? As explained in the footnote on page 6, in these units of time and distance units $c = 1$, so all those pesky c's go away. Let me express the idea a bit differently. The only equations we are using today are the LT equations and the definition of the interval I. In all those equations, *the c's always appear with the t's.* To illustrate, define the variable $T \equiv ct$. Then our equations are $x' = \gamma(x - \beta T)$, $T' = \gamma(T - \beta x)$, and $I \equiv T^2 - x^2 - y^2 - z^2$. No more c's *or* t's! In situations like this, it is natural to stop talking about time in seconds, and instead treat the combination *ct* as time. That has units of length, so let's just measure time in meters. (You can call it a "light-meter" if you prefer.) If we say "the kaon lived for 3 meters of time", what do we mean? Easy: if we need that value in actual seconds, we just put back the factor of $c \rightarrow$ we mean $3\text{ m} / c = 10^{-8}$ sec. In the rest of the unit we'll do exactly this: measure time t in the same units of length as we are using for x, y, and z. For our work below, we don't need any particular units of length (meters, feet, etc) so we will use the notation "**AU**" = **Arbitrary Units**. Just watch out: there is one more equation we'll need: the familiar $d = vt$. If we're measuring t and d in the same units, velocity v must be dimensionless. Our velocity equation thus becomes $\boldsymbol{d = \beta\, t}$, where $\beta \equiv v/c$ as always.

Exercise 3.3: Hyperbolae and Causality

So far, all our space-time graphs have been from the perspective of one given frame.
Let's now use our graphs to visualize how *different* observers see the same pair of events.

Consider a *collection* of observers, each moving with some different velocity v in the $\pm x$ direction relative to some "stationary" observer. All of the observers agree to synchronize their coordinate systems: they use some space-time event **A** to set their 0 of position and 0 of time. We'll drop the uninteresting coordinates y and z by considering only events at $y = 0$ and $z = 0$.

(a) Now consider some space-time event **B** that is separated from event **A** by a <u>timelike</u> interval $I = +4$ AU^2. The moving and stationary observers will see event B at different positions and times … but they all agree on the interval $I = +4$ between A and B → it's frame-invariant! Using this information, plot the *locus*[2] of space-time coordinates $(t,x)_B$ at which event B can occur in all possible moving frames.



The equation you should obtain for this locus of points describes a **hyperbola**. Calculate a few numerical values to find its shape.

Why did we plot this? → **Lorentz boosts** (transformations) from one frame to another always take you from one point to another along such a **hyperbola of constant *I***.

To really absorb the meaning of this important plot, we need to work some examples.

---

[2] Oxford definition of *locus*: "a curve or other figure formed by all the points that satisfy a particular equation or relation between coordinates."

Only two events are depicted on this plot. Let's pick a specific pair:

- Event A: Cape Canaveral launches a missile directly upwards into space.
- Event B: The missile destroys a satellite which is in danger of falling out of orbit.

All observers use event A (the missile launch) to define the origin $(t,x) = (0,0)$ of their coordinate systems, and define the $+x$ axis to points upward from Canaveral toward the satellite.[3]

(b) Let Cape Canaveral (Earth) be our S frame. In this frame, the position of the satellite is found to be $x_B = +1.5$ AU. Given that events A and B are separated by an interval $I = +4$ AU^2, calculate the time t_B at which the satellite was destroyed, and find the speed β_{missile} of the missile. Mark the coordinates $(t,x)_\text{B}$ in the Earth frame on your hyperbolic plot.[4]

(c) Consider another frame, S', in which A and B occur at the same position: $\Delta x'_{\text{B-A}} = 0$. Does such a frame exist? The hyperbolic plot tells us that it does. Since we're using event A as our origin, the frame we're looking for has event B also at the origin: $x'_\text{B} = 0$. What is the *time* t'_B of event B in this frame? To figure out t'_B, *use the invariant interval.* (FYI: this is a really useful trick that sometimes lets you solve frame-change problems faster than using the Lorentz Transformation!) Once you have the answer, mark the coordinates $(t',x')_\text{B}$ on your plot.

(d) What is this S' frame, by the way? (Clue: it's the frame of one of the objects in our story.)

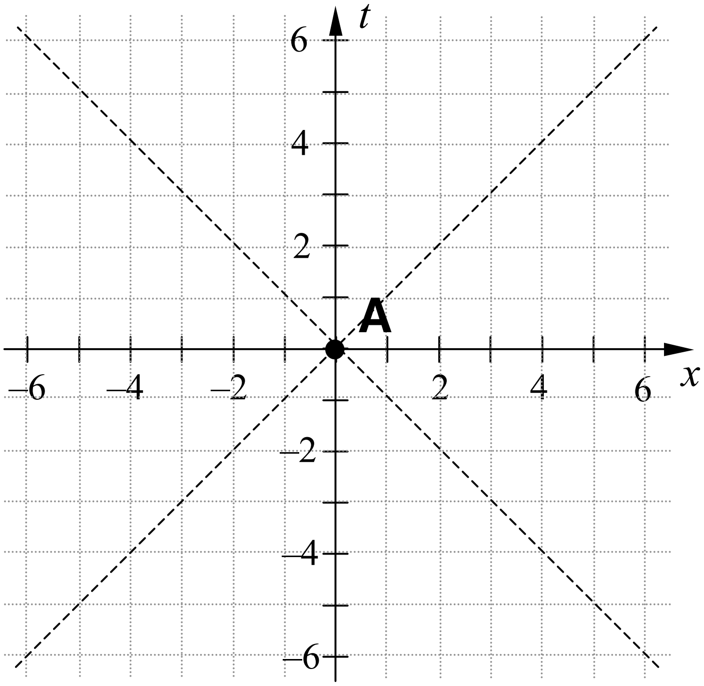
---

[3] That's the only axis of interest in this problem, and as you know, all observers must agree on the $+x$ direction if we are to use the LT equations.

[4] If you're really stuck on part (b), use the answer $(t,x)_\text{B} = (2.5, 1.5)$. Find β_{missile} from that, and keep going.

(e) We've thoroughly explored the boost hyperbolae for timelike-separated events. Now consider an event B that is separated from event A by a **spacelike interval** $I = -4$ AU^2. What is the locus of possible space-time coordinates $(t,x)_B$ for this event? (Again, we'll have all observers synchronize their origins to the event A.)

Hint: It is also a hyperbola, but it looks a bit different …



(f) Consider an inertial frame S in which some event A (x_A,t_A) occurs *before* some other event B (x_B,t_B) – i.e., where $t_A < t_B$. Here is a crazy statement:

> It is always possible to find another frame S' in which event A occurs after event B.

This statement seems ridiculous! But don't be too hasty … the statement is actually true *for a certain class of event pairs*. Have a look at your graphs on this page and the previous pages. Is this statement true when the interval between events A and B is *spacelike*, *timelike* … or both?

(g) Finally, we come to the issue of **causality**. In physics, we describe many principles where one event *causes* another. For example, lighting a fire under an icecube (event A) causes the icecube to melt (event B). But now we have special relativity, which is able to invert the time-ordering of events depending on which frame we are in! If the fire-melts-icecube casual relationship holds, it *cannot be* that some moving observer sees the icecube melt *before* the fire is lit! Fortunately, part (e) gives us an "out" and allows us to preserve the concept of causality. Make this explicit:

> If a physical theorem describes a *causal relationship* between event A and event B, what condition *must* exist on the invariant interval *I* between A and B?

(h) This finding has a profound consequence. Forces in nature, like gravity and electricity, are able to act at a distance. The moon, for example, exerts a gravitational pull on the earth's oceans which causes tides to occur. Now suppose Darth Vader's intergalactic super-ray suddenly vaporizes the moon! The destruction of the moon will affect the tides on earth … but will this change be felt *instantaneously*? We can think of the gravitational force between the moon and the earth as *carrying information*: the existence of the tides carries information about the presence of the moon. If the moon disappears, we will learn of it here on earth by a change in the tides. Given your work above, what is the **maximum speed that information can travel**, as carried by any force in nature?

Yep, it's the speed of light! I'm sure you've heard this famous physical law many times:

Nothing — not even information — can travel faster than light.

We have a century of experimentation to support this statement: we've never *found* anything that travels faster than light (FTL). But that's not a proof, and it's not the reason we are so confident about this law. It is the connection between *c* and causality that gives us the strongest evidence we have for *c* being nature's speed limit.

(i) Some day this law may be overturned, you never know … but if it does prove to be false, some very fundamental physics will have to be revised. Let's make sure we understand this by spelling out the precise sequence of arguments for *c* being nature's absolute speed limit.
Fill in the proof below, which proceeds by disproving the converse.

- **Axiom #1**: ______________________________ is the same in all inertial reference frames.
- **Axiom #2**: ______________________________ are identical in all inertial reference frames.
- Axioms 1 and 2 imply that the ______________________________ equations correctly relate the space-time measurements made by observers with different relative speeds.
- **Axiom #3**: Causal relationships exist in nature.
  In other words, it is possible to find in nature two events E_A and E_B where $\underline{E_A \text{ causes } E_B}$.
- **Hypothesis**: FTL travel **is** possible.
  In other words, there exists in nature some phenomenon X – a force, a physical object, or information of any kind – that is able to travel faster than the speed of light.
- If phenomenon X travels from space-time point A to space-time point B, then a causal relationship can be created between an event E_A at A and an event E_B at B such that $\underline{E_A \text{ causes } E_B}$. The only property X must have for this to be true is __________________________.
  (*This piece of the argument is extremely subtle, you can skip it … but any thoughts?*)
- If phenomenon X travels at faster-than-light speed from a space-time point A to a space-time point B, then the invariant interval $I_{A\text{-}B}$ is __________________.
- If $I_{A\text{-}B}$ is _________________ then we know from the ______________________ equations that it is possible to find a reference frame where ___________________________________.
- Therefore, it is impossible for ____________________________ → **hypothesis disproved**.

A Wiki-project for you: As it happens, there is a name for a hypothetical particle that travels faster than the speed of light: the **tachyon**. We have no evidence whatsoever that it exists, but that doesn't stop theorists from thinking about it. Try looking it up on Wikipedia, and check out the utterly goofy properties the tachyon would have to have to avoid the causality problem.

Challenge Zone: A few subtleties

Let's go back to the hyperbolae for *timelike*-separated events. If you were very attentive in sketching the equation in part (a), you may have noticed that it describes *two disconnected* hyperbolae. One lies in the upper part of the space-time continuum; the other is its reflection across the x axis. Can a space-time point along one of these hyperbolae be Lorentz-boosted to a point on the other hyperbola? No. If an event is on the upper hyperbola in one frame, then it will be on the upper hyperbola in all other frames.

(j) You can prove mathematically that the Lorentz Transformation *cannot* get you from one hyperbola to the other. Let's think physically first. Suppose you *could* find a frame where event B's coordinates lie on the lower hyperbola. What would the satellite-missile story look like to someone in such a frame?

(k) Now the rigorous proof: given an timelike-separated pair of events where A occurs before B ($t_A < t_B$, so B is on the upper hyperbola), try to find a boost speed β that will take you to a frame where $t_A > t_B$ → a necessary condition to get you onto the lower hyperbola. Hint: since you can always find a frame where $x_A = x_B$ for timelike-separated events, use that as your starting frame.

(l) So, you can *never* reverse the time order of events that are timelike-separated. Perfect: casual relationships are indeed safe for those types of events! ☺ But can you reverse the *positions* of timelike-separated events? Sure. The hyperbola says we can, and it's not hard to see how to accomplish it. For our satellite-missile story, consider a third frame S'' in which events A and B occur at reversed positions: $x''_A > x''_B$. (Since we're using A as our origin, this means $x''_B < 0$.) Try $\beta = 0.8$ for the speed of S'', calculate the coordinates $(t'',x'')_B$, and put them on your first hyperbola plot. *What kind of frame is this S''?* i.e., what short of spaceship would you have to be in to observe this spatial reversal of the missile launch and satellite destruction?