Physics 225

Relativity and Math Applications

Fall 2012

Unit 4

4-vectors, Tilted Axes, and the Expanding Universe

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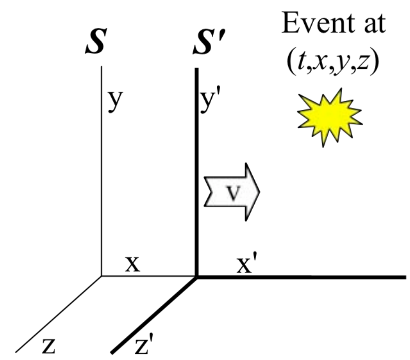
Exercise 4.1: Lorentz 4-Vectors



Here is our beloved Lorentz transformation (**LT**) yet again:

$$ct' = \gamma(ct - \beta x) \qquad x' = \gamma(x - \beta ct) \qquad y' = y \qquad z' = z$$

To get rid of all the Δ's, we use the now-familiar tactic of having observers S and S' agree to synchronize the space-time origins of their coordinate systems to the same event.[1]

As the LT shows, relativity *insists* that we treat space and time on an equal footing. A famous quote from Hermann Minkowski expresses this beautifully:

> *Henceforth, space by itself, and time by itself, are doomed to fade away into mere shadows, and only a union of the two will preserve an independent reality.*

We really should introduce some notation that removes the separation between space and time … and so we shall! We introduce the **space-time 4-vector x^μ**. It is the usual 3-vector of position (x,y,z) plus a "zeroth" component denoting time:

$$\boxed{x^\mu \equiv (ct, x, y, z)} \qquad \text{where the index } \mu = 0,1,2,3$$

The 4-vector x^μ provides a complete description of both *where* and *when* an event occurred. To put space and time on a truly equal footing, we set the time-component x^0 of our 4-vector to the combination ct instead of t → in this way, all components of x^μ have the same units: length.

With 4-vector notation in hand, it's time to introduce the elegance of **matrix notation**. A matrix is nothing more than a superb way to represent a system of linear equations. That's it. The Lorentz equations are perfect candidates. Have you learned how to do **matrix multiplication**? The rule is dead-easy: **"Row dot Column"**, where "dot" denotes the familiar dot-product operation between vectors of equal length. Instead of reading more words, just *stare* at the two versions of the Lorentz transformation below. Keep staring … it will be obvious in about 60 seconds how matrix notation and multiplication works:

Lorentz Transformation Equations:

$$\begin{aligned} ct' &= +\gamma\, ct \quad -\gamma\beta\, x \\ x' &= -\gamma\beta\, ct \quad +\gamma\, x \\ y' &= \qquad\qquad\qquad y \\ z' &= \qquad\qquad\qquad\quad z \end{aligned}$$

Matrix version of the exact same thing:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} .$$

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[1] If S and S' use some common space-time event to set their zero of position and zero of time, they will be measuring all other events *relative to that common origin*. We can get rid of the Δ's because all their measurements are Δ's!

The LT matrix usually goes by the symbol Λ. If indices are included it is $\Lambda^{\mu}{}_{\nu}$ where μ is the row index and ν is the column index. In this index notation, the matrix multiplication on the previous page is written as follows:

$$\Lambda^{\mu}{}_{\nu} \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x'^{\mu} = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu}\, x^{\nu}\,.$$

Please note: the μ and ν superscripts are ***not powers***! They are *indices*, denoting the 0^{th}, 1^{st}, 2^{nd}, and 3^{rd} elements of a 4-vector, or matrix-row, or matrix-column. (See footnote[2] for more info.)

Finally, since the y and z coordinates (x^2 and x^3) are never affected by a boost in the direction x (x^1), the most common way to see the Lorentz transformation is with only x^0 and x^1 included: →→→→→→→
The transformation is very easy to remember in this 2 × 2 form!

$$\begin{pmatrix} x'^0 \\ x'^1 \end{pmatrix} = \gamma \begin{pmatrix} 1 & -\beta \\ -\beta & 1 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \end{pmatrix}$$

The space-time 4-vector x^{μ} is not the *only* 4-vector in nature. Far from it, there are many more.

A Lorentz 4-vector is any 4-component quantity which transforms from a fixed frame to a moving frame via the Lorentz transformation matrix Λ .

We'll stick with the space-time 4-vector x^{μ} for today. But as we proceed through the course, pretty much everything you know will be recast in 4-vector form. ☺

(a) Some practice! Try a simple boost. Using either the full 4 × 4 LT matrix or the compact 2 × 2 version, boost the point $(ct, x, y, z) = (cT, 0, 0, 0)$ from the S frame into the S' frame. Self-check: your result for t' should be instantly recognizable as pure time-dilation.

(b) Self-check II: is it clear *why* that last problem gives pure time dilation?
… perhaps your response is "huh? what *else* would it give except time dilation?"
… or perhaps you're thinking "what do you mean by *pure* time dilation?"
If you are thinking anything like this, please talk! to! your! instructor! This is important.

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[2] You may be wondering why our 4-vector components are written with superscripts, $x^{\mu} = (x^0, x^1, x^2, x^3)$, instead of the more familiar subscripts, $x_{\mu} = (x_0, x_1, x_2, x_3)$. There is a very good reason, and I am very sad we don't have time to study it ☹. If you're curious, look up **contravariant vector** in Wikipedia; contravariant vectors get superscripts, covariant vectors get subscripts, and they are *not* the same type of vector. You can also wait until PHYS 435/436. ☺

(c) What is the *inverse* Lorentz transform in matrix notation? This one gets the symbol Λ^{-1}; what does it look like? (There's no need for any algebra here … remember: the inverse LT equations can be obtained *very easily* if you think *physically* for a minute …)

Matrices are *exceedingly* useful for performing **successive transformations**, i.e., one after the other. For example: if you boost a 4-vector x with the Λ matrix, then *reverse*-boost it back with the Λ^{-1} matrix, you had better get back the same 4-vector x! Here's that operation in symbolic matrix notation: $x = \Lambda^{-1}\,\Lambda\, x$. Well guess what: to perform this sequence of transformations, you can just multiply the transformation matrices together. That gives you a single matrix describing the combined operation "boost, then reverse-boost". Try it! Multiply your matrix Λ^{-1} from part (c) with the matrix Λ and see what you get. You'll soon discover if you got (c) correct.

In case matrix multiplication is unfamiliar, here's the "Row dot Column" rule once again, in more precise form: if you multiply a matrix A times a matrix B to get a matrix C, then

A[row *n*] dot-product **B**[column *m*] gives **C**[element at (row *n*, col *m*)]

(d) Ok, off you go → calculate the matrix product ($\Lambda^{-1}\,\Lambda$) !

Exercise 4.2: Graphical Boosts and Tilted Axes

We are now quite familiar with plotting events and trajectories on **Minkowski diagrams**, which are just plots of position (horizontal) versus time (vertical). In most of our examples, we have used such a plot to show the events and trajectories as seen by a single observer.

It is also possible – and very instructive – to show the observations made by *different* observers on the *same* plot. We saw one example of this last week with our "**boost hyperbolae**". If we pick one event A to set the origin (0,0) for all observers S, S', S'', …, our hyperbolae show all possible spacetime locations for another event B, as seen from the frames of all possible observers S, S', S'', …

Today we'll explore a second way of depicting multiple frames on one graph: **tilted axes**. A "stationary" observer S sees another observer S' moving at a velocity of <u>0.6 *c*</u> in the x direction. The blank plot on the next page shows the x and ct axes of the S coordinate system. Our goal is to display the ct' and x' axes of the "moving" observer S' on the *same plot*.

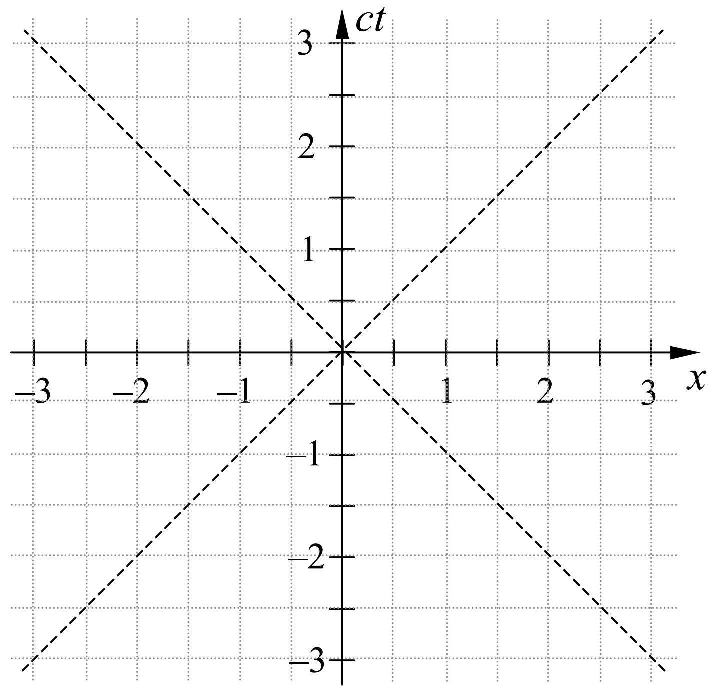
First, let's figure out where the x' axis runs. To do so, we need some points on this axis. That's easy: $(ct',x') = (0,0)$, $(0,1)$ and $(0,2)$ will do nicely. (The condition $t' = 0$ is what *defines* the x' axis.) All we need to do next is find out where those three points appear in the (ct, x) coordinate system of observer S; then we can plot them.

(a) What a great opportunity for some matrix practice! Your task is to Lorentz-boost the points $(ct',x') = (0,0)$, $(0,1)$ and $(0,2)$ from frame S' to frame S. You will need the *inverse* Lorentz transform for this. Write it down (in compact 2x2 format), then apply it to those three points to get their (ct, x) equivalents. Plug in the boost speed $\beta = 0.6$ at the very end to get numerical coordinates you can plot.

(b) Now for the figure! On the graph below,

- indicate the S frame x-axis points $(ct, x) = (0,0)$, $(0,1)$, and $(0,2)$ with solid circles;
- indicate the S' frame x'-axis points $(ct', x') = (0,0)$, $(0,1)$, and $(0,2)$ with open circles, using the S frame values you just calculated for these points;
- grab a ruler and connect the dots to draw the tilted x' axis

Quick check: do your three x'-axis points make a straight line? They must. As the LT is a *linear transformation* it will map any straight line in the S' frame onto a straight line in the S frame.



(c) Your work also reveals that the "tick marks" on the x' axis are equally spaced … but they are stretched with respect to those on the x axis. The stretch factor is $\sqrt{\frac{1+\beta^2}{1-\beta^2}}$. (You can figure that out in one line from the work you just did.) It equals 1.46 for $\beta = 0.6$. To make sure all went well so far, check that your drawing agrees with this stretch factor.

(d) On to the t' axis. We get the idea now, so we only need to calculate one point, e.g. $(ct', x') = (1,0)$. Reverse-boost that into the S frame, plot it, and you'll have the time axis of the S' frame. It has the same properties as the x' axis: tilted, and with equally spaced but stretched tick marks.

(e) Now clarify the S′ coordinate system by making a *grid*. Using thin lines, draw the grid lines $ct' = +1$ and $+2$, using the fact that lines of constant t' must be *parallel to the x' axis* ($ct' = 0$). Finally, draw in the grid lines $x' = -2, -1, +1$, and $+2$; these are parallel to the t' axis.

Major suggestion: Instead of believing me about what must be parallel to what, do this first: calculate then plot the S-frame coordinates of $(ct', x') = (1,1)$. Why? → This point is at the *intersection* of the grid lines $x' = 1$ and $ct' = 1$ (it's on both of them). By connecting (1,1) alternately to the (0,1) and (1,0) points you already have, you'll quickly see what the grid lines of a tilted coordinate system have to look like.

Time for some interpretation! Suppose the S observer is Bob, who is at home on earth. The S' observer is his twin sister Alice, who is on a spaceship traveling away from earth at 0.6c. Bob and Alice's common origin (0,0) is the place and time of her departure from earth. The t-axis thus represents Bob's worldline, while the t' axis is Alice's worldline. At the moment of Alice's departure, both of the twins are 30 years old. [3]

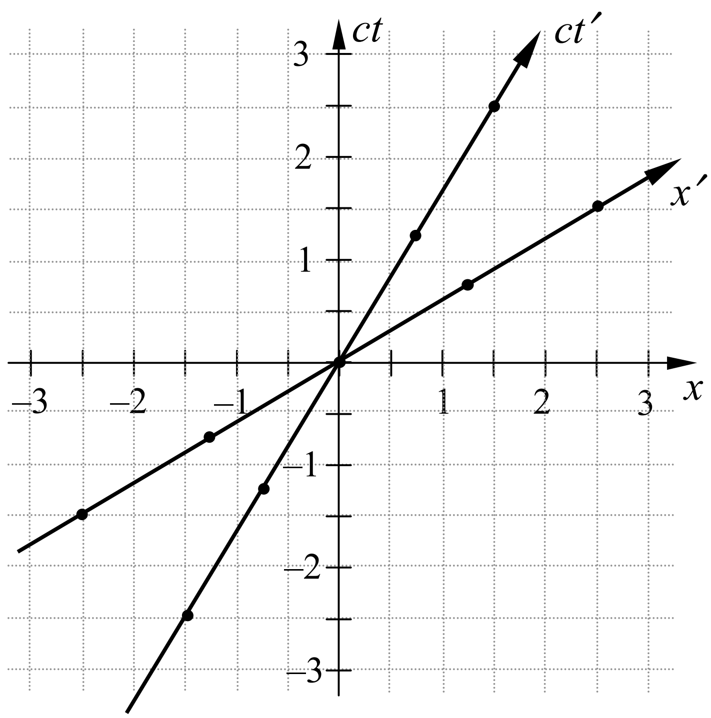
(f) For the purposes of this story, let's suppose the tick marks on your beautiful plot are spaced at intervals of 1 lightyear. Two years after Alice's departure, Bob celebrates his 32nd birthday. Here's a deceptively simple question:

"When Bob turns 32 years old, how old is Alice *according to Bob*?"

Don't use formulas, use the graph → it provides way more insight than plugging numbers into formulas. I've provided another copy below, with tilted axes included. The tick-marks on these axes are indicated with dots; it's up to you supply the grid lines. To solve this problem, you just need to think very carefully about the word "when" …

(g) … and you knew it was coming:

"When Alice turns 32 years old, how old is Bob *according to Alice*?"



Discussion: Has your brain (a) exploded, (b) imploded, or (c) relaxed into a place of meditative acceptance and oneness with the universe?

[3] Hmm, twins … could this possibly be related to the famous *Twin Paradox* you may have heard of? Why yes it is. ☺ The graph you built and the two questions on this page are key pieces in the final solution to this paradox. Patience, the final solution is on its way …

Exercise 4.3: The relativistic Doppler shift and the expanding universe

You are no doubt familiar with the **Doppler shift** of sound waves. In general, "Doppler shift" describes the change in frequency of a wave when the observer is moving relative to the wave's source and/or the medium through which the wave travels. When you hear a police car racing toward you on the street, the tones of its siren are higher-pitched; once it passes you and races away, the tones are lower-pitched. The tone you hear depends on the sound wave's frequency $f = 1/T$, where T is the time-interval between wave crests.

(a) To remind yourself how this works, explain *why* the siren sounds higher-pitched (higher frequency) when the police car is racing towards you, and lower-pitched (lower frequency) when it is racing away. Hint: a sketch often provides an easier explanation than words!

The Doppler shift for sound is called the **classical Doppler shift**. Sound is a pressure wave, and so it requires a *medium* in which to travel. The expression for the frequency-shift of sound is relatively complicated, as it involves the relative speeds of the source (the police siren), the medium (air), and the observer (you).

Light is also a wave. It is a combination of electric and magnetic fields that follow a periodic pattern in space and time, just like a sound wave. The **relativistic Doppler shift** is the change in the frequency of a light wave when the observer is moving relative to the source. The formula is actually simpler than for the classical Doppler shift as there is no medium[4] to worry about: light travels just fine through pure vacuum. Only one velocity is involved: the relative speed of the light-source and the observer. The derivation isn't difficult but it takes a bit of time. If you're zipping through this unit at a relativistic pace, you might get to it: it's on the last page.

[4] I strongly encourage you to read Chapter 2 of French's textbook. It provides an outstanding account of the exciting history behind special relativity, including the long and perplexing search for the **ether**, the hypothesized medium through which light travels. It took much thought and many challenging experiments before physicists finally accepted that the ether does not exist. Unlike sound waves and water waves, light needs no medium through which to propagate.

Here is the relativistic Doppler shift formula:

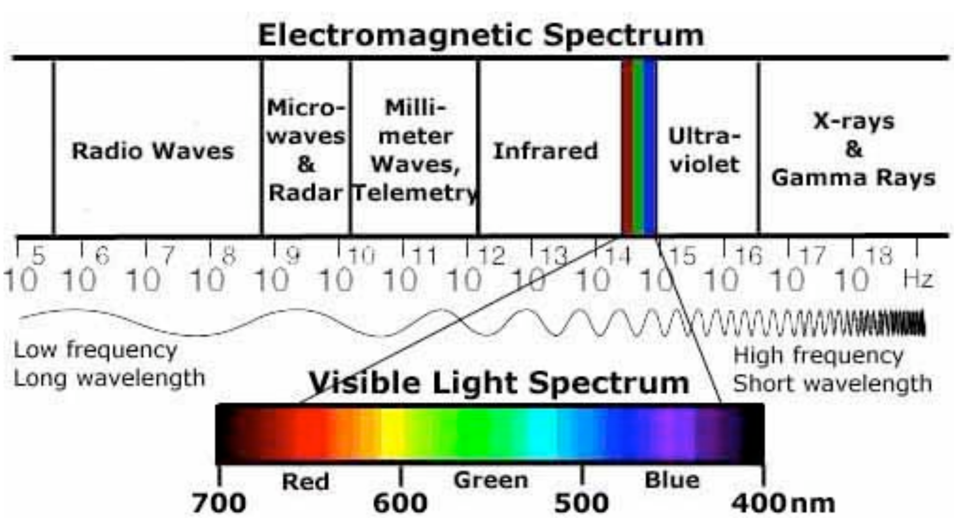
$$\frac{f'}{f} = \sqrt{\frac{1-\beta}{1+\beta}}$$

Elegant, eh?

(b) The one parameter $\beta \equiv v/c$ in this formula describes the relative speed v between the source and the receiver. But what does its sign represent? To figure it out, use your physical intuition: this formula must get the same qualitative effect (i.e. frequency shifted to higher or lower values) as the classical Doppler shift. Given that, does positive β in our boxed formula mean that the observer is *approaching* the light source or *receding* away from it?

(c) The speed of light is the same in all frames, and the speed of any wave is a strict function of its frequency and wavelength.[5] Using these facts, write down the Doppler shift formula for the *wavelength* λ' seen by the moving observer in terms of the design wavelength λ of the light source and the relative speed β between source and observer.

(d) Let's summarize the effect in terms of colors. If the light source is emitting green light and the observer is racing *away* from it, will the light be shifted toward red or toward blue?



[5] Are you stuck on the relation between the speed, wavelength, and frequency of a wave? Then turn to a familiar friend: *units!* They're not only useful for checking formulas but for building them as well. Thinking in terms of units, how do you combine a wavelength (distance) and a frequency (1/time) to get a speed (distance/time)?

Because of what you just discovered, the Doppler shift of light from receding sources is commonly called a **redshift**. Similarly, light from approaching sources is **blueshifted**. You may have heard of the term "redshift" in its most famous context: astrophysics. In 1919, Edwin Hubble collected a number of astronomical observations together, and discovered something remarkable. The measurements were made a couple of years earlier by astronomer Vesto Slipher. Slipher carefully measured the light from around 15 distant spiral nebulae, in particular looking for the distinctive "H and K" absorptions lines of ionized calcium. For a calcium ion at rest, these spectral lines are at a wavelength of 394 nm. Slipher found that the lines from the distant nebulae were shifted by various amounts. Let's see what the astronomers found …

(e) The wavelengths λ' that Slipher observed were all pretty close to the at-rest wavelength λ = 394 nm of the calcium spectral lines, which means that the speeds of the distant light sources were pretty slow: $\beta \ll 1$. Perfect opportunity to get rid of those annoying square roots in the Doppler formula! Use the now-familiar first-order Taylor approximation to obtain a simple formula for λ'/λ in the case of slow speeds[6] $\beta \ll 1$ (i.e. keep terms proportional to β and drop everything of higher order β^2, β^3, …). The simplicity of your expression will allow you to rearrange it easily and get a formula for β in terms of λ and λ'.

(f) Here are Slipher's actual measurements of the wavelength λ' of the H and K lines from five nebulae located in galaxies outside the Milky Way, along with Hubble's own measurements of the distance d to those galaxies. Recalling that the at-rest wavelength of the lines is 394 nm, calculate the speeds β of these distant galaxies relative to earth. Then, following Hubble, make a plot of the galaxies' speeds β vs their distance d from earth. What do you find ??

Galaxy in	**distance (10^8 light-years)**	**λ' (nm)**	**β**
Virgo	0.4	400	
Ursa Major	5.0	417	
Corona Borealis	7.0	426	
Boötes	13	450	
Hydra	20	479	

[6] Hubble actually used this simple formula … but not because of the approximation $\beta \ll 1$ that you made. He actually neglected to take relativistic effects into account! Good thing those wavelength shifts were indeed small!

Discussion: **the expanding universe**

The work of Slipher, Hubble and others led to **Hubble's Law**, which is a summary of the astronomical data you just analyzed: objects in the universe are *receding from us* at a rate which is *proportional to their distance from Earth*. Though the measurements and first interpretations were made before 1920, the law in this form was formulated by Hubble in 1929. It is the first observational evidence that our **universe is expanding** and remains one of the main experimental findings supporting the Big Bang theory.

The fact that spectral lines from all distant galaxies are redshifted, via the relativistic Doppler shift, immediately tells us that distant objects in the universe are moving away from us. (Or at least they *were* receding when the observed light left those galaxies several hundred million years ago!) It is not immediately obvious, however why the speed at which they are receding is proportional to their distance from us. When you first encounter this relation, it makes earth seem very special: why is it their distance from earth which matters?

(g) Think about that for a minute: what if the faraway galaxies were all receding from us at a *similar* speed – i.e., one that is independent of their distance from us? What would the motion of the galaxies look like relative to some *other* planet if this were the case?

Derivation of the Doppler Shift from Scratch

(a) Imagine a laser located at $x = 0$. It emits a beam in the $+x$ direction, with electromagnetic "wave crests" emerging every T seconds. The laser frequency is thus $f = 1/T$. Now imagine a moving observer – let's call him Captain Picard – traveling at speed $v = \beta c$ in $+x$ direction. (Positive β thus means that Picard is *receding* from the light source.) To set up the problem, first synchronize the coordinate systems: at time $t = 0$, Picard passes right by the laser, which is at $x = 0$; that's the S-frame origin, so Picard makes the event his origin too ($x' = 0$ and $t' = 0$). Great, now let's think about the laser beam. As Picard passes the laser (the origin), the beam is at one of its wave crests. In the laser's frame, the next wave crest emerges at time $t_1 = T = 1/f$. The question is: at what time t_2 and position x_2 does the second wave crest *reach Picard*? Calculate t_2 and x_2 in the (unprimed) frame of the laser.
Hint: Hopefully you're so sick of hearing *Make! A! Sketch!* that you've already been doing so while reading the problem … but let's expand a bit on this technique. This is a classic **intersecting trajectories** problem, and in such cases, the single most useful thing you can do is *Make! A! Plot!*, specifically a *space-time diagram* of the trajectories involved. If you plot the worldlines of the laser, of wave crest #1, of wave crest #2, and of Picard (all as seen from the laser's frame), this problem is a snap.

(b) Picard saw the first wave crest at $t' = 0$. When did he detect the second wave crest? The answer is t_2', which you can easily obtain from the (t_2, x_2) coordinates you just found.

(c) The time t_2' between the reception of the two wave crests determines the frequency f' of the light that Picard observes. What is f' in terms of the laser's design frequency f and the relative speed β between Picard and the laser?