

Physics 225
Relativity and Math Applications
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Unit 5
 $E = mc^2$

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Unit 5: $E = mc^2$

Relativistic kinematics and dynamics¹

We have learned a great deal about relativity: how space and time transform between frames of different speed, and how this leads to the strange phenomena of time dilation and length contraction. All of this came from Einstein's postulate that all observers see light traveling at the same speed c , no matter what the observer's motion is relative to the light source. But we have yet to encounter Einstein's most famous equation:

$$E = mc^2$$

In fact, this is probably the most famous equation in all of physics. ☺ It involves energy, and indeed, it's time to turn our attention from space and time to the dynamical concepts of **energy** and **momentum**. As we will shortly discover, we must make some changes to Newtonian mechanics, so let's remind ourselves of its basic principles:

Newton's First Law: the Law of Inertia

A particle will remain at rest or continue at a constant velocity unless acted upon by an external unbalanced force.

This tendency of an object to remain as it is is known as **inertia**. The law of inertia, which actually traces back to Galileo, can be more precisely stated as follows: "it is possible to select a *set of reference frames*, called **inertial frames**, in which a particle moves without any change in velocity if no net force acts on it". An inertial frame is literally "one in which Newton's first law holds". As we know, such frames are related to each other by constant relative speed. One more detail: do you recall how the law of inertia applies to an object of extended size?

It applies to a specific point called the object's **center of mass**, \vec{r}_{CM} . The formula

$$\vec{r}_{CM} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i}$$

is at right; the sums are over each piece i of which the object is composed.

The second pillar of mechanics is good old $\vec{F} = m\vec{a}$. That's the most famous form of Newton's second law ... but for today's work we'll find this equivalent version more useful:

Newton's Second Law

Observed from an inertial reference frame, the net force on a particle is proportional to the time rate of change of its linear momentum:

$$\vec{F} = \frac{d\vec{p}}{dt} \quad \text{where} \quad \vec{p} = m\vec{v}$$

These relations allow us to get quantitative and calculate an object's equation of motion. We will take a historical perspective today, discovering **relativistic mechanics** along with Einstein. There will be some changes to Newton's version ... but realize that what Einstein was trying to do was to *preserve* the fundamental principles of Newtonian mechanics in the face of Maxwell's theory of electromagnetism and its description of *light*.

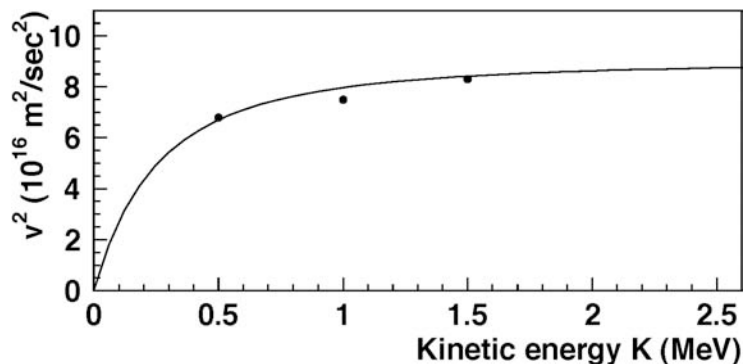
¹ Jargon busting! **Kinematics** denotes the *description* of moving objects. Examples of kinematic relations are $p = mv$ and $KE = \frac{1}{2}mv^2$; they relate energy, momentum, and speed to each other, but don't tell you how you actually get an object to a certain speed. **Dynamics** is the study of *forces* and their effect on moving objects.

Exercise 5.1: Something's wrong

For a first glimpse of how mechanics has to be modified when light or speeds close to c are involved, let's have a look at an actual experiment. This particular experiment was run in 1962 and filmed to make a documentary called *The Ultimate Speed*. The goal of the experiment was to use a linear accelerator ("linac" for short) to accelerate electrons to the highest possible speeds. As this is not Physics 212: Electromagnetism, we will skip the details of how the linac worked. All you need to know is that each time the experimental was run, the linac delivered a certain amount of energy, which we'll call K , to the electrons. This energy came from electrical sources. Just think of the linac as a giant battery, converting its internal electrical energy to the kinetic energy of the speeding electrons.

(a) Each time the accelerator was run, a different amount of total energy K was transferred to the electrons' kinetic energy. Using Newtonian mechanics, write down the formula for the final squared-speed v^2 of the electrons in terms of the energy K delivered and the electron mass m .

Now have a look at the experimental data. The plot below shows three actual measurements from 1962; the curve is a fit to the full data set. The energy K delivered by the linac is given in units of MeV (mega electron-volts) where $1 \text{ MeV} = 1.6 \times 10^{-13} \text{ J}$.



(b) Is this what you expected?? Draw directly on the plot the prediction of Newtonian mechanics. You'll need the mass of the electron, which is $9.1 \times 10^{-31} \text{ kg}$. Suggestion: calculate v^2 for the case when $K = 0.2 \text{ MeV}$, and use that to guide the Newtonian curve you draw.

(c) Something is definitely wrong! Newtonian mechanics predicts that the electrons' final speed will go up and up forever as we deliver more energy. What we see from the experiment, however, is that as the amount of kinetic energy delivered gets very large, it gets *harder and harder* to further increase the speed of the electrons. In particular, the speed v appears to approach an *asymptotic value*. It's when v approaches this value that the Newtonian expression for kinetic energy starts to break down. What is this asymptotic speed? Read it off the plot ... does it look familiar?

Our study of relativistic spacetime started with *light*, and its special property – predicted by Maxwell's equations – that its speed is a universal constant. Einstein accepted this fact, and concluded that time must therefore *not* be universal: it depends on one's frame of reference, as you have discovered yourself.

Our study of relativistic dynamics also starts with light. As you will see in Physics 212, Maxwell's equations describe what light is: it is a particular configuration of electric and magnetic fields propagating through space at speed c . Seriously, light is just *force-fields*! It hardly seems real at all ... but it is, and in fact, light behaves a lot like a beam of particles. Here's one example: Maxwell's equations tell us that light carries both momentum and energy. You can tell that light carries energy: if you sit in the sunlight for too long, your skin will turn a painful red, and it definitely takes energy to do that! Light also carries momentum: you need a very sensitive experiment to see it, but if you fire a laser beam at a thin suspended foil, the foil will be deflected → it will be pushed, just as if a beam of water instead of light had hit it.

Maxwell's equations provide the relation between the energy and momentum of light:

$$p_\gamma = E_\gamma / c \quad \text{for a light ray}$$

What is that γ subscript on the momentum and energy symbols? This use of γ has nothing to do with the γ -factor from the LT equations. It just so happens that γ is also the universal symbol for a **photon**, which is the quantum mechanical term for a packet of light. We'll use it to emphasize that the relation above only applies to light = photons.

Maxwell's energy-momentum relation is truly strange. Newtonian mechanics only has these relations for momentum and energy:

$$p = mv \quad \text{and} \quad E = \frac{1}{2}mv^2$$

They both involve mass. But a light is just force fields, it has no mass! It has no physical substance in any way we associate with normal objects.

Hmm.

In 1906, Einstein described the following thought-experiment². Consider a rectangular box of mass M and length L . It sits isolated in space, with no forces acting on it, and we view it from a frame in which it is stationary. Now imagine that a photon carrying an amount of energy E_γ is radiated from the left face of the box. It travels to the right, inside the box, at speed c until it hits the right face of the box, where it is reabsorbed.

(d) According to Maxwell and the experiments backing him up, the photon carries momentum. Momentum has to be conserved, so when the photon is emitted from the box's left face, the box acquires equal momentum in the opposite direction (to the left). Write down the speed at which the box moves to the left, in terms of E_γ , M , and c .

(e) When the photon is reabsorbed by the right face of the box, the photon's momentum is transferred back, and the box stops moving. Figure out the net distance Δx that the box moved while the photon was traveling at speed c along the length L of the box. (see footnote³)

(f) Now consider what all this looked like from an observer standing outside the box: the box suddenly started moving, then a short time later, it stopped. But no external force acted on it! What happened to Galileo's principle of inertia?? For a collection of objects, the law of inertia says that the *center of mass* of the collection should *remain at rest* if no external force is acting ... but the box's center of mass has shifted by Δx ! Einstein very much wanted to *preserve the Law of Inertia*, and so he proposed that the photon, though apparently insubstantial, carries an effective inertial mass m_γ . The box moved a distance Δx while the photon moved a distance L (see footnote⁴) in the opposite direction. What must the mass m_γ of the photon be to keep the total center-of-mass of the box+photon system stationary? Your expression should involve only the photon's energy and a constant.

² Such imagined experiments are often called "gedanken experiments", which is "thought experiment" in German.

³ You can just use $\Delta t = L/c \rightarrow$ to make this derivation simple, Einstein himself assumed a very massive box, so that its speed v would be very small compared to c . If you'd like to be more precise, you can: use $\Delta t = (L-\Delta x)/c$, and ...

⁴ If you decided to use $\Delta t = (L-\Delta x)/c$ in part (e) instead of Einstein's approximation L/c , you must use $(L-\Delta x)$ for the distance the photon moved in part (f) to be consistent. You'll get the same result as Einstein either way. ☺

You did it! You derived $E = mc^2$! ☺

Well, don't get too excited just yet. You derived $E = mc^2$ for one very special case: for light. But this first step is key: in order to preserve the Law of Inertia, Einstein postulated that something as insubstantial as light, which has no physical mass, must yet have an inertial mass proportional to its energy. What do we mean by "inertial mass"? We mean the mass that appears in Newton's laws → the mass that governs our ability to *accelerate* objects.

Einstein made the following leap: what if the **inertial mass m of any object is proportional to its total energy, via $E = mc^2$** ! Here he meant really-total: that m is proportional to the sum of *all* forms of energy *internal* to the object, including kinetic energy, heat energy, binding energy, etc.

(g) Think back to the linac experiment. If the electrons' inertial mass was not a constant but became *larger* as the electrons' kinetic energy increased, would it help to explain the data? i.e., do you think Einstein's idea will work? Explain why or why not in a few words / symbols.

(h) We could already have guessed that $KE = \frac{1}{2}mv^2$ has to break down somewhere. $KE = \frac{1}{2}mv^2$ says that as long as we keep delivering energy to a particle — converting energy from some other form to kinetic — the particle's speed will keep increasing, and so can reach arbitrarily large values. But we recently came up with a very strong argument that it must not be possible to accelerate an object to speeds faster than c . Can you briefly recap that argument?

(i) We've seen another example of Newtonian mechanics breaking down: the frame-dependence of velocity, a.k.a the **velocity addition rule**. Let $\vec{u}' = (u'_x, u'_y, u'_z)$ be the velocity of some object in frame S' , and let S' be moving relative to S with speed v in the $+x$ direction, as always. The velocity \vec{u} of the object in the S frame is given by the formulas at right. So far you've only worked with the first one, which we derived in lecture; that's all you need when the object and the S' frame are moving *parallel* to each other. You'll derive and work with the full 3D formulas on your homework this week. Just one quick question: I bet you expected that the y and z components of velocity would be unaffected by the frame change ... but they *are* affected! Can we understand why? Limiting cases to the rescue! → Consider the case $\vec{u}' = (0, u'_y, 0)$. (1) sketch this situation (2) write down the 3 formulas you get in this special case (3) stare at them do you see why $u_y \neq u'_y$?

$$u_x = \frac{u'_x + v}{1 + u'_x v / c^2}$$

$$u_{y,z} = \frac{u'_{y,z}}{\gamma(1 + u'_x v / c^2)}$$

Exercise 5.2: Relativistic mechanics

Our next step is to derive the basic relations for energy and momentum in relativity.

In Newtonian mechanics, we have $p = mv$ and the kinetic energy $E = \frac{1}{2} mv^2$. That gives $E = p^2/2m$. Our task is to derive a new relation between energy and momentum. When we're done, we'll see if we have come up with an alternative to Newtonian mechanics that successfully describes the linac experiment. That's how physics always works: you have a theory, you test it by experiment, you find that it fails, so you come up with a new theory to explain your data!

As Einstein did, we will use the following postulates:

1. First, we postulate Einstein's relation for the inertial mass of an object: $E = mc^2$ where E is an object's *total* energy.
2. Second, we preserve Newton's second law: $\vec{F} = \frac{d\vec{p}}{dt}$ where $\vec{p} = m\vec{v}$. We keep these relations as is ... the only change is that the mass m will no longer be a constant.
3. Third, we preserve the work-energy theorem: the work W done on an object by an external force is equal to the change in the object's energy, via the relation $W = \int \vec{F} \cdot d\vec{l} = \Delta E$.

Let's build our theory! We seek a new relation between energy and momentum that reflects this new idea of a variable mass.

(a) Combine postulates 1 and 2 to obtain an expression for v in terms of p , E , and c . We'll need it in the next step, to get rid of v . Also, work only with *magnitudes*: in this section, there is no need to carry the directions (vector symbols) around.

(b) Let's get forces involved. Consider a force F acting on an object in the direction x . We can then write the work-energy theorem in differential form as $dE = F dx$: this is the small change in the object's energy that a force produces when it acts on the object over a short displacement. Combine this with Newton's second law $F = dp/dt$ to get a relation for dE in terms of dp .

Your relation should involve speed v ... replace v with your result from (a) to get a relation between dE and dp that involves *only* energy, momentum, and the constant c .

Note: You'll have to *play with differentials* here, treating them like ordinary variables that you can rearrange and recombine to make new derivatives. Specifically, you'll have to rearrange dx and dt until you make the combination dx/dt ... which is v ! Is this legit?? Fear not: we'll discuss playing with differentials in great detail in a few weeks!

(c) Rearrange your result so that all the energy factors (E and dE) are on one side, and all the momentum factors (p , dp) are on the other side. Integrate both sides of the equation to get our new energy-momentum relation. But be careful of the *limits* of the integrals. This formula represents the change in momentum and energy that occurs when we apply a force for some amount of time. The upper limits of our integrals will just be the final energy E and momentum p of the object. For the lower limits, we want the conditions when we start applying the force. Use zero as the starting momentum: we will start with the object at rest⁵. But for the starting *energy* that the object has when it is at rest, use the symbol E_0 . This is a constant of integration representing the rest-energy of the object. We don't know what this E_0 is ... it might be zero, i.e. not needed. But since we don't know how our new theory works yet, we'll leave in the constant of integration to be as general as possible.

(d) You have your new energy-momentum relation! It doesn't look much like the old $E = p^2/2m$ relation for Newtonian kinetic energy, but that's ok: from postulate 1, this new relation involves the *total* energy of our particle. We don't quite know what that means yet, but onwards. What we hope to accomplish is an explanation for the linac experiment, where energy was transferred to electrons to accelerate them to speed v . Using your result from (a), rewrite your new formula to get an expression for energy E in terms of v . Your formula should only involve E , v , and constants.

⁵ There's no loss of generality here: we can always find a frame in which the object is at rest at any moment we choose.

(e) Our energy-speed formula has that constant of integration E_0 in it. What *is* this E_0 ? We never had any “rest energy” before, Newtonian mechanics only involves kinetic energy. To figure out what E_0 represents, we use a standard practice:

Make sure that the new theory agrees with the old theory in the
regime where the old theory worked!

Newtonian mechanics works just fine most of the time. Going back to the linac experiment, you see that the Newtonian kinetic energy formula $KE = \frac{1}{2} mv^2$ explained the measured data perfectly well *as long as v was small*. We must make sure we get this formula back in the regime $v \ll c$. Use the first-order Taylor approximation $(1 + \varepsilon)^n \approx 1 + n\varepsilon$ for small ε to rewrite your $E(v)$ formula from (d) as the sum of two terms.

(f) Only one of the two terms in your $v \ll c$ formula for energy E involves speed ... and it has exactly the same form as the old kinetic-energy formula! This contribution to the total energy E *is* the kinetic energy in the low-velocity limit. To match with our low-velocity Newtonian theory, what must the constant E_0 represent? Careful: can it be the same m as in postulate 1? Think for a minute before turning the page.

By matching to Newtonian kinetic energy in the low-velocity limit, the constant of integration E_0 is identified with mass times c^2 ... but this must be a *constant* mass, not the variable inertial mass we are working with in postulate 1. We label this constant mass m_0 and set $E_0 = m_0 c^2$. What it represents is **rest mass**: it is the particle's mass at zero velocity. Before the 20th century, applications of mechanics were confined to speeds much smaller than c , and so we never before saw anything *but* this constant rest mass. The rest mass is what we *normally* call the mass of an object. When you look up the mass of the Uranium-235 nucleus in a reference book, for example, what's listed is the rest mass.

We now have an interpretation for the constant-of-integration E_0 and we can replace it everywhere with the rest mass m_0 times c^2 . Everything is in hand to write down the fundamental relations of relativistic mechanics.

(g) We seek the equations of full relativistic mechanics, so go back two pages (to before you applied the $v \ll c$ approximation). Using your result from part (c), write down the new energy-momentum formula: the relation for the energy E of an object in terms of its momentum p and the constants m_0 and c . Put this relation in a box!

(h) Combine postulate 1, $E = mc^2$, with your energy-speed relation from (d) to obtain the formula for the inertial mass m in terms of velocity v (and the constants m_0 and c). Put this formula in a box too! This is the first time we have a formula for a variable mass, so let's inspect it. What values does the inertial mass m have in the limits of very large and very small velocity?

(i) Those two relations, plus the four formulas from our postulates, give a *complete* description of mechanics. That's it! ☺ However, you can combine them a bit to produce the following very convenient relations ... derive:

- $m = \gamma m_0$
- $\vec{p} = \gamma m_0 \vec{v}$
- $E = \gamma m_0 c^2$

... and from those, obtain these super-useful relations:

- $\beta = \frac{pc}{E}$
- $\gamma = \frac{E}{m_0 c^2}$

Trust me, these are worth remembering!

Exercise 5.3: Kinetic energy and rest mass

We see from our new formulas that when a particle's velocity is zero,

- its inertial mass m is equal to its rest-mass m_0
- its total energy E is $m_0 c^2$

This **rest energy** $m_0 c^2$ is a constant, underlying piece of any object's total energy. It is new: we never saw it Newtonian mechanics. One reason we've never seen this energy term before is because mechanics usually deals with *changes* in energy, and this term (usually) doesn't change.

(a) To complete the return to our starting point, recall Maxwell's energy-momentum relation for a photon (light): $p = E/c$. Given your full relativistic energy-momentum relation, what is the **rest mass of the photon**?

With that knowledge in hand, we now have a theory of mechanics that isn't confined to "ordinary" massive particles → it successfully incorporates light into its framework as well! That was one of our two original goals ... so mission 1 is accomplished.

Before we complete mission 2, let's get **kinetic energy** out of the way. (You may be wondering what has happened to it.) When you apply a force to an object, the object accelerates and acquires kinetic energy, same as always⁶. This kinetic energy is added on to the rest energy to give the total energy: $E = m_0 c^2 + KE$. Unlike in Physics 211, kinetic energy does not play a big role in relativistic mechanics; in relativity, we generally work in terms of total energy E instead. Kinetic energy is basically just a definition:

$$KE = E - m_0 c^2 .$$

KE is the portion of an object's energy that arises from the object's speed (motion); it's just the total energy with the rest energy (no motion) subtracted off.

On to our second mission: explaining the linac data with our new mechanics! But first ...

⁶ The dictionary definition of the word "kinetic" explains the concept perfectly: kinetic means "related to or resulting from motion". Kinetic energy is precisely the energy resulting from an object's motion.

Before you embark on the explanation of the linac data, please read these **three important tips** for performing calculations in the unfamiliar world of relativistic mechanics ... *and in general!*

• **Tip #1: Pick the right formula for the job.**

One of the most confusing aspects of relativistic mechanics is the *large number of kinematic variables* we use to describe a moving object. Even if we ignore direction, we have eight: v , p , E , β , γ , KE , m , and m_0 . The key to keeping this mess under control is to realize that only two of them are *independent*. The others are all calculable using the formulas we've derived. These formulas are collected on the back page for your convenience → keep it handy. Example: if you give me m_0 and v (or p and E , or v and KE , ...), you've told me everything there is to know about the particle's kinematics. So here's the tip → as much as you possibly can, use formulas that involve only the quantities you know and/or the ones you want to know. If you make a poor choice from that big formula collection on the back page, you'll introduce variables you don't need, and you'll have to do extra work to get rid of them.

• **Tip #2: AVOID VELOCITY.**

In your study of Newtonian mechanics, you learned to work with v , m , and KE as your primary kinematic variables. Your instinct is to “run to velocity” in your calculations, because v is the most familiar thing there is about a moving object, and because v is in so many 211 formulas. In Einstein's mechanics, however, v is a *horrible* quantity → it is wrapped up in gamma factors most of the time (square roots! ack!), it requires ugly formulas to transform from frame to frame (the velocity addition rules, yech), and it's not even conserved (more on that next week). Get used to relying on these variables instead: energy E , momentum p , and rest mass m_0 . Just trust me for now. It takes some getting used to, but you will quickly see that *avoiding velocity if at all possible* make relativistic calculations much easier to handle!

• **Tip #3: Dimensionless terms are awesome.**

We introduced this tip in lecture, and I'm repeating it here as it's very helpful for the calculation on the next page. Whenever you have terms added together, like $(a + b)^n$, it is good practice to “factor out the units”, then rearrange the rest into the form $(1 + \text{<dimensionless term>})^n$. This practice greatly facilitates both *unit-checking* and the study of *limiting cases*.

To illustrate: $(a + b)^n$ will often be more revealing and easier to work with if you write it as $a^n (1 + b/a)^n$. Unit-checking: b/a has to be dimensionless, and the a^n out in front has to have the correct overall units for your result. Limiting cases: if you know in advance which term (a or b) is the dominant one, or if you're trying to see what happens when one of them is dominant, that's the one to factor out → “factor out the big term”. e.g. If you know $b \gg a$, or want to test that situation, go with the form $b^n (1 + a/b)^n$. You now immediately see the approximate form of the solution (b^n), and you see how good that approximation is (fractional correction $\approx na/b$).

Just try it. You'll see that grouping things into dimensionless terms is *awesome*. ☺

The Explanation of the Linac Data

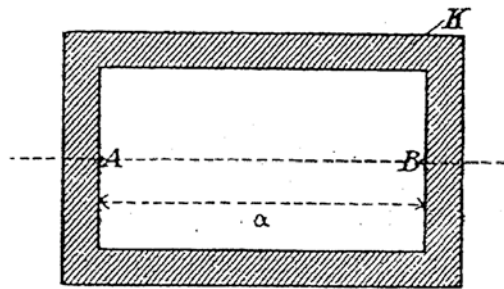
(a) This is our first real problem in relativistic mechanics, so it may be a bit weird at first. Using your various relations – all collected on the back page – calculate the final squared-speed v^2 of an electron that was given a kinetic energy K by the linac. Your formula for v^2 should involve only K , m_0 , and c .

(b) This is a truly new result and we need to get a feeling for what it implies. To do so, check the behavior of your formula in two limiting cases. First, show that it gives the non-relativistic result $v^2 = 2K / m_0$ when $v \ll c$. (Hint: use our beloved friend $(1 + \varepsilon)^n \approx 1 + n\varepsilon$ with the dimensionless quantity $\varepsilon = K / m_0 c^2$, which is very small when $v \ll c$.)

(c) Now show that your formula explains the measured behavior $v \rightarrow c$ in the limit $K \rightarrow \infty$. If you succeed, mission 2 is accomplished as well, and we have a theory we can believe.

Einstein's actual paper: On the next page is the light-box thought experiment from Einstein's 1906 paper in the journal *Annalen der Physik*. The paper is in German, with the imposing title *Das Prinzip von der Erhaltung der Schwerpunktsbewegung und die Trägheit der Energie*, and can be found at <http://www.wikilivres.info>. Using a combo of Google translator, Babel Fish, and some tweaking, you get the amusing but comprehensible pseudo-English on the next page. ☺

“The principle of conservation of gravity movement and the inertia of energy” by A. Einstein (1906)



In last year published work [1] I have shown that the Maxwell electromagnetic equations lead in connection with the principle of relativity and the principle of energy to the conclusion that the mass of a body changes by the change of its energy content, be any kind that energy change. It was shown that an energy change of the size ΔE must correspond to a change the same direction in the mass of the amount $\Delta E/V^2$, where V means the speed of light.

In this work I will now show that this proposition is the necessary and sufficient condition that the law of conservation of momentum holds true of the center of gravity (at least in first approximation) for systems, which occur in non-mechanical and electromagnetic processes. Although the simple formal considerations, which must be accomplished for the proof of this statement, are in the main already in a work by H. Poincaré [2], I for the sake of clarity will nevertheless not rely on that work.

§ 1 A special case.

K is a resting rigid hollow cylinder freely floating in the area. In A a mechanism is, in order to send a certain quantity S of radiant energy through the cavity to B . While sending that radiation dosage, a radiation pressure affects the left inner wall of the hollow cylinder K , which lends a certain to the left arranged speed to the latter. The hollow cylinder has the mass M , [628] is so that speed, as easy to prove from the laws of radiation pressure, equal to $(1/V)(S/M)$, where V means the speed of light. This speed keeps K so for a long time, until the radiation, whose physical size relative to the cavity K is very small, is absorbed in B . The duration of the movement of the hollow cylinder is (up to higher order terms) equal to α/V , where α is the distance between A and B . After absorption of the radiation in B the body K rests again. In the considered radiation process K has shifted to the left by the distance $\delta = (1/V) (S/M) (\alpha/V)$.

In the cavity K a for simplicity massless body k is present together with a (also massless) mechanism, for the body k , which is initially in B may find, between B and A to move back and forth. After the amount of radiation S is included in B , this amount of energy going to k transmitted, and then k to A moves. Finally, the energy quantity of S to A again from the hollow cylinder K was added and k again after B moved back. The whole system has now gone through a complete cycle, which one can imagine arbitrarily often repeated.

Assuming that the transport body k is even massless when the amount of energy S has taken, the one must also assume that the return motion of the energy quantity of S is not connected with a position change of the hollow cylinder K . The success of the whole cyclic process is therefore described only in a shift δ of the whole system to the left, which shift can be made arbitrarily large by repeating the cycle. We thus get the result that a system originally at rest, without external forces acting upon it, can change [629] its center of gravity as much as desired, and without that the system would suffer a continuing change.

It is clear that the obtained result contains no internal contradiction, but it contradicts the fundamental laws of mechanics, according to which an originally resting body, which other bodies do not affect, no translational motion can be driven.

Assuming, however, that the inertia E/V^2 comes to any energy E , then the contradiction with the elements of mechanics disappears. Following this assumption, the transport body, while it transports the amount of energy S from B to A , possesses the mass of S/V^2 ; and because the focus of the whole system must rest during this operation, after the main block, the hollow cylinder K during the same throughout a shift S' to the right of the size $\delta' = \alpha (S/V^2) (1/M)$.

A comparison with the result found above shows that (at least in first approximation) $\delta = \delta'$; it is thus that the position of the system before and after the cycle is the same. Thus the contradiction with the elements of mechanics is eliminated.

Summary of Relativistic Mechanics

The formulas for relativistic mechanics are derived from the following four postulates:

$$\textcircled{1} \vec{F} = \frac{d\vec{p}}{dt} \quad \textcircled{2} \vec{p} = m_{\text{inertial}} \vec{v} \quad \textcircled{3} W = \int \vec{F} \cdot d\vec{l} = \Delta E \quad \textcircled{4} E = m_{\text{inertial}} c^2$$

The first three formulae are preserved from Newtonian mechanics but with a twist: the mass in $p = mv$ is now interpreted as a particle’s *speed-dependent* inertial mass. To emphasize this reinterpretation, I’ve explicitly labeled it m_{inertial} . The fourth postulate is Einstein’s and was motivated by trying to reconcile Maxwell’s photon – light – with the law of inertia. From these hypotheses, you yourself derived our new master relation between an object’s total energy E , momentum p , and rest mass m_0 :

$$E = \sqrt{(pc)^2 + (m_0c^2)^2}$$

I’ve put this particular formula in a box for two reasons: (1) it is the one you are most likely to *avoid* because it is unfamiliar and has a nasty square root, and (2) it is generally the *most* useful of all the relations on this page! So don’t avoid it, run to it with open arms! ☺ The following relations are useful combinations of those above:

$$m_{\text{inertial}} = \gamma m_0 \quad \vec{p} = \gamma m_0 \vec{v} \quad E = \gamma m_0 c^2 \quad \beta = \frac{pc}{E} \quad \gamma = \frac{E}{m_0 c^2}$$

where $\gamma \equiv 1 / \sqrt{1 - (v/c)^2}$ and $\beta \equiv v/c$ as always. Finally, kinetic energy is defined to be the difference between an object’s total energy and its rest energy:

$$KE \equiv E - m_0 c^2$$

With that, we’ve successfully patched Newtonian mechanics: (a) to apply to speeds close to c , while never exceeding c ; and (b) to incorporate the photon and its strange energy-momentum relation $E_\gamma = p_\gamma c \rightarrow$ in our new formalism, it fits in perfectly as a particle with zero rest mass.

For convenience, here also is the complete, 3D set of formulae for transforming velocities between stationary and moving frames (often called the “velocity addition rules”):

$$u_x = \frac{u'_x + v}{1 + u'_x v / c^2} \quad u_{y,z} = \frac{u'_{y,z}}{\gamma(1 + u'_x v / c^2)}$$

These formulae are separate from the other relations: they’re the only ones that involve different frames, and they come from the Lorentz Transformation, not from our new mechanics. We will soon learn how to boost the other kinematic quantities too.