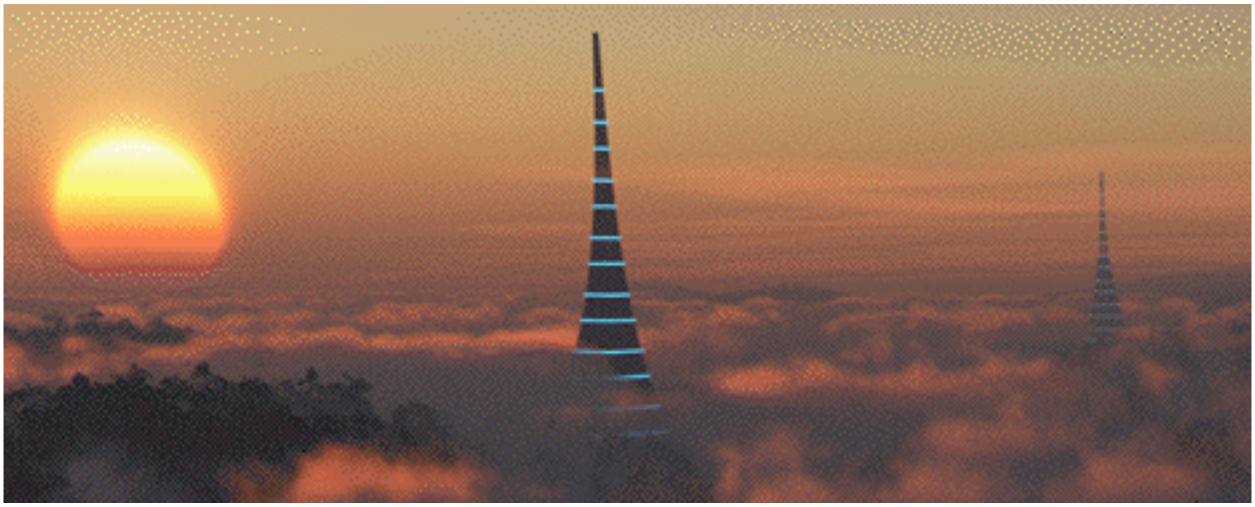

Physics 225

Relativity and Math Applications

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Unit 7

The 4-vectors of Dynamics

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Unit 7: The 4-vectors of Dynamics

Recap of 4-vectors from Unit 4

A Lorentz <u>4-vector</u> is <u>any</u> 4-component quantity which transforms from a fixed frame to a moving frame via the <u>Lorentz transformation matrix</u> $\Lambda^{\mu}{}_{\nu}$.

In unit 4, we introduced and worked with the Lorentz transformation (LT) matrix Λ and the space-time 4-vector x^μ. Here they are for reference:

$$\Lambda^{\mu}{}_{\nu} \equiv \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$x^\mu \equiv (ct, x, y, z)$$ where the index $\mu = 0,1,2,3$

Remember how to use these objects? To boost a 4-vector into another frame, you write it in column form (making it a 1-column matrix), then hit it with the matrix Λ. Matrix multiplication is summarized by the simple phrase "**row-dot-column**". The result is the 4-vector in the new frame, also in column form. All is explained by staring at the following:

Lorentz Transformation Equations:

$$\begin{aligned} ct' &= +\gamma\, ct \quad -\gamma\beta\, x \\ x' &= -\gamma\beta\, ct \quad +\gamma\, x \\ y' &= y \\ z' &= z \end{aligned}$$

Matrix version of the exact same thing:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}.$$

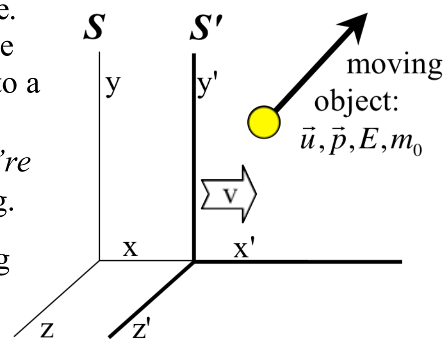
The 4-vectors of dynamics

In dynamics, the main quantities of interest are energy, momentum, velocity, and mass.
How do we transform those quantities between frames? → We figure out their <u>4-vector form</u>.
Our strategy for building new 4-vectors is to <u>combine a known Lorentz 4-vector with a known Lorentz scalar</u> in some way. A **Lorentz scalar**, a.k.a. a **Lorentz invariant**, is a quantity that is *unchanged* by a Lorentz boost. When you combine a Lorentz 4-vector (boosts with the Λ matrix) with a Lorentz scalar (doesn't boost at all), the result is a new 4-vector (also boosts with the Λ matrix). Here are the two main 4-vectors of dynamics: the **4-velocity** η^μ and **4-momentum** p^μ of a moving object:

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma_u\left(c, u_x, u_y, u_z\right)$$

$$p^\mu \equiv m_0 \eta^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

Notice how momentum and energy *appear together* in the 4-momentum → they will *mix* under Lorentz boosts, just like position and time do! Now to explain their derivation.



1. Two Speeds: Dynamics is the physics of moving objects, i.e. objects with speed. Our new 4-vectors will allow us to boost the properties (E, p, m_0, etc) of moving objects from a fixed frame to a moving frame. So watch out → there are two speeds involved when boosting dynamical quantities: the speed of the *object we're observing* and the speed of the *S′ frame* to which we're boosting.

- We reserve the letter ***v*** to denote the speed of the moving frame. That's the boost speed: the one that will appear in the γ and β factors of the LT.
- We use ***u*** and ***u′*** for the speed of the object we're describing (as measured in the S and S′ frames respectively).

This is precisely the same convention that we used in our velocity addition formulae.

2. The Differential of Proper Time: To construct the 4-velocity η^μ describing an object's speed, we must take some derivative of position with respect to time. (That's what velocity is!). We have a 4-vector that includes position: x^μ. To build 4-velocity, we will take the derivative of the 4-vector x^μ with respect to a Lorentz-invariant version of time → the result will be a *new* 4-vector, i.e. a new 4-component object that boosts with the *same* LT equations as x^μ.
The Lorentz scalar we will use is the **differential of proper time, $d\tau$**. Remember proper time?[1] The proper time interval $\Delta\tau \equiv \sqrt{I}/c$ between two events is the "watch time" measured by an observer who is physically at both events. So *whose watch* are we talking about in this situation? → *The object's*. Why? Because it's the only relevant one in the problem! We're building a 4-vector version of the object's speed $\vec{u} = d\vec{r}/dt$, so it only makes sense to use the moving object's own proper time $d\tau$. Now, to relate proper time, $d\tau$, and frame-dependent time, dt …

Consider some observer who is watching the object move and determines its speed to be $\vec{u} = (u_x, u_y, u_z)$. Take two points on the object's trajectory that are separated in time by dt. The separation between those points in space is necessarily $(dx, dy, dz) = (u_x\,dt, u_y\,dt, u_z\,dt)$. The *proper time* interval between the points is therefore:

$$d\tau = \frac{1}{c}\sqrt{(c\,dt)^2 - (dx^2 + dy^2 + dz^2)} = \frac{dt}{c}\sqrt{c^2 - (u_x^2 + u_y^2 + u_z^2)} = dt\sqrt{1 - \frac{u^2}{c^2}} = \frac{dt}{\gamma_u}$$

This Lorentz-invariant version of time is perfect for the study of moving objects! Any observer S can form $d\tau$ by taking his clock time, dt, and multiplying it by $\sqrt{1 - u^2/c^2}$, where u is his own measurement u of the object's speed. Another observer S' can do the same with her time and speed measurements, forming the combination $dt'\sqrt{1 - u'^2/c^2}$. As you proved yourself, proper time is the same in all frames, so the differentials $d\tau$ they form are *exactly the same!*

[1] … or more correctly the French "propre temps", meaning "one's own time"

Summary:

For a moving object of speed u, $\boxed{d\tau = \frac{dt}{\gamma_u}}$ where $\gamma_u = \frac{1}{\sqrt{1-(u/c)^2}}$.

$d\tau$ is a time interval on the moving object's wristwatch, dt is the corresponding time that elapses on some observer's watch, and u is the speed of the object according to that same observer. Please note carefully the subscript u on the time-dilation factor $\gamma_u \rightarrow$ it indicates that the object's speed u must be used in the gamma factor, *not* the letter-v speed of some *other* observer S' that hasn't even entered the picture yet. (So far, we're just describing one observer S watching a moving object; as emphasized above, we reserve the letter v for a boost to some other frame S′).

3. Those new 4-vectors: Now we can form our new 4-vectors. Take the derivative of the Lorentz 4-vector x^μ with respect to the Lorentz scalar $d\tau$ to get **4-velocity**:

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma_u \frac{dx^\mu}{dt} = \gamma_u \left(\frac{d(ct)}{dt}, \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) = \gamma_u \left(c, u_x, u_y, u_z \right)$$

That's a weird version of velocity, with that gamma factor in there and all, but at least we know how it transforms from frame to frame. Next, we combine our object's 4-velocity with its *rest mass* m_0. Rest mass is definitely a scalar: all observers must agree on an object's rest mass since its very *definition* refers to a particular frame = the object's own rest frame. The product of (scalar mass) times (velocity 4-vector) is — you guessed it — **4-momentum**:

$$p^\mu \equiv m_0 \eta^\mu = m_0 \gamma_u (c, u_x, u_y, u_z) = \left(E/c, p_x, p_y, p_z \right)$$

Derivations can't be fully absorbed in one sitting. Let's work with these new 4-vectors first … then come back to these derivations in a couple of days, and their significance will be much more clear. Trust me. ☺

Section 7.1: Exploring our new 4-vectors with the Scalar Product

The invariant interval from which proper time arises is $I = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$.
Rewriting that in 4-vector notation, we get $I = (\Delta x^0)^2 - (\Delta x^1)^2 - (\Delta x^2)^2 - (\Delta x^3)^2$.
As you proved in homework, the Lorentz invariance of this quantity comes directly from the fact that Δx^μ transforms via the LT matrix Λ. Well, every other 4-vector *also* transforms via the Λ matrix! We formalize the invariance of our magic combination of components by defining the **scalar product $a^\mu \cdot b^\mu$** as follows:

for any 4-vectors a^μ and b^μ, $\boxed{a^\mu \cdot b^\mu \equiv a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3}$

This is the <u>4-vector version of the dot product</u>. The dot product $\vec{a} \cdot \vec{b}$ of two 3-vectors is invariant under rotations; <u>the scalar product of two 4-vectors is invariant under both rotations and boosts</u>. Of particular interest is the dot product of a vector with itself: as you know, $\vec{a} \cdot \vec{a}$ is what we call the **magnitude** (squared) of the vector $\vec{a}$. As a magnitude should, it makes no reference to any frame. The scalar product of a 4-vector with itself, $a^\mu \cdot a^\mu$, is also considered the magnitude (squared) of that 4-vector.

(a) So every 4-vector a^μ has a frame invariant quantity associated with it: its magnitude $\sqrt{a^\mu \cdot a^\mu}$. Relativity says that physics *itself* is frame invariant, so invariant quantities must have physical significance! Let's check it out for our new 4-vectors. What is the invariant $\sqrt{\eta^\mu \cdot \eta^\mu}$ associated with an object's 4-velocity? Work it out explicitly using the four components of $\eta^\mu = \gamma_u \left(c, u_x, u_y, u_z\right)$. It is a *very* recognizable invariant! Hints: don't forget to substitute in the form of γ_u; once you do, you can simplify $\sqrt{\eta^\mu \cdot \eta^\mu}$ down to a *single letter*.

(b) Now calculate the invariant $\sqrt{p^\mu \cdot p^\mu}$ associated with an object's 4-momentum $p^\mu = \left(E/c, p_x, p_y, p_z\right)$. What property of the object does it represent?

Section 7.2: Boosted photons

Our wonderful new 4-vectors η^μ and p^μ at last allow us to transform speed, momentum, and energy from a stationary frame to a moving frame! Here is some **slick notation** you may find convenient: it is common to <u>write the last three components as a 3-vector</u>, e.g., $\eta^\mu = \gamma_u \left(c, \vec{u}\right)$ and $p^\mu = \left(E/c, \vec{p}\right)$. The scalar product between any two 4-vectors $a^\mu = (a^0, \vec{a})$ and $b^\mu = (b^0, \vec{b})$ can then be written very compactly:

$$a^\mu \cdot b^\mu \equiv a^0 b^0 - \vec{a} \cdot \vec{b}$$

(a) First, let's make absolutely sure we understand this 4-vector business. Consider a particle with energy E and momentum components p_x, p_y, p_z as seen from a stationary frame S. A second frame S' is moving with speed $v = \beta c$ in the $+x$ direction, as always. By applying the Λ matrix to p^μ, write down the LT equations that give E', p_x', p_y', and p_z' of the particle in the S' frame.

So we finally know how to boost energy and momentum, awesome! Now, we practice. Let's start with our old friend the **relativistic Doppler shift**. Forgot the formula? No problem! 4-vectors will give it to you in a second. You just need one more piece of information.

Einstein discovered many things, not just relativity. When he was awarded the Nobel Prize in 1921, it was actually not for relativity, but for a phenomenon called the **photoelectric effect**. He described "quanta" of light (photons) as behaving very much like ordinary massive particles, each with an energy proportional to its frequency. His relation for the energy of a photon is in the box:

$$\boxed{E_\gamma = hf}$$

Here f is the photon's frequency and $h = 6.63 \times 10^{-34}$ J·s is **Planck's constant**. We are already accustomed to treating photons like particles: they behave like particles of zero rest mass, which gives us the relation $E_\gamma = p_\gamma c$. Einstein's $E_\gamma = hf$ adds a new formula to our photon collection.

(b) Usual setup: we have a "stationary" observer S and a second observer S' moving with velocity $+\beta c\,\hat{x}$. Now suppose a laser in the S frame emits a photon of frequency f in the $+x$ direction. Write down the 4-momentum p^μ of this photon, expressing your answer entirely in terms of the photon's frequency f and physical constants. Once you've got it, read the footnote[2].

(c) Next, boost the photon's 4-momentum into the S' frame to obtain $p^{\mu\prime}$.

(d) What frequency f' does the observer in S' detect? Your result should give you the Doppler shift formula for f'/f. *Much* easier than our previous derivation! ☺

[2] Do your p^μ components have any *factors in common*? (They should …) If so, **factor them out**! For example, the 4-vector ($27c$, 0, $9c$, $9c$) can be written *much* more elegantly as $9c$ (3,0,1,1). Is this legitimate? Good question, and yes it is: scalar multiplication works the same for 4-vectors as for 3-vectors, e.g. $3\vec{v} = 3(v_x, v_y, v_z) = (3v_x, 3v_y, 3v_z)$

(e) On to the **searchlight effect** that you calculated in homework. Suppose an observer in S' sends off a photon of frequency f' in the $+y'$ direction (i.e., perpendicular to the relative motion of S and S'). According to an observer in S, however, the photon's path is *not* perpendicular to the path of S', it is tilted. *In which direction?* Calculate the angle θ that the photon makes with the x-axis in the S frame, then check your answer: is it greater than 90°, less than 90°, or can it have any value? (The correct answer reveals why this tilt is called the "searchlight effect".)

Finally, guess what: the elegant Doppler shift formula $f'/f = \sqrt{1-\beta}/\sqrt{1+\beta}$ is not the end of the Doppler shift story. That formula is only true when the observer S' and the photon are moving *parallel* to each other. In part (e), you worked with a photon that *wasn't* moving parallel to $S' \rightarrow$ easy to do with 4-vectors! So let's derive the **general Doppler shift** formula for a photon traveling in *any* direction relative to the two frames of interest, S and S'. As ever and always, S' is moving with velocity $+\beta c\,\hat{x}$ relative to the S frame.

The calculation is on the next page; if you find yourself totally stuck, you can flip back here and look at the Footnote of Helpfulness[3] below.

[3] First, write down the 4-vector p^μ for the photon in the S frame, expressing it entirely in terms of its frequency f and the angle θ. Without loss of generality, you can take the photon's path to be in the xy plane, i.e. set p_z to zero. Then boost the photon to the S' frame, and get the Doppler-shifted frequency f' from the energy (0^{th}) component of p'^μ.

(f) In the S frame, a photon has frequency f and its momentum makes an angle θ with the x-axis. Calculate the photon's frequency f' in the S' frame in terms of f, θ, and β.

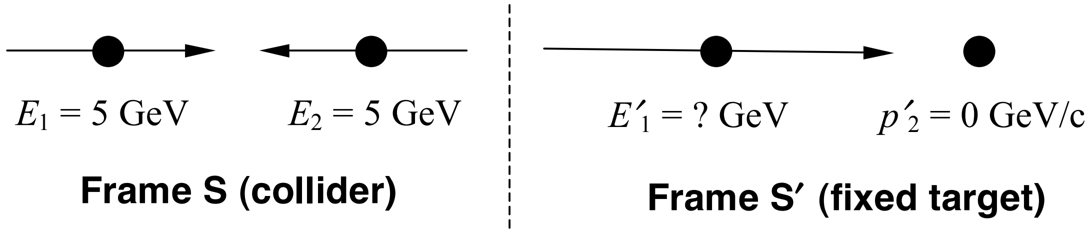
Section 7.3: Boosts in particle physics

Particle physics experiments collide particles at high energies to explore their structure and to learn about nature in general. In **fixed-target experiments**, a high-energy beam is directed at a target that is at rest in the laboratory frame; in **collider experiments**, two beams crash into each other. As it happens, *much* higher energy collisions can be produced in collider mode. To see the enormous gain, consider a collider where two beams of protons (m_p = 1 GeV/c^2) are each accelerated to an energy of E = 5 GeV and fired headlong into each other. One can perform the exact same experiment in fixed-target mode by colliding a proton beam of energy E' into a proton target at rest. The question is: how high would E' have to be?

(a) Quick warmup: in the collider setup, what is the speed β of the 5 GeV protons? (We'll need it shortly.) No need for calculators, leave your answers in irreducible form (e.g. $\sqrt{3}$ or $5/17$).

(b) We want to find the beam energy E' we would need in fixed-target mode to replicate the 5-on-5 GeV collider experiment. The difference between the two experiments is nothing more than a Lorentz transformation between frames – "laboratories" – moving at different speeds. This will make no sense unless you Draw! A! Sketch! Show how the experiment looks in the two frames, which we'll call S = Collider frame and S' = Fixed-Target frame. As always, *label* your sketch with *what you know* and *what you want to know*, and provide well-chosen *symbols* for everything. Don't do any calculations here, please, just sketch, label, then flip the page.

rest mass m_0 = 1 GeV/c^2 for all particles



Your sketch should look pretty close to the one above. If you see something you missed, make a mental note for next time → a good sketch with a clear summary of what's-known and what's-wanted and a good choice of symbols for those quantities is more than half the battle.

(c) Our goal is to find E_1', the fixed-target beam energy. To find it, we will boost the 4-momentum of proton 1 from frame S to frame S′. First we need the boost speed: what is the relative speed $\beta = v/c$ between frame S and frame S′? Think visually: to turn the collider experiment into the fixed-target experiment, you = the lab will have to *move along with one of the protons*. Be careful: both the speed and the *direction* of the boost are important in this problem, so don't forget to <u>*draw the boost vector β on the sketch*</u>.

(d) Now you have everything you need to solve the problem. What beam energy E_1' is needed in fixed-target mode to reproduce the 5 GeV + 5 GeV collider experiment?

(e) In scattering experiments like this, an important quantity is the **<u>total 4-momentum</u>** p^{μ}_{TOT} **of the system**. It is obtained by adding the 4-momenta of all particles involved, where the addition is done component-by-component (as with familiar 3-vector addition). In our example, we have only two particles: the beam proton and the target proton. In case something went wrong in your previous calculation, I'll give you the 4-momentum of the beam in the fixed target frame: $p_1'^{\mu} = (49,\ 10\sqrt{24}, 0, 0)$ in GeV/c.

Write down the other three 4-momenta — p_1^{μ}, p_2^{μ}, and $p_2'^{\mu}$ — and sum them to get $p^{\mu}_{\text{TOT}} = p_1^{\mu} + p_2^{\mu}$ in the collider frame and $p'^{\mu}_{\text{TOT}} = p_1'^{\mu} + p_2'^{\mu}$ in the fixed target frame.

(f) Important fact: **the sum of 4-vectors is also a 4-vector**.[4] Well, with any 4-vector a^{μ} comes an invariant: its magnitude $\sqrt{a^{\mu} \cdot a^{\mu}}$, so what is the invariant associated with p^{μ}_{TOT}? Calculate the magnitude of p^{μ}_{TOT} in both frames. (You'd better get the same answer!)

Discussion: The collider frame is special here: in that frame, the total 3-momentum $\vec{p}_{\text{TOT}}$ of the system is zero, so the total 4-momentum only has an energy component: $p^{\mu}_{\text{TOT}} = (E_{\text{TOT}}, 0, 0, 0)$ with $E_{\text{TOT}} = 10$ GeV. This is called the **center-of-mass (CM) frame** of the system (or less commonly, the center-of-momentum frame). In the CM frame, the invariant $\sqrt{p^{\mu}_{\text{TOT}} \cdot p^{\mu}_{\text{TOT}}}$ is simply E_{TOT}. In part (f), you found that $\sqrt{p'^{\mu}_{\text{TOT}} \cdot p'^{\mu}_{\text{TOT}}}$ in the fixed target frame is *also* 10 GeV (it has to be, it's an invariant). The quantity $\sqrt{p^{\mu}_{\text{TOT}} \cdot p^{\mu}_{\text{TOT}}}$ is a very important invariant in relativity and is called the **center-of-mass energy** of a system. (That's what it is: the total energy in the CM frame.) As physics is frame-independent, two experiments with the same CM energy are completely equivalent. To understand the energy scale of a scattering experiment, it's the CM energy that matters; the individual energies of beam and target are just frame-dependent details.

[4] This follows immediately from the *linear* nature of the Lorentz transformation. But if you don't believe it, try it out! Boost the sum of any two 4-vectors and compare it to the sum of the boosted vectors. You'll find that they are identical because the algebra you perform in the two calculations is exactly equivalent.

Section 7.4: Velocity addition from 4-vectors

Ready for another piece of 4-vector elegance? Remember the general formulas for relativistic velocity addition? Boosting a velocity $\vec{u} = (u_x, u_y, u_z)$ from a stationary frame S into a frame S' moving with speed $\vec{v} = v\hat{x}$ gives you the relations at right. These are pretty nasty. The chance you're going to remember them a year from now (or on, say, a Ph.D. qualifying exam ☺) is pretty much zero. Can we rederive them using the elegant 4-velocity $\eta^\mu \equiv dx^\mu / d\tau$? (That's easy to remember! Seriously, what else could it be but $dx^\mu / d\tau$? ☺)

$$u'_x = \frac{u_x - v}{1 - u_x v / c^2}$$

$$u'_{y,z} = \frac{u_{y,z}}{\gamma(1 - u_x v / c^2)}$$

(a) Let's do it! Boost the 4-velocity $\eta^\mu = \gamma_u (c, \vec{u})$ to find η'^μ in the S' frame, then interpret your result to obtain the complete velocity addition formulas above.
<u>Major hint</u>: You will need $\gamma_{u'}$ in the S' frame to interpret η'^μ and extract $\vec{u}'$. Oh no! Do you need to figure out the full magnitude u' of the boosted speed to get $\gamma_{u'}$? (Eek!) Fortunately not: you can grab $\gamma_{u'}$ directly from *one of the components* of η' …

If you managed that, you are officially a **Grand Master of Four Vectors**.