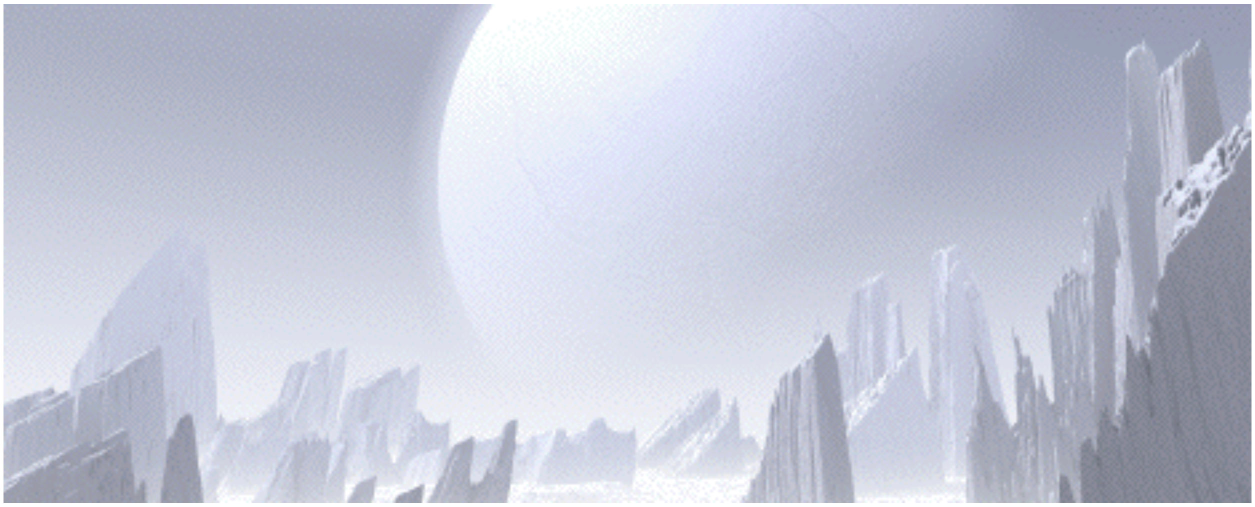

Physics 225

Relativity and Math Applications

Fall 2012

Unit 8

Differential Calculus: now available in 3D

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Unit 8: Differential Calculus … now in 3D!

We're done with Special Relativity and we're moving on. Our overarching goal for the second half of the course is to study **electromagnetism** in the way it was meant to be studied: using **3D differential calculus**. Along the way, we will encounter many mathematical tools, introducing them always in a physical context. Our destination is to see **Maxwell's equations** in their most powerful form — **differential form** — and learn how to use them. When we reach our goal, I promise you: you will be so familiar with differential calculus that you will be able to read these beautiful equations like English sentences. ☺

Exercise 8.1: Fields

Death to jargon!

"Jargon" means "a technical term that is used to conveniently represent an idea in some specific area of expertise". To experts in that area, jargon is great: everyone knows what it means, so you can use one word of jargon to convey a complex idea that would otherwise take a whole sentence to explain. But *until* you become an expert, *jargon is miserable* and can be *very dangerous!* When an expert-sounding person permeates their speech with jargon whose meaning is unclear to you, you must ask. "What does word X mean?" All too frequently, people (not just students!) are intimidated by jargon. People are generally scared of sounding ignorant, so they don't ask what word X actually means because they think they should know. Pretty soon, they've heard word X enough to figure out *roughly* what it means, so they start constructing sentences using word X (basically by copying what they've heard other people say). Before long, this becomes a *terrible* habit. You can find yourself far down the road of your studies, routinely using terms whose meaning was *never* clear to you in the first place. Building your knowledge on jargon you don't really understand is like building a house on sand: it gets shakier and shakier as it grows, and that is one very uncomfortable house to live in.

You know what? Most of the time, the fancy-sounding jargon that makes folks sound so expert and seems so intimidating isn't complicated at all. You just have to ask … and ask early!

Here is our first piece of Jargon Assassination for today: **fields**. Electric fields, force fields, gravitational fields, magnetic fields … Fields start popping up all over the place in 212 and they sound like something very deep and profound. We'll they're not.

A **field** is a physical quantity that has a value for each point (x,y,z,t) in space-time.

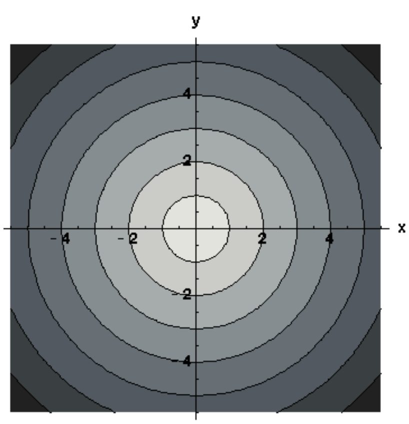
That's it: a field is a **multidimensional function**. It is a *map* of a physical quantity versus position and time. Now a field doesn't have to depend *explicitly* on all four coordinates of spacetime. For example, we very often work with fields that have no time-dependence, like the electric field $\vec{E}(x,y,z)$ of a static charge distribution. We may also talk about fields in a restricted space of only two dimensions, e.g. the electric potential $V(x,y)$ in the narrow gap between two capacitor plates. Nevertheless, the term "field" is usually reserved for quantities that extend throughout spacetime, even if they are zero or irrelevant in large parts of it. An example of a multi-D function that would *not* typically be called a field is the surface charge density $\sigma(x,y)$ on a capacitor plate. That quantity is only *defined* on the plate. In contrast, the electric potential $V(x,y)$ between two plates is still called a field because electric potential is certainly *defined* throughout all of space and time — we are simply *ignoring* it outside the 2D gap of interest when we write $V(x,y)$. More jargon:

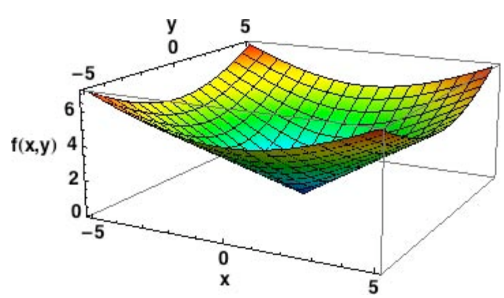
- A **scalar field** is a field that describes a scalar quantity, i.e., one **without direction**. Example: the temperature throughout this room is a scalar field $T(x,y,z,t)$.
- A **vector field** is a field that describes a vector quantity, i.e., a quantity **with direction**. Example: the velocity of the air currents in this room is a vector field $\vec{v}(x,y,z,t)$

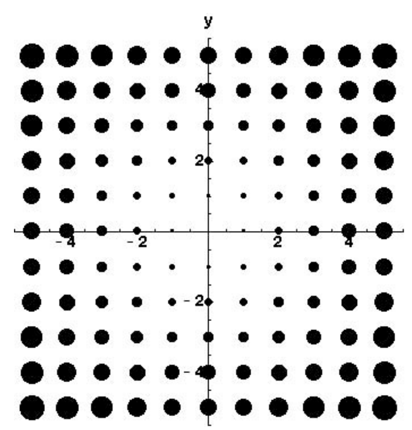
(a) Come up with two more examples of a scalar field and two more examples of a vector field, all of which could be measured in this room.

Half of our course title is "math applications". *Physics **IS** math applied to nature.* To do physics, we must be able to translate things we observe into mathematical expressions, and vice versa. With fields, that takes geometric thinking, so let's engage our physical intuition with some visualizations.

Here are three plots of the same scalar field $f(x,y)$ in the 2D space of the (x,y) plane. The one on the left is a density plot with contours. Here *color* is used the represent our map of $f \rightarrow$ dark means larger values of the field f at a given point, while light means smaller (possibly negative) values. The middle plot is a 3D visualization where we use the z axis to denote the value of f at each point. In this sort of plot, our 2D scalar field becomes a surface. The right-hand plot uses the size of dots to show the value of f at a sample of points.



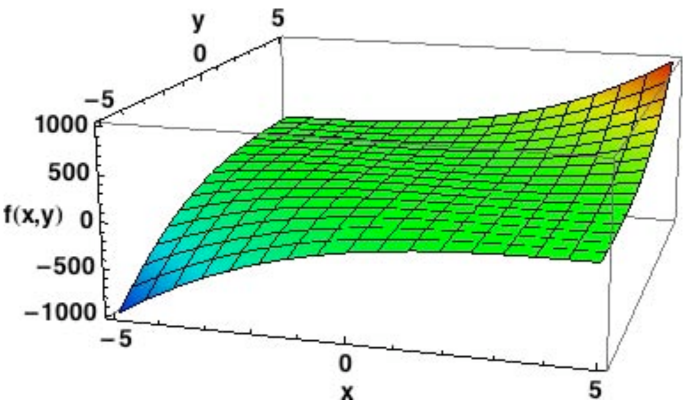


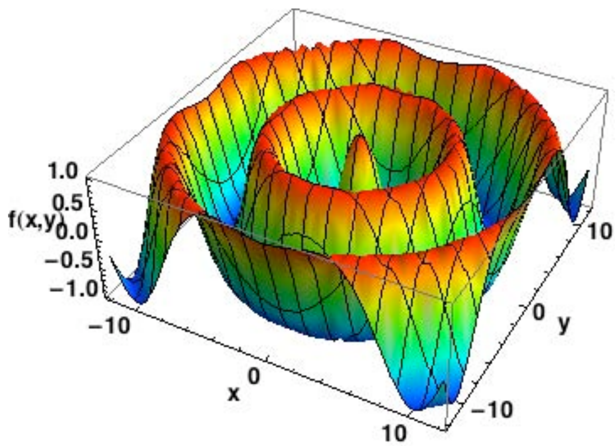


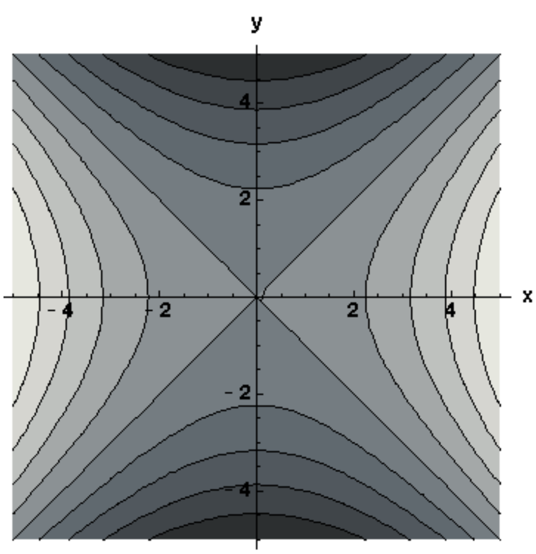
(b) What field *is* this? Take your best guess and write down a mathematical expression for $f(x,y)$.

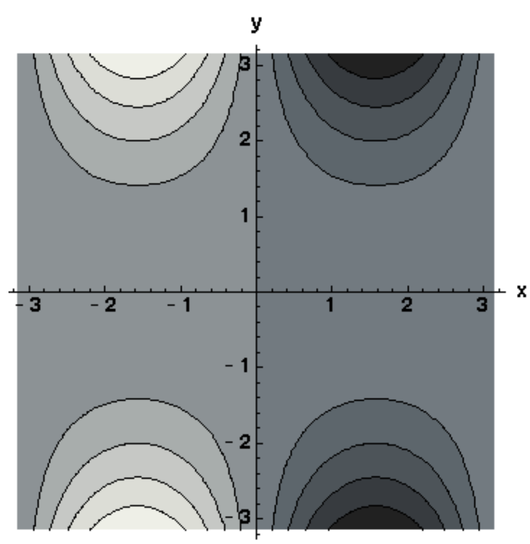
(c) Now the reverse: Let's say your calculations have resulted in a scalar field $f(x,y) = y^2 - x^2$. *What does it look like*? Sketch this field. Suggestion: I find the dot representation the easiest one to sketch, using closed circles for positive numbers and open circles for negative numbers.
(As before, use the size of the circles to indicate the absolute magnitude of f at each point.)

(d) A few more: what mathematical functions are these? (Each plot shows a different field.)









Exercise 8.2: Partial Derivatives

On to the next Jargon Assasination! Here is some notation you may not have seen before: the expression $\frac{\partial f}{\partial x}$ denotes a **partial derivative**. Partial derivatives appear in multi-dimensional calculus where you are dealing with fields — i.e., with functions of several variables.
If *f* is a function of three variables, such as *f*(*x*,*y*,*z*):

> The **partial derivative** $\frac{\partial f}{\partial x}$ means **differentiate *f*(*x*,*y*,*z*) with respect to *x*** while **holding all the other variables constant** (*y* and *z* in this case)

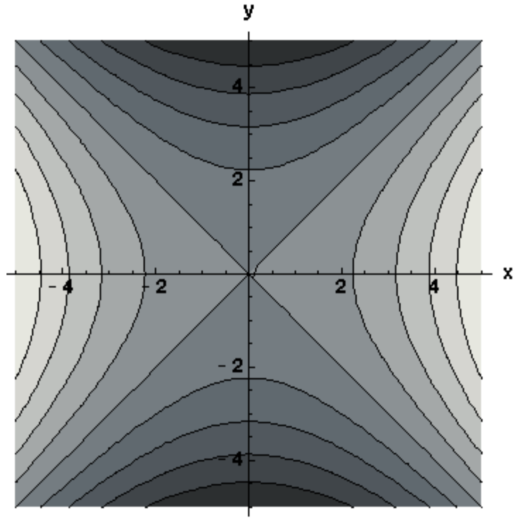
It is basically the *x*-derivative you would take if you were not told to do otherwise. ☺ Here is an example to clarify how straightforward this is:

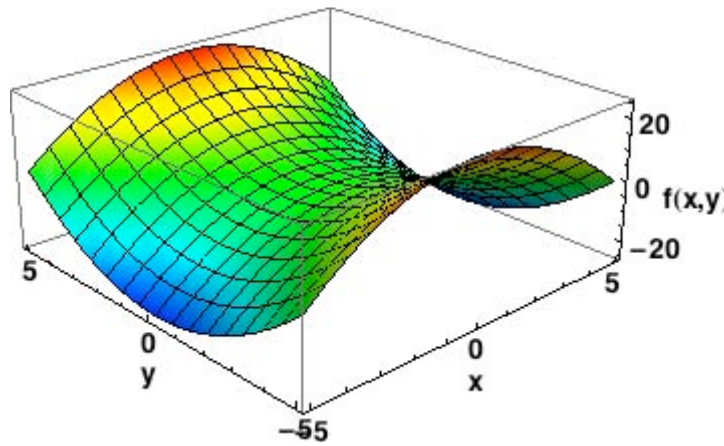
If $f(x,y,z) = \frac{x}{y} + 3z$ then: $\frac{\partial f}{\partial x} = \frac{1}{y}$, $\frac{\partial f}{\partial y} = -\frac{x}{y^2}$, and $\frac{\partial f}{\partial z} = 3$

(a) Your turn: if $f(x,y,z) = 3ze^{2y} - 5x^3y^2$, what are the three partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$?

All clear? Cool. Now here's the physical interpretation of a partial derivative:
A derivative is a slope. But multi-D fields have *more than one slope*. The partial derivatives give you the slopes of the function *f*(*x*,*y*,*z*) along each of the Cartesian directions $\hat{x}$, $\hat{y}$, and $\hat{z}$.

(b) Here's one of our 2D scalar fields, in two graphical representations:





Below are three points. Using only the graphs and your understanding of what partial derivatives mean, figure out if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are positive, negative, or zero at each point.

- at $(x,y) = (3,0)$:
- at $(x,y) = (-3,0)$:
- at $(x,y) = (2,2)$:

(c) Now check your intuition against a calculation. The function depicted above is (drumroll …) $f(x,y) = y^2 - x^2$. Calculate $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$, evaluate them at the three (x,y) points from the previous questions, and see if your guesses were correct!

- at $(x,y) = (3,0)$:
- at $(x,y) = (-3,0)$:
- at $(x,y) = (2,2)$:

(d) One more exercise. Can you sketch a 2D scalar field $f(x,y)$ that has $\frac{\partial f}{\partial x} = y$? What does such a thing look like? Try building the function graphically, using whatever representation you prefer.

(e) Now that you've visualized the meaning of the condition $\frac{\partial f}{\partial x} = y$, see if you can determine the general solution[1] $f(x,y)$ for this partial differential equation.[2]

Exercise 8.3: The Gradient and its Physical Meaning

The ABCs of 3D differential calculus are three operations called **gradient**, **divergence**, and **curl**. The "derivative" of a function is no longer a unique concept in 3D: a scalar field $V(x,y,z)$ has three partial derivatives, while a vector field $\vec{E}(x,y,z)$ has nine (three for each component). Grad, div, and curl are specific *combinations* of these partial derivatives that have *physical meaning*. We will become very good friends with these three operations in the next couple of weeks.

Today we'll study the **gradient**. First, we introduce the **gradient operator,** $\vec{\nabla}$:

$$\vec{\nabla} \equiv \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

Just like the familiar 1D derivative d/dx, $\vec{\nabla}$ is an **operator**, which means that it *acts on functions*. If you apply it to a scalar field $V(x,y,z)$, you get the **gradient of that scalar field**:

$$\vec{\nabla}V = \hat{x}\frac{\partial V}{\partial x} + \hat{y}\frac{\partial V}{\partial y} + \hat{z}\frac{\partial V}{\partial z}$$

Our job today is to figure out what the gradient of a scalar field *means*. Next week we'll study divergence and curl, which arise from applying the operator $\vec{\nabla}$ to *vector* instead of scalar fields.

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[1] The phrase "general solution" means "a mathematical expression that covers all possible solutions".

[2] Haven't had a course in partial differential equations yet? Not a problem! Your intuitive understanding of what these partial derivatives mean is all you need to figure out the mathematical solution.

Before we continue, check your answer to that construction exercise (d) on the previous page → the solution is on the back page of this packet. ☺ All good? Then on we go!

Let's try out our new gradient operation. Consider the following scalar field:[3]

$$V(x,y,z) = (x - y^2)^2 + 2z^3$$

(a) Calculate the gradient $\vec{\nabla}V$ of this mathematical function. Remember, you will obtain a vector field as an answer so be sure to obtain all three components. We'll be playing a lot with both this function V and its gradient $\vec{\nabla}V$ over the next few pages.

(b) Now evaluate the gradient $\vec{\nabla}V$ at the four specific points given below, all of which are in the xz-plane. Remember, you should get four *vectors*, each with three components.

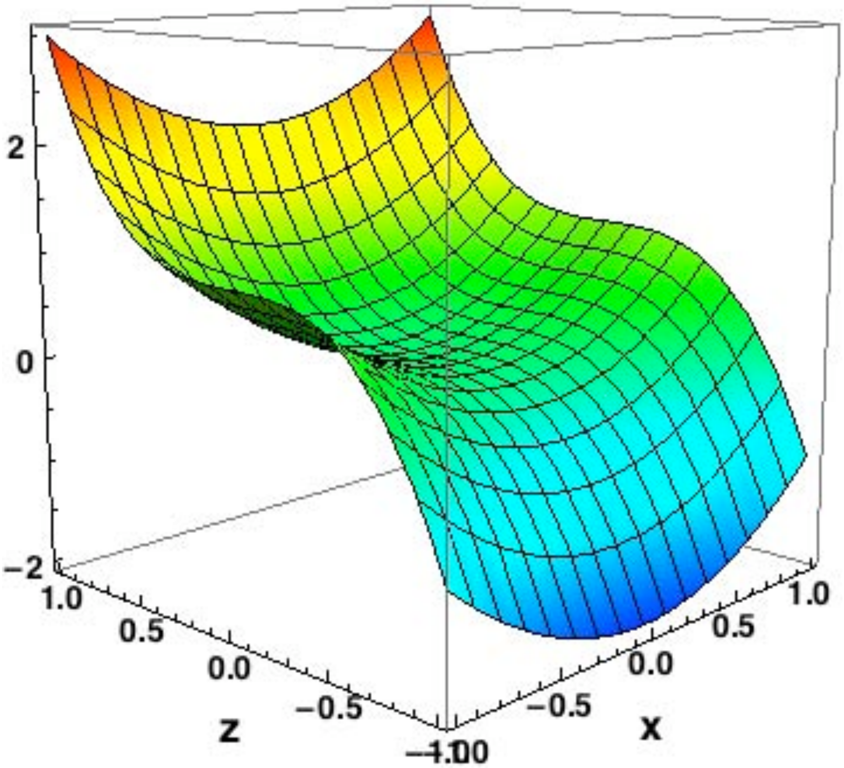
- at $(x,y,z) = (0,0,1)$:
- at $(x,y,z) = (0,0,-1)$:
- at $(x,y,z) = (1,0,0)$:
- at $(x,y,z) = (-1,0,0)$:

What could the gradient operation possibly **mean**? What does the vector field $\vec{\nabla}V(x,y,z)$ **tell us** about the scalar field V at the point (x,y,z)? Well, have a look at the plot on the next page, which shows what our function V looks like in the xz-plane …

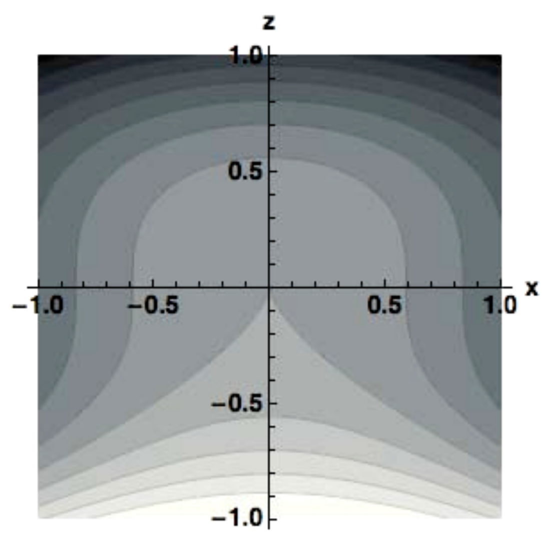
[3] Hopefully you are recoiling in revulsion at this expression → $(x - y^2)$ is mixed units! The horror, the horror! Well, just this once, we'll tolerate it: this part of the unit is just math, so we will treat all the coordinates as dimensionless values. We could introduce constants to fix up the units, like $(ax - y^2)$, but they will just get in our way, so fine. ☺

Self-check: The gradient field is $\vec{\nabla}V(x,y,z) = 2(x-y^2)\,\hat{x} - 4y(x-y^2)\,\hat{y} + 6z^2\,\hat{z}$ and its values at those four points are $\vec{\nabla}V(0,0,\pm1) = 6\hat{z}$ and $\vec{\nabla}V(\pm1,0,0) = \pm2\hat{x}$. Got it? If not, find your error.

(c) Mark on each plot below the four points where you evaluated the gradient of V.
Next, draw arrows on the plots to show the *direction* of $\vec{\nabla}V$ that you obtained at each point.



V(x,z) at y = 0



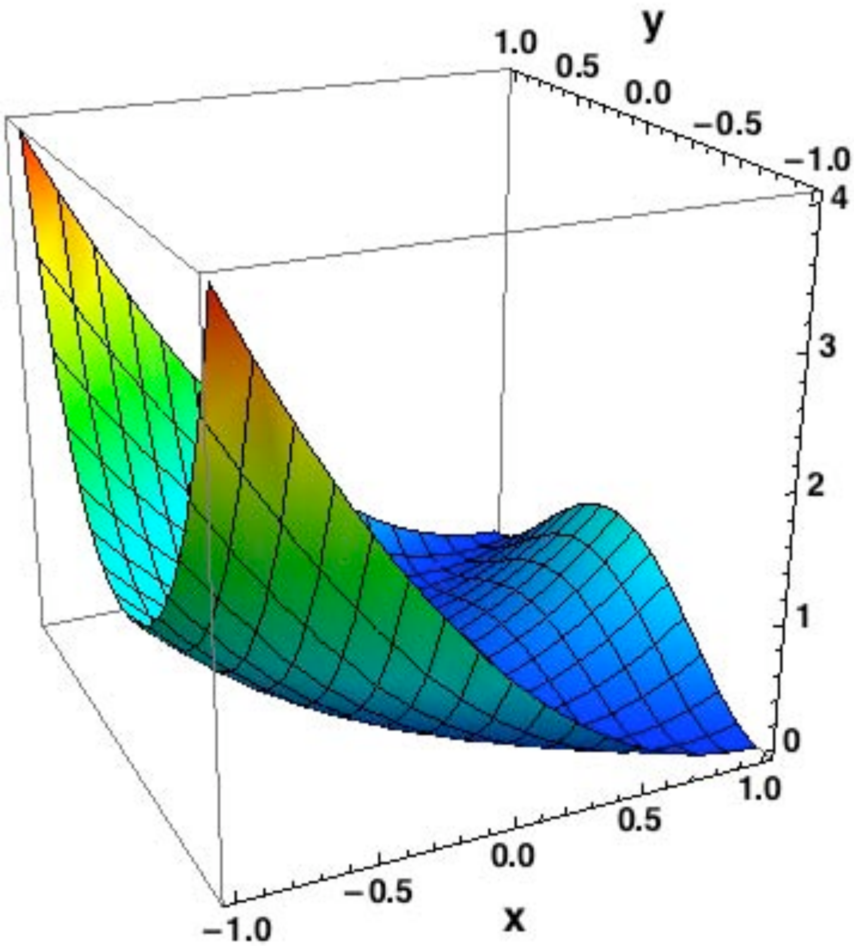
(d) Have a look at those arrows, and at the plots. Can you guess: what does the **direction** of $\vec{\nabla}V$ tell us about the function V? You could try evaluating the gradient at a couple more points to confirm your theory (e.g. try it at the top two corner points (±1,0,1) …)

(e) Can you also guess: what does the **magnitude** of $\vec{\nabla}V$ tell us? You obtained larger gradients at some points than others … what's different about the function V at those points? To help you figure it out, you could try evaluating the gradient at the origin (0,0,0) …

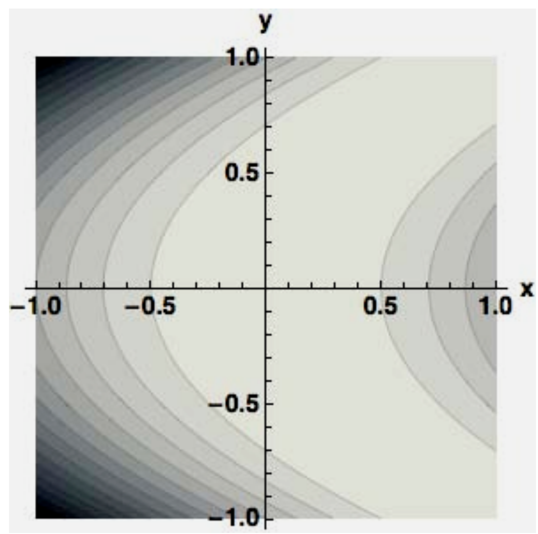
Epiphany! Did you figure it out? I'm sure you did!

- The gradient of a scalar field always points "**uphill**"
  → in the direction of the **maximum rate-of-increase** of the field.
- The gradient's magnitude tells us the **slope** of the function **along the uphill direction**.

CHECK THE TIME: If there are ≤20 minutes left, please jump to page 14 → physics!



V(x,y) at z = 0



(f) The plots above show the behavior of our function V in the xy-plane. Use these plots to **predict** the direction and relative magnitude of $\vec{\nabla}V$ at these four points:

	direction of $\vec{\nabla}V$: ($+\hat{x}$, $-\hat{y}$, ...)	relative magnitude of $\vec{\nabla}V$: (smallest, largest, 2nd largest, …)
at **A** = (1,0,0):		
at **B** = (–1,0,0):		
at **C** = (1,1,0):		
at **D** = (–1,–1,0):		

(g) Let's test your reading of the plot. Using your gradient formula from part (a),

$$\vec{\nabla}V(x,y,z) = 2(x-y^2)\,\hat{x} - 4y(x-y^2)\,\hat{y} + 6z^2\,\hat{z}\,,$$

evaluate $\vec{\nabla}V$ at the points **A** through **D**. Were your guesses accurate?

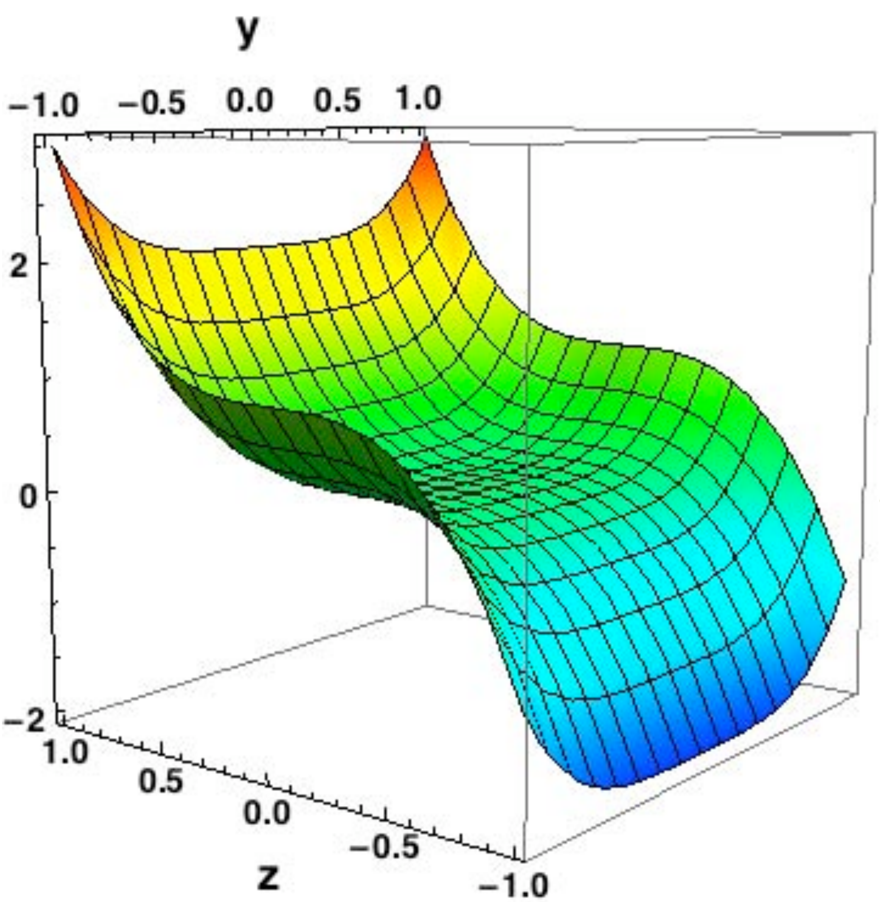
at **A** = (1,0,0):

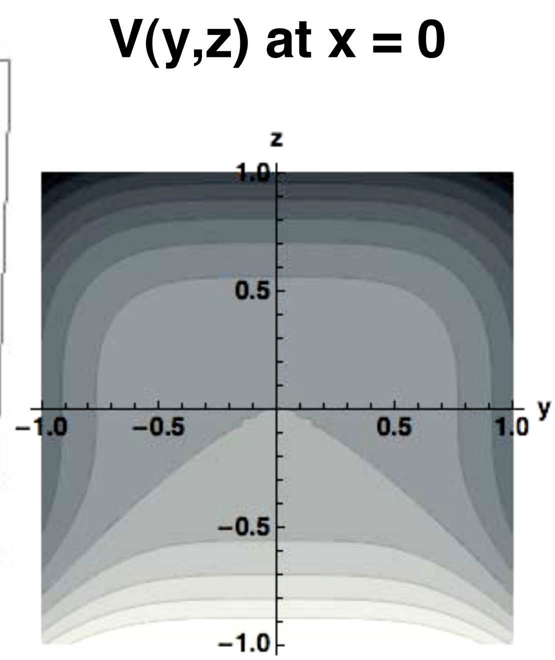
at **B** = (–1,0,0):

at **C** = (1,1,0):

at **D** = (–1,–1,0):

(h) Finally, let's look at the *yz*-plane. Again use the graphical representations below to **<u>predict</u>** the direction and relative magnitude of the gradient at these two points: (0,0,–1) and (0,–1,0).

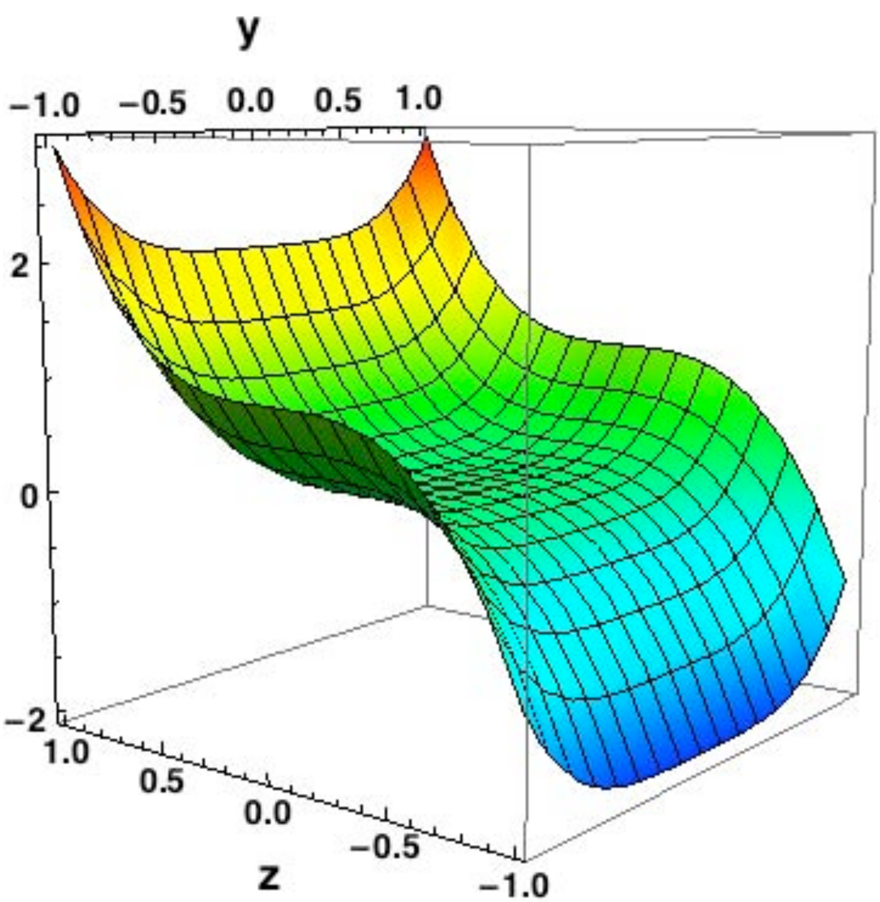


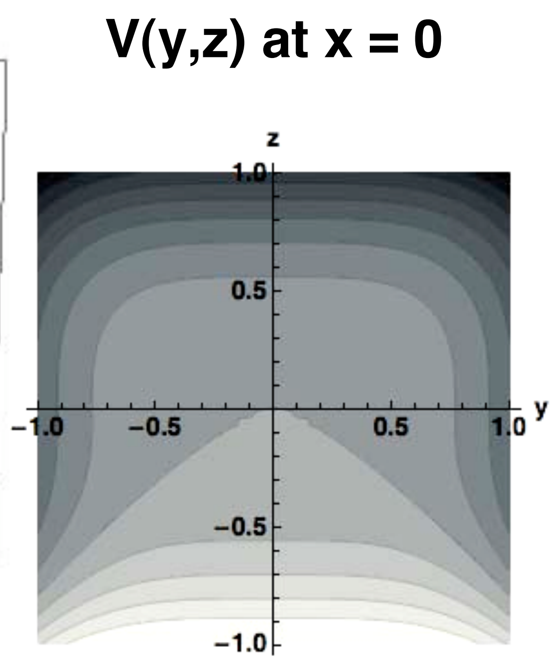


(i) Now evaluate those gradients! $\vec{\nabla}V(x,y,z) = 2(x-y^2)\,\hat{x} - 4y(x-y^2)\,\hat{y} + 6z^2\,\hat{z}$

- at $(x,y,z) = (0, 0, -1)$:
- at $(x,y,z) = (0, -1, 0)$:

(j) I'm certain that your guesses from part (h) were *nearly* correct. However, I'm also pretty sure that you did not guess the direction of the full gradient at the point (0, –1, 0). Please think about this question for a moment: is it possible *at all* to determine the direction of that gradient using *only* the plots on the previous page (repeated below)? What if you were to consult all *three* plots we've seen so far, could you determine it then? Can you explain why or why not? This is an exercise in 3-dimensional thinking which isn't trivial …





Exercise 8.5: The Gradient in Physics

One of the many things the gradient does for us in physics is to describes the relation between potential energy $U(x,y,z)$ of some object and the force $\vec{F}(x,y,z)$ it experiences at different locations. Here's the connection:

$$\boxed{\vec{F} = -\vec{\nabla} U}$$

(a) Now that we know exactly what a gradient tells us about a mathematical function, so back to physics. Please try to explain in words how the relation $\vec{F} = -\vec{\nabla} U$ between force and potential energy <u>makes sense</u>. (The "simple" exercise of expressing something in words is a very powerful way to refine our understanding!)

Rather than working with a potential energy map $U(x,y,z)$, which is always specific to *some particular object* sitting in a force field, in physics we prefer to work with **potential**, which describes the nature of the *force field alone.* In 212, you work with **electric potential** $\boldsymbol{V(x, y, z)}$. If you stick a particle of charge q in such a potential, you can immediately find its potential energy: $U = q\ V$. Similarly, one can define **gravitational potential** $\boldsymbol{\Phi(x, y, z)}$: it is just the potential energy of a mass m placed at the point (x,y,z) with the mass divided out $\rightarrow U = m\ \Phi$.

The **electric field** $\vec{\mathbf{E}}(\mathbf{x,y,z})$ and **gravitational field** $\vec{\mathbf{G}}(\mathbf{x,y,z})$ are related to force $\vec{F}$ in the same way that electric and gravitational potential are related to potential energy U: if you stick a charge q in an electric field, it experiences a force $\vec{F} = q\vec{E}$, and if you stick a mass m in a gravitational fieldit experiences a force $\vec{F} = m\vec{G}$.

(b) Write down the relations for obtaining the electric field $\vec{E}(x,y,z)$ from the electric potential $V(x,y,z)$ and for obtaining the gravitational field $\vec{G}(x,y,z)$ from the gravitational potential $\Phi(x, y, z)$.

Those formulas belong in boxes! … or do they? They're obvious, really. ☺

(c) Let's calculate something with our physics knowledge and multi-D math skills. What is the gravitational potential $\Phi(x,y,z)$ in this room? You need to set up a coordinate system first → take the spot where the center of your paper is sitting to be the origin, then pick some axes.

(d) Given that potential field, what is the gravitational field $\vec{G}(x,y,z)$ in this room?

(e) What is the gravitational field $\vec{G}$ at the point 2 m directly above your paper?
(I'm not giving you an (x,y,z) point because I have no idea what axes you selected. ☺)
You can get a complete numerical answer … but what are its *units*?

(f) What is the gravitational potential Φ at that same point? (This might be a trick question …)

(g) What is the gravitational potential energy U at that same point? (This might also be a trick question …)

The last two *were* trick questions. ☺ Here are the reasons:

- **For (f)**: If someone asks "What is the potential at point X?", and hasn't specified anything else about this potential, your response should be: "Potential ***relative to WHAT ??***"
  Before any meaningful answer can be given, a **reference point** must be specified, e.g.
  "What is the potential Φ at point X relative to the center of the table?", or equivalently,
  "What is the potential Φ at point X given that $\Phi \equiv 0$ at the center of the table?"
  Potential on its own has no physical significance at all. Only *changes* in potential — i.e., the *derivatives* of potential — have physical significance, via relations like $\vec{G} = -\vec{\nabla}\Phi$.

- **For (g):** Same issue as for (f): without a reference point, potential energy is as meaningless as potential. But even with a reference, the answer to "What is the potential energy at point X?" is "Potential energy ***of WHAT ??***" Potential is a property of space, but potential energy is a property of an **object**. As you can see from $U = m\Phi$ or $U = qV$, you need the mass or charge of an object to find its potential energy in a gravitational or electric field respectively.

If either of these important points is unclear, please ask your instructor!

(h) Now suppose the surface of your work table carries a uniform surface charge density of $+5\ \mathrm{C/m^2}$. What is the electric potential $V(x,y,z)$ in the space right above your desk?
You'll likely want to make some sort of approximation here → go for it! Just specify what it is.
And another suggestion: $\vec{E} = -\vec{\nabla}V$ means $E_x = -\frac{\partial V}{\partial x}$, $E_y = -\frac{\partial V}{\partial y}$, $E_z = -\frac{\partial V}{\partial z}$. If you know the three components of $\vec{E}(x,y,z)$, that's three little differential equations you can use to find $V(x,y,z)$, just like you did before with that $\partial f / \partial x = y$ construction problem.

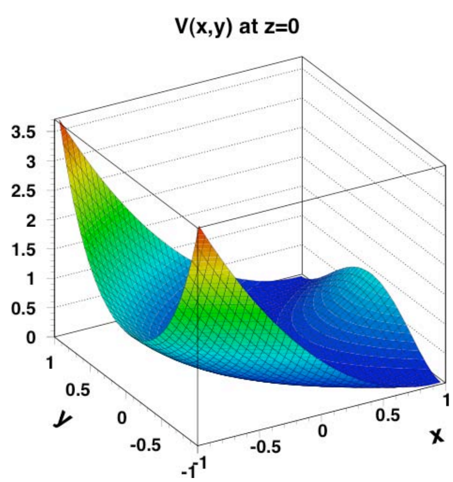
(i) Suppose your eraser carries a negative charge of –2 C. What force would it experience if you placed it 10 cm above your paper?

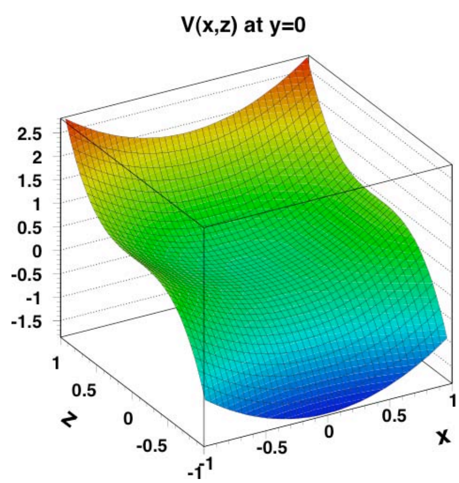
(j) What is the likelihood that you could hold your charged eraser in place 10 cm above your table? To figure out how big the force actually is, calculate how much mass the eraser would need to experience a gravitational force equal to the electrical force you just foundb.[4]

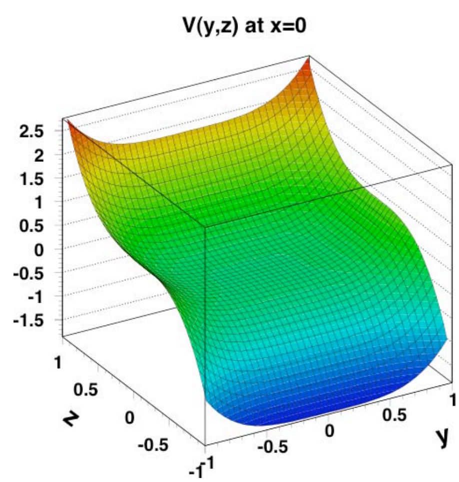
[4] If you can actually hold that charged eraser in place, you are spending way too much time at the gym.

(k) Suppose our big scalar function from before, $V(x,y,z) = (x - y^2)^2 + 2z^3$, represents an actual electric potential caused by an actual charge distribution. Now suppose you placed a *positive test charge* at the origin (0,0,0). Would that charge be in stable equilibrium, unstable equilibrium, or not in equilibrium at all? Snapshots of the three plots we've seen so far are collected at the bottom of this page to help you determine the answer.

(l) Finally, suppose you instead placed a *negative* test charge at the origin. What would be the equilibrium condition of that charge?







Answer to part (d) on Page 8: To satisfy the condition $\frac{\partial f}{\partial x} = y$, any function of this form will do: $f(x,y) = xy + g(y)$ where g is any function that depends only on y. The simplest choice is $f(x,y) = xy$ of course. Here are several graphical representations of this function … how did you do with your sketching?