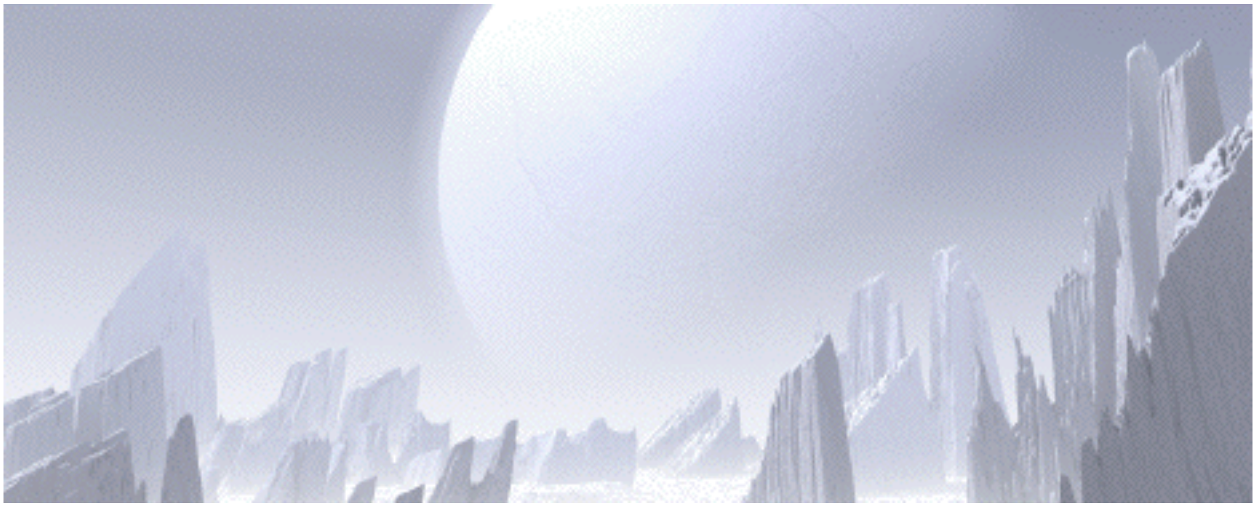

Physics 225

Relativity and Math Applications

Fall 2012

Unit 9

Vector Fields and Curvilinear Coordinates

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Unit 9: Vector Fields and Curvilinear Coordinates

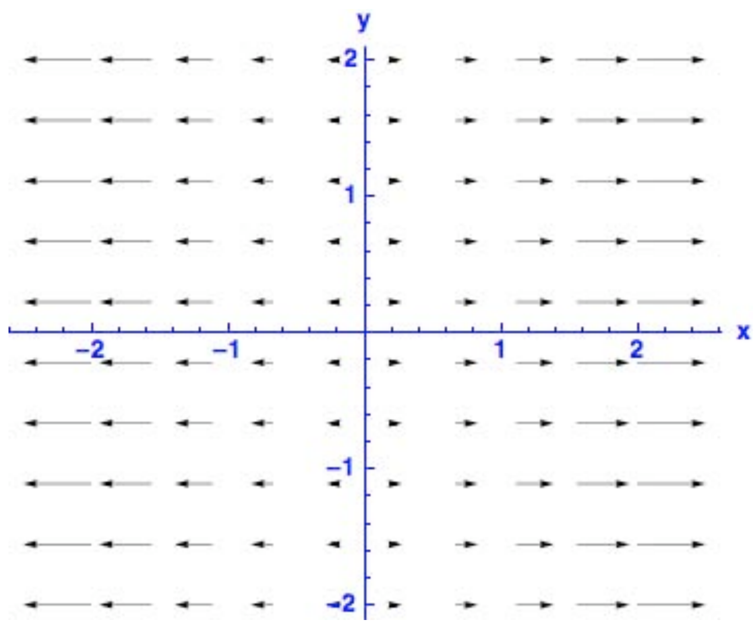
Section 9.0: Vector Field Warmup

Quick recap from last week:

- A **field** is basically a multidimensional functional; it is a **map** of a physical quantity that varies with position and time.
- A **scalar field** is a field that describes a scalar quantity, i.e., one **without direction**. Example: the temperature throughout this room is a scalar field $T(x,y,z)$.
- A **vector field** is a field that describes a vector quantity, i.e., a quantity **with direction**. Example: the velocity of the air currents in this room is a vector field $\vec{v}(x,y,z)$

Last week, we got lots of practice visualizing scalar fields. Let's do the same with vector fields via a few examples.

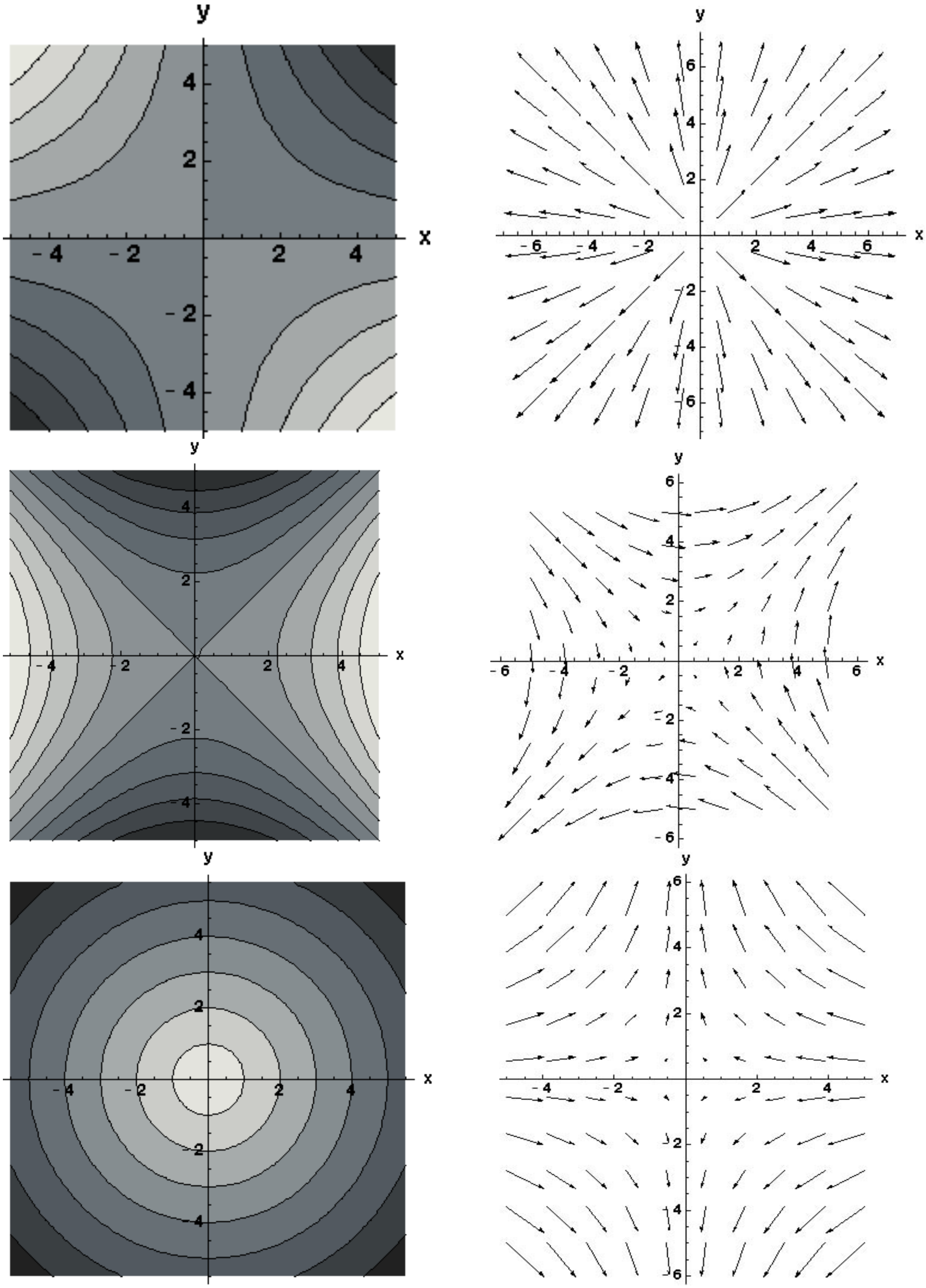
Vector fields are rather easy to plot. Last week we studied three different ways to represent a scalar field graphically. For vector fields, there are only two such representations, and we'll skip the second one as it is too subtle for our introduction.[1] The easiest way to draw a vector field is a simple sampling method: just draw little arrows at a sample of points in your space. The size and direction of the arrows shows the size and direction of the field at the arrow's location. Nothin' to it.



(a) At right is a picture of a vector field $\vec{E}(x,y)$ in 2D space. What field *is* this? Take your best guess and write down a mathematical expression for $\vec{E}(x,y)$.

[1] That second representation is *field lines*. They are used quite a bit in electromagnetism, but they are only practical as a drawing tool for highly symmetric fields.

(b) For any scalar function $V(x,y,z)$, the gradient $\vec{\nabla}V(x,y,z)$ is a vector function with a *special meaning*. To make sure you remember that meaning, here's a matchup exercise for you. On the left are contour maps of three scalar fields $V(x,y,z)$, with darker colors indicating larger values; on the right are their gradient fields $\vec{\nabla}V(x,y,z)$. Which goes with which?



For part (a), I bet you got the correct answer for the 2D field shown, and I bet you wrote it like this: $\vec{E}(x,y)=(x,0)$. That is perfectly fine, your notation specifies the field's two components very clearly: $E_x(x,y)=x$, $E_y(x,y)=0$. In case you hadn't quite realized it, notice how a vector field in 2D is actually *two independent scalar fields*. In 3D, a vector field is three independent scalar fields, one for each component. Now here is an alternative notation that may not be as familiar to you:

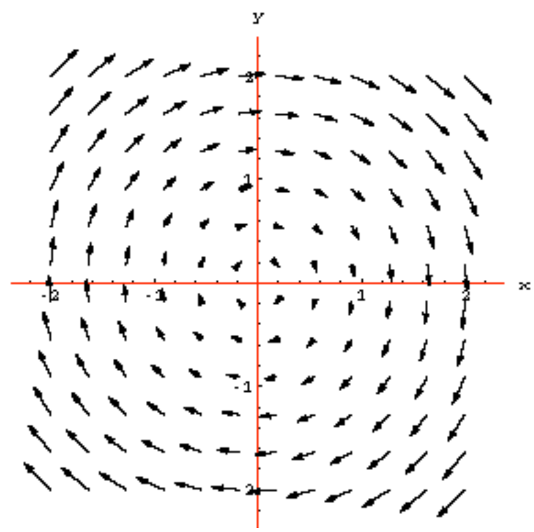
$$\vec{E}=(E_x,E_y,E_z) \qquad \text{can also be written} \qquad \vec{E}=E_x\hat{x}+E_y\hat{y}+E_z\hat{z}$$

This second form uses **unit vectors**, $\hat{x}$, $\hat{y}$, $\hat{z}$.[2] Here are the defining properties of a unit vector:

For any coordinate u of your coordinate system, the corresponding **unit vector** $\hat{u}$: ❶ points in the "+u direction", which is the direction you move when you increase u ❷ has a length of 1 ❸ is dimensionless

Unit vectors are pure directions. As you will see in your continuing studies, the unit vector notation is both flexible and powerful. It is vastly preferred over (E_x, E_y, E_z) and we will use it exclusively.

(c) One more guess-the-function: what is this vector field? Express your answer in unit vector notation.



(d) Finally, create your own plot. Let's say your calculations have resulted in a vector field $\vec{E}(x,y)=\sin(x)\ \hat{y}$. *What does it look like*? Sketch this fabulous vector field!

[2] The most common unit vector notation is the "hat" notation used here, but you will sometimes see the vector ***e*** used instead, with a subscript to denote the relevant coordinate: e.g. $\boldsymbol{e}_u$ instead of $\hat{u}$. Marion & Thornton's mechanics text (often used for Physics 325) uses this "***e***" notation. Also, you've probably seen $\hat{i}$, $\hat{j}$, $\hat{k}$ at some point. This notation is synonymous with $\hat{x}$, $\hat{y}$, $\hat{z}$ but is rarely used in physics as it can't be extended to any non-Cartesian coordinate system.

Curvilinear Coordinate Systems and the Five Keys that unlock them

So far, we have only studied fields in Cartesian coordinates: good old (x,y,z). Well, sticking with Cartesian coordinates forever will cripple us in our study of physics, much as the use of I, II, III, IV, V, etc … crippled the ancient Romans 2,000 years ago in their pursuit of scientific progress. Instead we must learn how to work with vector calculus in **spherical** and **cylindrical** coordinate systems. These beautiful **curvilinear** coordinate systems do have their pitfalls, some of which we will tackle today (e.g. curvilinear unit vectors, which can be confusing!). But curvilinear coordinates offer an advantage that greatly exceeds the modest effort required to get used to them: they follow the natural symmetries of many systems, both natural and man-made.

I will use the notation r_i, where the index *i* runs from 1 to 3, to denote a generic system of coordinates in 3D space. The spherical coordinate system, for example, is $(r_1, r_2, r_3) = (r, \theta, \phi)$. If I want to refer to the Cartesian system (x, y, z) specifically I will use the symbol x_i.

5 key elements must be studied to unlock the full power of any system of coordinates, r_i :

1. **The Definitions** of r_i: the transformation equations to/from Cartesian coordinates x_i
2. **The Unit Vectors** $\hat{r}_i$: the trickiest part of curvilinear coordinate systems!
3. **The Position Vector** $\vec{r}$: driving directions to the spacepoint described by r_i
4. **The Line Element** $d\vec{l}$: turning coordinates into distances
5. **The Gradient** $\vec{\nabla}$: and other differential operators in 3D

We will study each of these key properties for both the spherical and cylindrical coordinate systems. The resulting formulas are often found on the inside covers of physics texts, that's how much they are used. But we will make sure we know how to *derive* all those formulas, so we know what they mean. With a good grasp of 3D geometry and calculus *you can derive all the key elements from element #1*. Most of our derivations today will be based on visualization: doing the relevant 3D geometry graphically, so you *picture* the mathematical objects you're working with, in your head or on paper. The derivations can also be done mathematically, using a tidy formalism that we will cover in lecture.

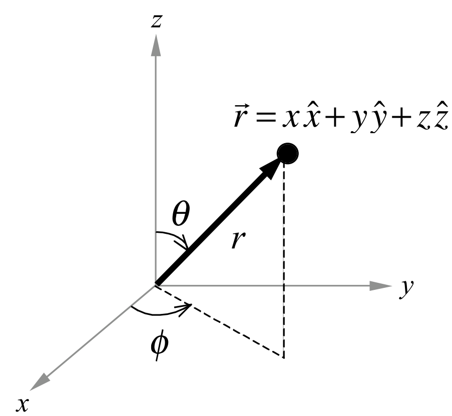
Section 9.1: **The First Key → The Definitions of r_i**

The purpose of any coordinate system is to mathematically describe any **position in space**. We use the generic symbol $\vec{r}$ to refer to such a random position. To build a coordinate system, we first define an origin, namely some fixed reference location. Then we select three directions and call them $\hat{x}$, $\hat{y}$, and $\hat{z}$ (the unit vectors we introduced before). Why three? Because we happen to live in 3-dimensonal space. That is the nature of our strange universe. ☺

To complete our coordinate system, we select the <u>three independent variables</u> needed to describe the location of an arbitrary spacepoint $\vec{r}$. The most intuitive choice is **Cartesian coordinates,** where our three variables are *x*, *y*, and *z*. Here is their significance: they give you explicit "driving directions" for finding a given spacepoint :

- start at the origin
- then travel a distance *x* in the $\hat{x}$ direction
- then travel a distance *y* in the $\hat{y}$ direction
- and finally travel a distance *z* in the $\hat{z}$ direction

That brings you exactly to the spacepoint $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$.



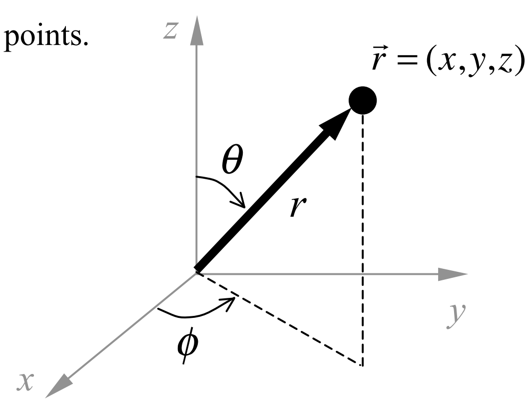
(a) **Spherical coordinates** represent a different choice for the three variables we need to define an arbitrary location $\vec{r}$ in space relative to our chosen origin and our chosen set of directions $\hat{x}$, $\hat{y}$, $\hat{z}$. The spherical-coordinate choice is (r, θ, ϕ) rather than (x, y, z). <u>The diagram above *defines* these coordinates</u>.[3] I'm serious, we will derive everything there is to know about spherical coordinates from that diagram. We start with the *mathematical* definitions of r, θ, and ϕ. Using only the diagram, and your expertise at trigonometry, write down expressions for

- the spherical coordinates r, θ, and ϕ in terms of the Cartesian coordinates *x*, *y*, and *z*
- the Cartesian coordinates *x*, *y*, and *z* in terms of the spherical coordinates r, θ, and ϕ

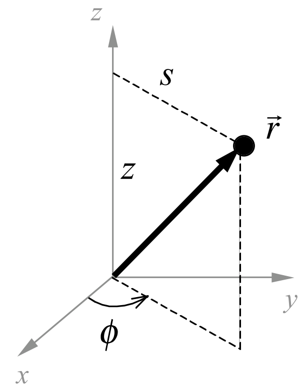
(b) Let's develop our intuition. Here are three spacepoints expressed in Cartesian coordinates:

A: $(x,y,z) = (2,0,0)$ **B**: $(x,y,z) = (0,-2,0)$ **C**: $(x,y,z) = (0,1,-1)$

Determine the spherical coordinates (r,θ,ϕ) of these three points. We're after intuition here, so try <u>not</u> to use the formulas you just wrote down. Rather, <u>draw</u> the points on the diagram and figure out (r,θ,ϕ) from that.



[3] Notation alert: the physics convention for θ and ϕ is the *reverse* of the convention used in math. ☹ In physics, θ is the *polar* angle that runs from 0 to π while ϕ is the *azimuthal* angle that runs from 0 to 2π (or –π to +π, if you prefer).



The other major curvilinear coordinate system is that of **cylindrical coordinates** (s, ϕ, z). Cartesian z is back, ϕ is the same as in spherical, and we introduce the radial coordinate $s = \sqrt{x^2 + y^2}$. The use of s for this variable is probably unfamiliar, you've likely seen it as r, R, or ρ before[4]. The crucial point is that s is <u>not</u> the same as the spherical radial coordinate $r = \sqrt{x^2 + y^2 + z^2}$, so we must use a different letter!

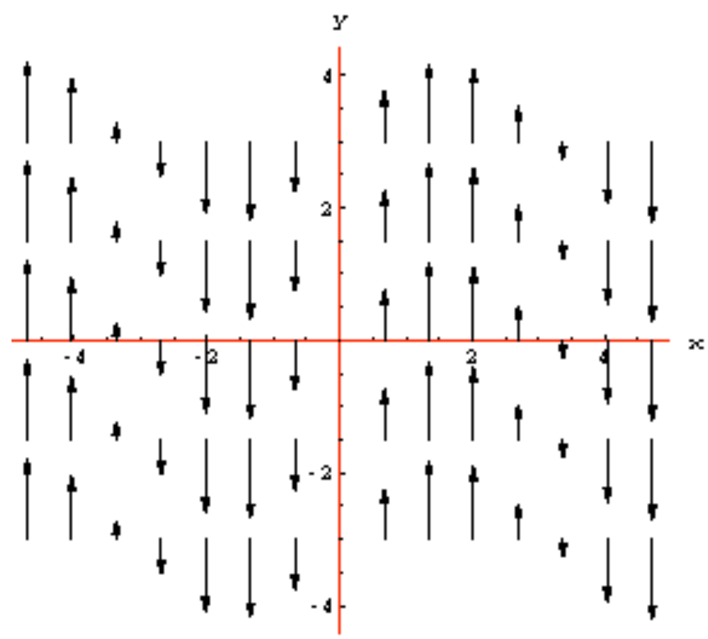
Cylindrical coordinates are the familiar 2D polar coordinates (s, ϕ) extended to include the z direction. Cylindrical coordinates are a piece of cake. ☺

(c) Using the defining diagram above, write down the equations for
- the cylindrical coordinates s, ϕ, z in terms of the Cartesian coordinates x, y, z
- the inverse of that

OK, so you can skip the z coordinate, it's the same.

We have the definitions of our curvilinear coordinates! Onwards.

… so what is this weird plot, you ask? →



It's the answer to part (d) on page 5. ☺
How did you do with your sketch?

[4] The use of s to denote the cylindrical radial coordinate was introduced by David Griffiths in his wildly popular textbook on electromagnetism.

Spherical coordinate definitions:

$$x = r\sin\theta\cos\phi \qquad r = \sqrt{x^2+y^2+z^2}$$
$$y = r\sin\theta\sin\phi \qquad \theta = \tan^{-1}(\sqrt{x^2+y^2}\,/\,z)$$
$$z = r\cos\theta \qquad \phi = \tan^{-1}(y\,/\,x)$$

Cylindrical coordinate definitions:

$$x = s\cos\phi \qquad s = \sqrt{x^2+y^2}$$
$$y = s\sin\phi \qquad \phi = \tan^{-1}(y\,/\,x)$$
$$z = z \qquad z = z$$

Section 9.2: The Second Key → The Unit Vectors

We defined unit vectors before. This definition is the heart of the 2nd key so let's repeat it:

> The ***unit vector*** $\hat{r}_i$ has unit length, is dimensionless, and points in the ***direction you move*** when you ***increase the coordinate*** r_i

That seems pretty straightforward. The Cartesian unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$ obviously follow this simple definition. But the spherical unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are not so familiar. You'll get to apply the definition to r, θ, ϕ in just one second, but first, let me prepare you for what you'll find:

> The ***unit vectors*** $\hat{r}$, $\hat{s}$, $\hat{\theta}$, and $\hat{\phi}$ are ***position-dependent***.

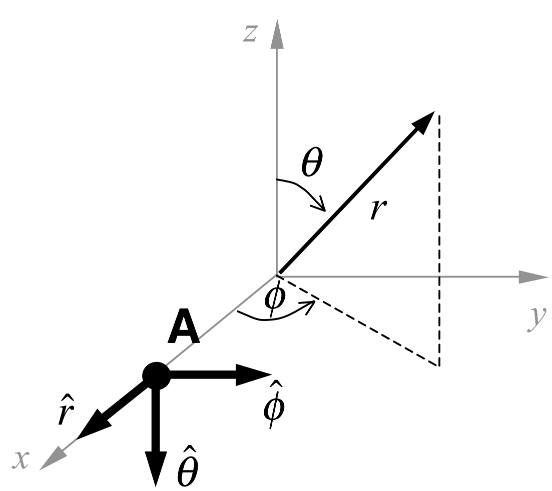
Cartesian unit vectors always point in the same direction, but curvilinear unit vectors *do not*.
This single fact is responsible for 99% of the complications in spherical and cylindrical coordinates.

(a) Consider the same three spacepoints as before:

A: $(x,y,z) = (2,0,0)$ **B**: $(x,y,z) = (0,-2,0)$ **C**: $(x,y,z) = (0,1,-1)$

Using the definition given above, determine the spherical unit vectors $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ at these three points. Draw them first, then express your answers in terms of the Cartesian unit vectors $\hat{x}$, $\hat{y}$, $\hat{z}$.
To get you started, here is the result for the point **A**:

- At spacepoint **A**: $\hat{r} = +\hat{x}$, $\hat{\theta} = -\hat{z}$, $\hat{\phi} = +\hat{y}$



Next factoid: All three of our unit-vector triplets are **orthonormal**, meaning that they are perpendicular to each other (ortho-) and of unit size (-normal). Consequently, we have relations like these

$$\hat{x} \times \hat{y} = \hat{z}, \qquad \hat{y} \times \hat{z} = \hat{x}, \qquad \hat{z} \times \hat{x} = \hat{y}$$

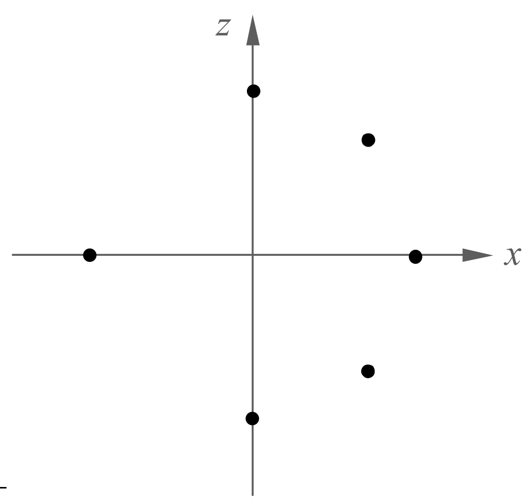
for all three unit-vector sets. Further, (x, y, z) is called a **right-handed** coordinate system. You can always determine $\hat{z}$ from $\hat{x}$ and $\hat{y}$ using the right-hand rule: $\hat{x} \times \hat{y} = \hat{z}$ *in that order*. "(x,y,z) is right-handed" means "$\hat{x}$ cross $\hat{y}$ is $\hat{z}$".

(b) Is (r, θ, ϕ) – in that order – a right-handed coordinate system? What about (s, ϕ, z)?

(c) Spherical unit vectors appear all over the place in physics. Consider this electric field:

$$\vec{E}(r,\theta,\phi) = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\ \hat{r} + \sin\theta\ \hat{\theta}),$$

where p is a constant. This is a very well-known field; you can find this formula in any E&M textbook. Can we *read* the formula? Let's figure out what it represents by sketching this E field. To make your sketch, pick a few points in coordinate space and determine the direction and rough magnitude of the E field at each one. Once you have a few field vectors drawn up, see if you can connect them in a smooth way to form continuous electric field lines. Once you see the pattern, do you recognize it? What could be causing this field? Guidance: This is a tricky field, so I've indicated below some good points to plot (the dots). Note that they're all in the xz plane → you only need a 2D drawing to visualize this field. The footnote[5] explains how I chose these points.



[5] Picking good points to plot: (1) Look for symmetry: Our mystery field *does not depend* on the angle ϕ → it is *azimuthally symmetric*. You can therefore confine your attention to a *single value of* ϕ when picking your spacepoints, without any loss of information. $\phi = 0$ is the $(+x,\pm z)$ plane; that's all we need to draw. (2) Look for dependences that factor out: The dependence of our mystery field on r is just an *overall factor* of $1/r^3$. That behavior is easy to visualize, so no there's no need to plot it → pick points at constant r for simplicity.

Section 9.3: The Third Key – The Position Vector $\vec{r}$

The symbol $\vec{r}$ has a universal meaning that is independent of any coordinate system. It is called the **position vector** and here is its definition:

> The **position vector** $\vec{r}$ is the vector pointing ***from the origin*** to an ***arbitrary point*** $(r_1, \ldots, r_n)$ in your coordinate system. As it depends on position, it is actually a ***vector field***: $\vec{r}(r_1,\ldots,r_n)$

In Cartesian coordinates, we have this familiar representation: $\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}$. That is indeed the vector that points from the origin to the generic point (x,y,z). How do you represent $\vec{r}$ in spherical or cylindrical coordinates? I have seen many students write down expressions like these:

$\vec{r} = r\,\hat{r} + \theta\hat{\theta} + \phi\hat{\phi}$ (spherical) and $\vec{r} = s\,\hat{s} + \phi\hat{\phi} + z\,\hat{z}$ (cylindrical)

Such expressions come from thinking about the (r, θ, ϕ) and (s, ϕ, z) coordinates as "driving directions", in the Cartesian vein → e.g., to get to $\vec{r}$, you move a distance r along $\hat{r}$, then a distance θ along $\hat{\theta}$, and finally a distance ϕ along $\hat{\phi}$. This is a *very* easy mistake to make – which is why I'm bringing it up now – but those expressions makes **no sense whatsoever** for several reasons.

(a) Can you explain why the expressions above are hopelessly wrong? There's more than one reason.

(b) What *are* the correct spherical and cylindrical representations of the position vector $\vec{r}$?
The **driving directions** idea is perfect: the position vector *does* give you step-by-step directions to get from the origin to the point (r,θ,ϕ) or (s,ϕ,z) … just not the ones above …

Why this matters: Before you flip the page, let me explain why the position vector $\vec{r}$ is so important → because it appears in *countless physics formulas*. Scratch that … it appears in *every* major physics formula describing a position-dependent quantity. E.g., if you look up the electric potential of an electric dipole with moment $\vec{p}$ in any text, you'll find this: $V(\vec{r}) = \vec{p}\cdot\vec{r}\,/\,4\pi\varepsilon_0 r^3$. (You'll work with it on your homework.) This is the **coordinate-free form** of the formula, meaning that it works in *any coordinate system*. Clearly, this is the best way to present a physics formula → physics doesn't care about coordinate systems, so our formulas shouldn't either! (Put differently, we shouldn't force a particular coordinate system on the problem if we can avoid it.) Coordinate-free form is standard and works beautifully in whatever coordinates you choose … as long as you know what to put in for $\vec{r}$!

… and here they are, the position vectors of all three major coordinate systems:

Cartesian: $\boxed{\vec{r} = x\,\hat{x} + y\,\hat{y} + z\,\hat{z}}$ Spherical: $\boxed{\vec{r} = r\,\hat{r}}$ Cylindrical: $\boxed{\vec{r} = s\,\hat{s} + z\,\hat{z}}$

Is this what you got? If these results are not completely and utterly clear to you, *please ask!*

Next complication: **adding vectors** in curvilinear coordinate systems.

(c) Here are two position vectors for you:

$\vec{r}_A = 3\hat{z}$ and $\vec{r}_B = 4\hat{x}$.

Make a sketch: make a plot of the *xz*-plane in the space at right, and draw in the two vectors and the points A and B to which they point.

(d) Add and subtract these vectors. You know how this goes in Cartesian coordinates: *vectors add by components*. Great! Write down Cartesian expressions for the sum $\vec{r}_A + \vec{r}_B$ and the difference $\vec{r}_A - \vec{r}_B$. Also, calculate the magnitude of your two results, and be sure to draw both $\vec{r}_A + \vec{r}_B$ and $\vec{r}_A - \vec{r}_B$ on your plot.

(e) Over to spherical! What are the coordinates $(r,\ \theta,\ \phi)_A$ and $(r,\ \theta,\ \phi)_B$ for the two points A and B?

(f) So now: how do you add and subtract $\vec{r}_A$ and $\vec{r}_B$ in spherical form?
Try things! Try adding by components … try using the position vectors … try anything …
You have already figured out the correct result for $\vec{r}_A + \vec{r}_B$ using Cartesian coordinates, and you've got the result up there on your plot. This is the perfect sandbox in which to try any method that seems reasonable to you and see if it works. Take 5 minutes or so, not more, but *do try*.
I've seen countless different attempts to add and subtract vectors in spherical coordinates. If a method occurs to you now, it can occur to you during an exam, so figure out right now if it works!

Success?

Nope! It is ***not possible*** to add/subtract vectors at different locations in curvilinear coordinates without ***transforming back to Cartesian unit vectors***. Let's see why.

The only reasonable things to add are the position vectors, and they are $\vec{r}_A = 3\hat{r}$ and $\vec{r}_B = 4\hat{r}$. What completely useless expressions! How can one possibly add those? They look the same!

The problem is, of course, the infamous position-dependence of the unit vectors. Here is a more helpful way to write those two position vectors: $\vec{r}_A = 3\hat{r}_A$ and $\vec{r}_B = 4\hat{r}_B$. This time I've labeled the unit vectors to indicate the position at which they're being evaluated. If we also have the coordinates $(r,\theta,\phi)_{A,B}$ for the two points, we know where they are so the unit vectors $\hat{r}_A$ and $\hat{r}_B$ can be interpreted. But to add them, there's no choice but to ***transform back to Cartesian unit vectors*** → those are the *only* unit vectors that are independent of position.

(g) Before you throw out these spherical unit vectors in disgust, hold on: there are some situations when you ***can*** add spherically or cylindrically expressed vectors by components. Those situations involve the addition of two vectors ***at the same location***, which gets around the position dependence problem. Here is a very typical example: consider a region of space occupied by two electic fields,

$$\vec{E}_1(\vec{r}) = \frac{kq}{r^2}\,\hat{r} \qquad \text{and} \qquad \vec{E}_2(\vec{r}) = \frac{2kq}{a^2}(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}),$$

where *k, q,* and *a* are all constants. Add those fields to obtain the total field $\vec{E}_{1+2}(\vec{r})$. Is it totally clear why you do ***not*** have to switch back to Cartesian-anything in this case? If not, ask! ask!

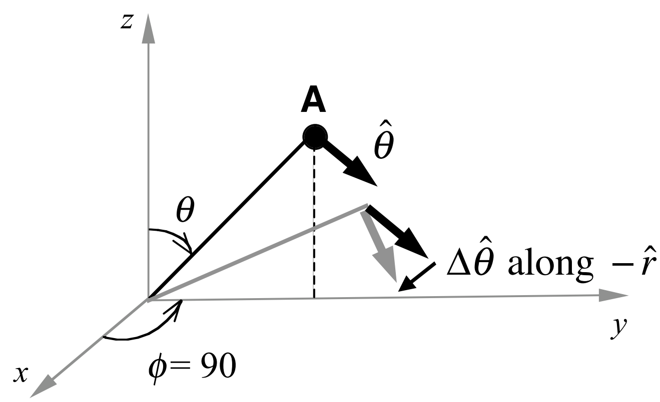
(h) What is the magnitude $\left|\vec{E}_2(\vec{r})\right|$ of the second field? Is it clear how to calculate this in spherical? Hint: *orthonormal, orthonormal, they're all orthonormal ...*

Section 9.4: The Unit Vectors Strike Back

As $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$, and $\hat{s}$ point in different directions depending on where you are in space, guess what:

Curvilinear unit vectors have <u>derivatives</u>

Yep, and we need to know them. The drawing at right shows how to visualize one example: the partial derivative of $\hat{\theta}$ with respect to itself. The thick black arrow shows $\hat{\theta}$ at some point **A** in the *yz*-plane. If the polar coordinate θ of this point is then *increased* by a little bit, notice how the unit vector $\hat{\theta}$ turns. This change in the unit vector $\hat{\theta}$ is radially inward, i.e. along $-\hat{r}$. Thus, $\frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}$.



(a) Using similar visualization, can you fill in the following table? This is the complete set of <u>derivatives of the spherical unit vectors</u>. The missing 5 answers are given at the bottom, you simply have to figure out which ones go where.

$$\frac{\partial\hat{r}}{\partial r} = 0 \qquad \frac{\partial\hat{r}}{\partial\theta} = \_\_\_\_\_\_ \qquad \frac{\partial\hat{r}}{\partial\phi} = \_\_\_\_\_\_$$

$$\frac{\partial\hat{\theta}}{\partial r} = 0 \qquad \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r} \qquad \frac{\partial\hat{\theta}}{\partial\phi} = \_\_\_\_\_\_$$

$$\frac{\partial\hat{\phi}}{\partial r} = \_\_\_\_\_\_ \qquad \frac{\partial\hat{\phi}}{\partial\theta} = \_\_\_\_\_\_ \qquad \frac{\partial\hat{\phi}}{\partial\phi} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta$$

➡ **Answers**: 0, 0, $\hat{\theta}$, $\sin\theta\,\hat{\phi}$, and $\cos\theta\,\hat{\phi}$ $\rightarrow$ where do they go?

(b) The <u>derivatives of the cylindrical unit vectors</u> are a piece of cake. Out of the 9 partial derivatives available, only <u>two</u> of them are non-zero. Go for it! Identify the two non-zero partial derivatives of $\{\hat{s}, \hat{\phi}, \hat{z}\}$ in terms of $\{s, \phi, z\}$ and figure out their values. (It's easy, requiring only the 2D geometry of the *xy*-plane.)

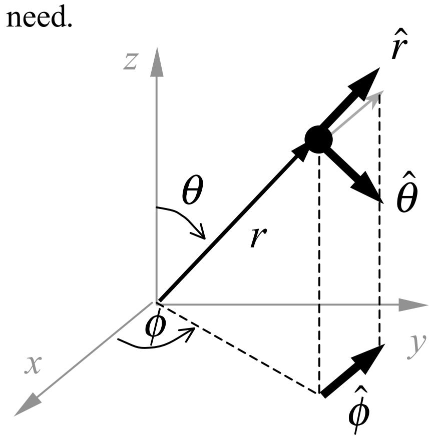
Level-Up!

It would be nice to calculate those unit-vector derivatives mathematically, wouldn't it? I do think visualization is the way to go intuition-wise … but indeed, visualizing $\partial\hat{\phi}\,/\,\partial\phi$ is pretty hard. ☺ On to some formalism! What we need are mathematical expressions for the unit vectors.

They are actually easy to obtain, as long as you are aware of the boxed formula below. It shows how to express any vector $\vec{v}$ in terms of any complete, orthonormal set of unit vectors $\hat{r}_i$. The formula may look unfamiliar, but the concept is not. The terms $(\vec{v}\cdot\hat{r}_i)$ are the components of $\vec{v}$ in our chosen coordinates r_i. These components are often labeled v_{ri}, as in $\vec{v} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z}$ or $\vec{E} = E_r\,\hat{r} + E_\theta\,\hat{\theta} + E_\phi\,\hat{\phi}$. This formula shows you how to get those components: by calculating $(\vec{v}\cdot\hat{r}_i)$ which is precisely the projection of $\vec{v}$ onto the $\hat{r}_i$ axis.

$$\vec{v} = \sum_{i=1}^{3}(\vec{v}\cdot\hat{r}_i)\;\hat{r}_i$$

(a) Apply this formula to the vectors $\vec{v} = \hat{r},\, \hat{\theta},\, \hat{\phi}$ in turn to obtain their Cartesian components. Your result will be 3 formulas: one for each of $\hat{r},\, \hat{\theta},\, \hat{\phi}$, expressed in terms of the Cartesian unit vectors $\hat{x},\hat{y},\hat{z}$ but with the components still written with the spherical coordinates r,θ,ϕ. How to do those dot-products? It's pure geometry, so a sketch is all you need.



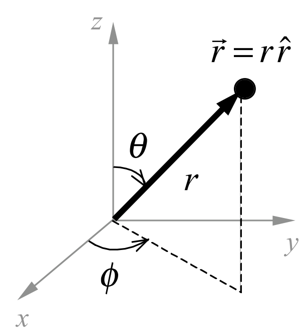
(projected onto xy-plane for visibility)

(b) And the reverse: get the formulas for $\hat{x},\hat{y},\hat{z}$ in terms of spherical coordinatesand unit vectors. Massive hint: the 9 dot products are *the same* as in part (a), just *reordered.*

Check! These crucial unit-vector translation formulas are tabulated on the next page. The unit-vector derivatives you derived on the previous page are there too, check those as well.

(c) You can now calculate all the spherical unit-vector derivatives explicitly! ☺ Let's do it: calculate the derivative $\partial\hat{r}/\partial\theta$.

- Take the derivative $\partial/\partial\theta$ of the $\hat{r}$ expression you found in (a) → this Cartesian unit-vector version can be easily differentiated as $\hat{x},\hat{y},\hat{z}$ don't have any derivatives.
- Then use your formulas from (b) to put everything back in terms of spherical unit vectors.



Summary

Spherical coordinates

$$x = r\sin\theta\cos\phi \qquad \hat{x} = \sin\theta\cos\phi\ \hat{r} + \cos\theta\cos\phi\ \hat{\theta} - \sin\phi\ \hat{\phi}$$
$$y = r\sin\theta\sin\phi \qquad \hat{y} = \sin\theta\sin\phi\ \hat{r} + \cos\theta\sin\phi\ \hat{\theta} + \cos\phi\ \hat{\phi}$$
$$z = r\cos\theta \qquad \hat{z} = \cos\theta\ \hat{r} - \sin\theta\ \hat{\theta}$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \hat{r} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}$$
$$\theta = \tan^{-1}(\sqrt{x^2 + y^2}/z) \qquad \hat{\theta} = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}$$
$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$$

	∂_r	∂_θ	∂_ϕ
$\hat{r}$	0	$\hat{\theta}$	$\sin\theta\hat{\phi}$
$\hat{\theta}$	0	$-\hat{r}$	$\cos\theta\hat{\phi}$
$\hat{\phi}$	0	0	$-\sin\theta\hat{r}$ $-\cos\theta\hat{\theta}$

Cylindrical coordinates

$$x = s\cos\phi \qquad \hat{x} = \cos\phi\ \hat{s} - \sin\phi\ \hat{\phi}$$
$$y = s\sin\phi \qquad \hat{y} = \sin\phi\ \hat{s} + \cos\phi\ \hat{\phi}$$
$$z = z \qquad \hat{z} = \hat{z}$$

$$s = \sqrt{x^2 + y^2} \qquad \hat{s} = +\cos\phi\ \hat{x} + \sin\phi\ \hat{y}$$
$$\phi = \tan^{-1}(y/x) \qquad \hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$$
$$z = z \qquad \hat{z} = \hat{z}$$

	∂_s	∂_ϕ	∂_z
$\hat{s}$	0	$\hat{\phi}$	0
$\hat{\phi}$	0	$-\hat{s}$	0
$\hat{z}$	0	0	0