

Physics 225

Relativity and Math Applications

Fall 2012

Unit 10

The Line Element and Grad, Div, Curl

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Unit 10: The Line Element and Grad, Div, Curl

Section 10.1: Two More Basic Pieces

Our first task today is to cover some pieces of multidimensional or calculus-related math that we use all the time in physics but tend to fall through the cracks between your math and physics courses. Homework 9 started this “**Basic Pieces**” collection with two vector-handling skills:

Piece #0: Vector Magnitudes & Dot Products

① $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}}$ Tools for Dot Products, ● **distribute** the dot ● **geometry** $\rightarrow A B \cos \theta$
esp of vector sums: ● **orthonormal components** $\rightarrow \sum_i A_i B_i$ & Pythagoras

Piece #1: Expressing Vectors in Component Form

① $\vec{v} = \sum_{i=1}^3 (\vec{v} \cdot \hat{r}_i) \hat{r}_i$ for any complete, orthonormal set of unit vectors $\hat{r}_1, \hat{r}_2, \hat{r}_3$

... and now, the new stuff ...

Piece #2: Playing with Differentials

Today’s two basic pieces are related to calculus rather than geometry.

Check the clock: spend no more than 6 minutes total on the following 2 questions, ok? Here we go:

(a) In upper-level physics texts, kinetic energy is referred to as T (not KE). In Physics 325, you’ll see a relation like this, written without comment: $dT = m v dv$. Where does this come from? Any idea? Hint: this would be in the *non-relativistic* part of 325. ☺

(b) Now try this one, which I grabbed at random (in a slightly simplified, i.e., 1D form) from Taylor’s mechanics text: A particle of mass m moves in the x direction under the influence of a force F that also points in the x direction. A useful relation for this system is $dT = F dx$. Can you derive *that*? Start with the relation $dT = m v dv$ from (a), even if you didn’t manage to derive it.

Success? Here are the derivations, please read them carefully:

- for (a): $T = \frac{1}{2}mv^2 \rightarrow \frac{dT}{dv} = mv \rightarrow dT = mv dv$
- for (b): $dT = m v dv = m v dv \frac{dt}{dt}$... now rearrange $\rightarrow dT = m \frac{dv}{dt}(v dt) = ma(dx) = F dx$

Hold. The. Phone. *Is this legal??* We are merrily rearranging differentials like dx and dt just like they were normal variables. Don't differentials appear in *derivatives only*? Can we really break up a derivative like $v = dx / dt$ and start playing around with dx and dt on their own?

Yep! In physics that is. And there's a good reason for it: physics is smooth. In the large majority of situations, the functions we work with in physics are well-behaved, continuous functions because *nature* is generally well-behaved and continuous. Also, remember what a derivative *is*: $x'(t) = dx / dt$ is just a ratio. It's the ratio $\Delta x / \Delta t$, run to the limit where both Δx and Δt are vanishingly small. Since you can take that limit at any time, go ahead and treat differentials df exactly like finite numbers Δf while you are doing your calculation. When you're done with your algebra, mentally turn all the Δ 's back into d 's and take the limit that they're infinitesimally small.

Here's Piece #2 in a box, for reference:

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$\frac{df}{dx} = g$ is equivalent to $df = g dx$	as long as the functions f and g are smooth
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In words: As functions of physical interest are almost always smooth (continuous), you can treat infinitesimal differentials df like finite numbers Δf \rightarrow you can manipulate them algebraically as you like – rearrange, regroup, etc – then mentally return them to d 's and derivatives when you're done.

(c) A particle moves along the x -axis with a position-dependent acceleration $a(x) = Cx$, where C is a constant with the appropriate units. At time $t = 0$ the particle is at rest at position $x = 0$. Calculate $v(x)$ = the particle's velocity as a function of position.

TACTICS: The second bullet at the top of the page shows a key trick in the “playing with differentials” game \rightarrow We had a differential dv that we didn't want in our answer ... to get rid of it, we multiplied by dt / dt (which is 1, so nothing changes), then regrouped the differentials into new derivatives (e.g. dv / dt) corresponding to quantities we *did* want ($dv / dt =$ acceleration a).
Tactic summary: **Multiply by $d<blah> / d<blah>$, then regroup.** Apply this trick to solve part (c)!

Self-check! The answer is $v(x) = \sqrt{C} x$. If that's not what you got, give it another try.

Piece #3: The 3D Chain Rule

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$$df(x_1, \dots, x_n) = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$$

where the variables x_1, \dots, x_n are all *independent*

In words: The change df that occurs in a function f when the independent coordinates x_i on which it depends are changed by amounts dx_i is just sum of each change dx_i times the relevant slope $\partial f / \partial x_i$.

(d) This one is best explained by going straight to an example. A section of a particle accelerator is occupied by an electric field that produces an electric potential $\phi(x, y, z) = Ax + Bxy + Cz$, where A , B , and C are constants with appropriate units. (Phi for potential?? See¹) A proton is sent through the accelerator along a specific trajectory: the proton is confined to the plane $y = 2$ and its speed is $\vec{v}(t) = v_x(t) \hat{x} + v_z(t) \hat{z}$, where $v_x(t)$ and $v_z(t)$ are known functions of time. Calculate $\phi(t)$ = the electric potential seen by the proton as a function of time. Your answer will be an *integral* over time, with the functions $v_x(t)$ and $v_z(t)$ in the integrand. (It will be an integral that you could do if you knew those functions.) If you're really stuck after a couple of minutes, hints are below².

Back to the 5 Keys of Vector Calculus

With those tools in hand, we can now return to our study of 3D calculus and its use with the **Cartesian**, **spherical**, and **cylindrical** coordinate systems. Our roadmap is to cover in turn the five key elements of all coordinate system. We did keys 1,2,3 in Unit 9 and Lecture 10 ... onwards!



1. **The Definitions** of r_i : the transformation equations to/from Cartesian coordinates x_i
2. **The Unit Vectors** \hat{r}_i : they're position-dependent in curvilinear coordinate systems!
3. **The Position Vector** \vec{r} : driving directions to the spacepoint with coordinates (r_1, r_2, r_3)
4. **The Line Element** $d\vec{l}$: turning coordinates into distances
5. **The Gradient** $\vec{\nabla}$ and the other 3D differential operators, div and curl

¹ We're using ϕ for electric potential instead of V to avoid confusion with velocity v . I know ϕ makes you think of an angle, but it is actually used for potential quite often in advanced texts.

² Hint 1: Start by applying the 3D chain rule to the scalar field $V(x, y, z)$ and see what that gives you.

Hint 2: We need to get *time* involved \rightarrow use the "*multiply by $d<blah> / d<blah>$* " tactic from Basic Piece #2!

Section 10.2: The Fourth Key → The Line Element $d\vec{l}$



The **line element** $d\vec{l}$, also called the **infinitesimal displacement vector**, is defined as follows:

The **line element** $d\vec{l}$ tells you how far and in which direction you move when you start at a point (r_1, r_2, r_3) of your coordinate system then shift all your coordinates r_i by a little bit dr_i .

Here is the Cartesian line element:

$$d\vec{l} = dx \hat{x} + dy \hat{y} + dz \hat{z}$$

In words:

“If you move from some point (x, y, z) to the point $(x + dx, y + dy, z + dz)$, your spatial displacement is $d\vec{l} = dx \hat{x} + dy \hat{y} + dz \hat{z}$.”

Remember the **position vector** \vec{r} (the Third Key)? It provides driving directions from the origin to the point (r_1, r_2, r_3) of your coordinate system. The line element provides driving directions for *changes* in your coordinates (dr_1, dr_2, dr_3) . Then why on earth are we calling it $d\vec{l}$ instead of $d\vec{r}$?

This is important:

$$d\vec{l} \text{ is } d\vec{r}, \text{ in every sense.}$$

The symbol $d\vec{l}$ is chosen because it avoids the overused letter r , which would cause us serious notational trouble in about 20 seconds.

We must find the spherical & cylindrical line elements. They all have this form:

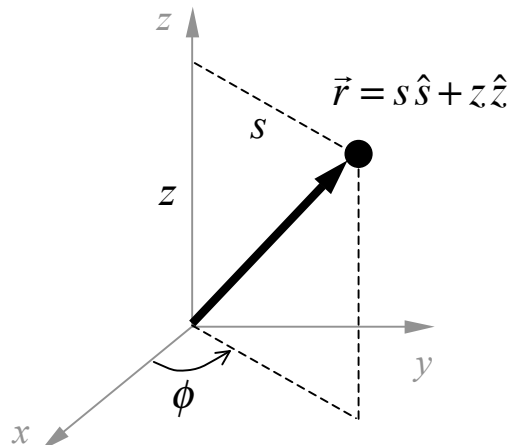
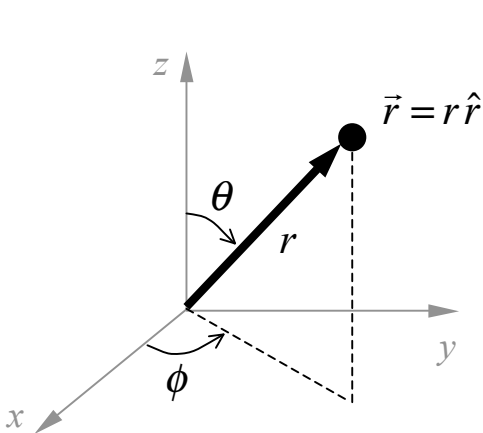
$$d\vec{l} = \sum_{i=1}^3 dl_i \hat{r}_i$$

In words: the line element’s components dl_i specify the **distance you travel** when you **change coordinate i** by dr_i (from r_i to $r_i + dr_i$).

We’ll figure out these dl_i geometrically. Key point: in general, dl_i is **not** dr_i . The vector $d\vec{l}$ does mean “ $d\vec{r}$ ” = differential change in position. However, its components dl_i are *physical distances* while the symbols dr_i are *coordinate changes*, and not all coordinates have units of distance.

(a) Using geometry, fill in the blanks to complete the spherical and cylindrical line elements.


- Spherical: $d\vec{l} = \underline{\hspace{2cm}} dr \hat{r} + \underline{\hspace{2cm}} d\theta \hat{\theta} + \underline{\hspace{2cm}} d\phi \hat{\phi}$
- Cylindrical: $d\vec{l} = \underline{\hspace{2cm}} ds \hat{s} + \underline{\hspace{2cm}} d\phi \hat{\phi} + \underline{\hspace{2cm}} dz \hat{z}$



(b) The terms “ $______ dr_i$ ” in those expressions are precisely dl_i : the components of the line element. It’s very important that you got the correct values! Please run through these checks:

- Here are the answers for the first two terms: $dl_r = dr$ and $dl_\theta = r d\theta$. Is that what you got?
- If you got a different result for dl_θ , ponder the correct answer $r d\theta$... then look back at the picture and think about the concept of arc length. Does it make sense now?
- The “ $______$ ” prefactors for dl_θ and dl_ϕ are not the same; is that what you found?
- Try the limiting cases $\theta = 0$ and $\theta = \pi/2$. Do your answers make sense at these angles?

If any of these checks fail, revise your answers to (a) as necessary ... we’re gonna need them!

Section 10.3: The Fifth Key → The Gradient $\vec{\nabla}$ and its 3D friends Div and Curl 

(a) The fundamental differential operator in 3D is the gradient $\vec{\nabla}$. Do you recall its physical significance? When you apply it to a scalar field $V(\vec{r})$, what does it tell us about that field?

- The direction of $\vec{\nabla}V$, evaluated at any point \vec{r} , tells us the _____ of V at the point \vec{r} .
- The magnitude of $\vec{\nabla}V$, evaluated at any point \vec{r} , tells us the _____ of V at the point \vec{r} .

In Cartesian coordinates, the gradient operator is:

$$\vec{\nabla} = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z}$$

(b) Now suppose your scalar field is expressed in spherical coordinates: $V(r, \theta, \phi)$. How do we calculate $\vec{\nabla}V$?

Here is what everyone guesses at first: if the Cartesian gradient is what’s in the box above, then the

spherical gradient must be $\vec{\nabla}V = \hat{r} \frac{\partial V}{\partial r} + \hat{\theta} \frac{\partial V}{\partial \theta} + \hat{\phi} \frac{\partial V}{\partial \phi}$. Right ?

Take a hard look at this “obvious” expression. Do you see why it cannot possibly be correct?

(c) The gradient is the 3D version of “slope”. Here is a symbolic representation of the i^{th} component

of that slope: $(\vec{\nabla}V)_i = \lim_{\Delta r_i \rightarrow 0} \frac{\Delta V}{\Delta l_i}$, where Δl_i is the **distance** you travel when you change the

coordinate r_i by an amount Δr_i . That’s what a slope is: the change in a function divided by the change in position that produced it. With your new knowledge about the line element, can you figure out the correct form of the gradient in both spherical and cylindrical coordinates?

Drum roll ... the correct **gradient** formulas are:

$$\vec{\nabla} = \hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi}$$

$$\vec{\nabla} = \hat{s} \frac{\partial}{\partial s} + \frac{\hat{\phi}}{s} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial z}$$

Time to put Keys 4 and 5 together. Here are the correct answers for the **line elements** in spherical and cylindrical coordinates:

$$d\vec{l} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}$$

$$d\vec{l} = ds \hat{s} + s d\phi \hat{\phi} + dz \hat{z}$$

Did you get them correct too? And do you see how they work together? → The three components of the line element are exactly the *spatial displacements* you need to make a 3D slope, so they provide the *denominators* of the gradient operator.

Discussion: It's crucial that all four of the above formulae make *complete intuitive sense* to you. If you're the slightest bit unclear about the *utter obviousness* of these four expressions, **ask!!!**

Two weeks ago, you learned the most common physics application of the gradient: obtaining the force $\vec{F}(\vec{r})$ on an object at a location \vec{r} from its potential energy map $U(\vec{r})$: $\vec{F} = -\vec{\nabla}U$. Dividing both sides of this relation by the charge q on a test charge, we get the relation between electric field and electric potential: $\vec{E} = -\vec{\nabla}V$.

(c) In Physics 212, you learn that the electric potential caused by a point charge q located at the origin is $V(\vec{r}) = kq/r$. Using the gradient in spherical coordinates, obtain the electric field due to this point charge. Piece of cake!

That's it for the gradient! On the next page, we'll be taking partial derivatives of vector fields, to construct two more differential operations. It's crucial that you realize what a vector field is: it is three scalar fields — one for each component — that are each multiplied by a unit vector, then added together. In Cartesian, e.g., a vector field is written like this:

$$\vec{E}(x,y,z) = E_x(x,y,z) \hat{x} + E_y(x,y,z) \hat{y} + E_z(x,y,z) \hat{z}$$

When we take partial derivatives of \vec{E} we'll have to consider $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ of *each component* E_x , E_y , and E_z → that's 9 partial derivatives in all. At this point, you might appreciate some **shorthand notation**: $\partial_x f$ is often used as shorthand for $\partial f/\partial x$. Instead of considering all 9 partials $\partial_i E_j$ independently, we'll work with them in two groups: 3 of them will appear in an operation we call the **divergence** $\vec{\nabla} \cdot \vec{E}$ and the other 6 will appear in the **curl** $\vec{\nabla} \times \vec{E}$. You'll see!

The **divergence** of a vector field \vec{E} is obtained by taking the dot product of the gradient operator and the field: $\vec{\nabla} \cdot \vec{E}$. In Cartesian coordinates, we have

$$\vec{\nabla} \cdot \vec{E} = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot (E_x \hat{x} + E_y \hat{y} + E_z \hat{z}) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

In our slick shorthand notation, the formula is nice and compact:

$$\vec{\nabla} \cdot \vec{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z$$

Like any dot product, the divergence produces a scalar result.

(d) Calculate the divergence $\vec{\nabla} \cdot \vec{E}$ of this vector field: $\vec{E}(x,y) = y^2 \hat{x} + 2xy \hat{y} + z^2 \hat{z}$.

On to spherical coordinates! In this system we express our field via its r , θ , and ϕ components:

$$\vec{E} = E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi},$$

where each component E_i is a separate function of (r, θ, ϕ) . Before we get to the divergence, let's make sure we know what such an expression means: let's write some familiar fields in this notation.

The Physics 212 formula sheet has several electric and magnetic fields, created by charge and current distributions that have sufficiently high symmetry to be calculated using Gauss' Law or Ampere's Law in integral form. Here are five of them:

$$E = \frac{q}{4\pi\epsilon_0 r^2} \quad E = \frac{\sigma}{2\epsilon_0} \quad E = \frac{\lambda}{2\pi\epsilon_0 r} \quad B = \frac{\mu_0 I}{2\pi r} \quad B = \mu_0 nI$$

(e) Oh dear! These are only magnitudes ... and the variable "r" is being horribly abused, meaning different things in different formulae! → Fix up the five familiar³ fields above with a good choice of coordinate system and with unit vectors to turn them into proper vector fields $\vec{E}(\vec{r})$ or $\vec{B}(\vec{r})$.

(f) Now for the **divergence** in spherical coordinates. Based on the Cartesian example at the top of the page and your knowledge of the gradient in spherical coordinates, take your best guess for what $\vec{\nabla} \cdot \vec{E}$ is in terms of the partial derivatives $\partial_r, \partial_\theta, \partial_\phi$ of the components E_r, E_θ, E_ϕ .

³ Not so familiar any more? The 5 fields refer to (*not* in order): an infinite wire, and infinite line of charge, an infinite sheet of charge, a charged sphere or point, and an infinitely long solenoid. I'm sure you can spot which is which.

Check out the correct formula:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 E_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta E_\theta) + \frac{1}{r \sin \theta} \frac{\partial E_\phi}{\partial \phi}$$

!!!!!!! How can this be? The gradient is just a dot-product of two vectors ... where did those additional r^2 and $\sin \theta$ factors come from?? I bet you have already figured out the reason: the spherical unit vectors we used to write $\vec{E} = E_r \hat{r} + E_\theta \hat{\theta} + E_\phi \hat{\phi}$ are position-dependent, and therefore have derivatives. You see the problem? The derivatives in the gradient operator must be applied to *both* the field components E_i and the unit vectors! Aieee ...

Mastery: You have all the skills necessary to derive the bizarre-looking formula above. On this week's homework, you'll derive the divergence formula in cylindrical coordinates, which requires the same procedure but is a bit shorter. I urge you to try it out for the spherical formula as well some time! When you succeed, you will truly be a **grand master of 3D vector calculus**. ☺

The third differential operation of 3D calculus is the **curl**. It is also formed by applying the gradient operator to a vector field, but this time using a cross-product instead of a dot product $\rightarrow \vec{\nabla} \times \vec{E}$.

As with $\vec{\nabla} \cdot \vec{E}$, the component-by-component form of the curl in Cartesian coordinates is exactly what you'd expect, while in spherical and cylindrical coordinates it's a giant mess. In fact, it's a longer mess: the curl produces a vector result, which means three components. You can derive these expressions in the same way as indicated above — using the derivatives of the unit vectors, which you now know how to obtain — but there are slicker ways to obtain their forms. We'll come to those, and to the *physical significance* of the divergence and curl, in the coming week.

The most important thing to take away from this page is the following:

Div and curl in spherical & cylindrical coordinates are not obvious

In this one and only one case, my advice to you is **never guess, always look them up**.

If you flip to the back page of the unit, you will find a tear-off sheet \rightarrow our master formula sheet for 3D calculus. Have a look: you'll find grad, div, curl, the line element, the derivatives of the unit vectors, and more. We haven't derived everything on there yet, but we're getting very close!

Section 10.4: Velocity in Spherical & Cylindrical Coordinates

Here's a great exercise to develop your mastery of 3D calculus: calculate the velocity $\vec{v} = d\vec{r} / dt$ of a moving particle in spherical coordinates. This formula appears on the Physics 325 formula sheet without derivation → perfect example of the purpose of this course. With the particle's path described in spherical coordinates by the functions $r(t)$, $\theta(t)$, and $\phi(t)$, its velocity is:

$$\vec{v} \equiv \dot{\vec{r}} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta} + r \sin\theta \dot{\phi} \hat{\phi}.$$

No box around this one → you're going to derive it! In this expression, we've introduced **dot-notation**: a dot over a function denotes its **time derivative**. It's very useful shorthand in mechanics where most of the derivatives of interest *are* time derivatives.

There are multiple ways to derive this relation, and they're all instructive. You'll need several formulas from our growing collection (e.g. unit vector definitions and derivatives), so tear off the back page: they're all there. You may also need to consult the Basic Pieces from this unit's intro.

(a) The first derivation is the faster one, and definitely the one to use if you don't have a formula sheet handy. Think about the meaning of velocity: it's *change in position* over change in time. "Change in position" makes us think of the *line element*, doesn't it? Figure out how to use the line element to construct the velocity vector. If you've mastered those Basic Pieces, this derivation is very fast. Go go go!

(b) The second derivation is more formal: start from the definition $\vec{v} = d\vec{r} / dt$ and calculate that time derivative explicitly. You'll need to put in something sensible for the \vec{r} in the numerator⁴, you must remember how spherical unit vectors behave⁵, and you'll need the 3D chain rule as the spherical unit vectors are position-dependent *vector fields*. Ready? Set? Go! Derive that formula!

⁴ The Third Key = the position vector!

⁵ The Second Key = the unit vectors ... many of which are *position-dependent* and therefore have *derivatives*. You'll find all the unit vector derivatives in compact tables on the back page of our 3D calculus formula sheet.

(c) Using any method you like, determine the formula for velocity in cylindrical coordinates.

(d) Calculate the angular momentum vector $\vec{L} = \vec{r} \times \vec{p}$ in cylindrical coordinates for a moving particle of mass m whose path is described by the functions $s(t)$, $\phi(t)$, $z(t)$.

As with your previous answers, this one must be written entirely in terms of the particle's coordinates s , ϕ , and z and/or their time-derivatives (as well as the particle's mass m in this case). The cylindrical unit vectors will also appear, of course.

(e) That is one complicated formula! To understand it, let's try a simple limiting case: a particle moving in a circle of radius a in the plane $z = z_0$ with constant angular velocity ω . This motion is described by the coordinate functions $s(t) = a$, $\phi(t) = \omega t$, $z(t) = z_0$. What does your angular momentum expression give you in this special case?

(f) Even for such a simple path, your \vec{L} will have two components. Can you interpret them? To figure out why there are two, ask yourself this question: how could you simplify the particle's path even further to reduce \vec{L} to a *single* cylindrical component?