Physics 225
Relativity and Math Applications
Fall 2012

Unit 11
Multidimensional Integration I

N.C.R. Makins

University of Illinois at Urbana-Champaign

©2010

Unit 11: Multidimensional Integration

Today, we turn from the differential to the integral side of 3D calculus. We will learn how to set up line, surface, and volume integrals based on a verbal description of a physical problem. Multidimensional integration is a universal tool that appears in every area of physics.

First, notation and definitions. All multi-D integrals look like this $\rightarrow$ $\int_{\mathbb{R}^n} F(r_i) \otimes d\mathbb{R}^n$
Here's what this generic notation means:

- $\mathbb{R}^n$ is the **region of integration**. The superscript n (it's a label, not a power) indicates the <u>dimensionality</u> of the region $\rightarrow \mathbb{R}^1$ means a line, $\mathbb{R}^2$ a surface, $\mathbb{R}^3$ a volume.

- $d\mathbb{R}^n$ is the **differential of length/area/volume** over which you will integrate:
    $d\mathbb{R}^1$ means differential of path-length, $\vec{dl}$; its direction is <u>tangent</u> to the path.
    $d\mathbb{R}^2$ means differential of area, $\vec{dA}$; its direction is <u>perpendicular</u> to the surface.
    $d\mathbb{R}^3$ means differential of volume, dV; it is always a scalar (no direction).

- $F(r_i)$ is the **function** you want to integrate over the region $\mathbb{R}$. It can be a scalar field or a vector field. The (r_i) argument just indicates that the function is expressed in a particular **coordinate system** r_i where $i = 1,2,3$.

- The symbol $\otimes$ denotes a **generic multiplication**, of an unspecified type. If F or $d\mathbb{R}^n$ is a scalar, $\otimes$ is ordinary multiplication; if they're both vectors, $\otimes$ is a dot- or cross-product.

- The integral itself is, in general, a **multiple integral**. Path integrals are single integrals $\int_{\mathbb{R}^1} ...$, area integrals are double integrals $\iint_{\mathbb{R}^2} ...$, volume integrals are triple integrals $\iiint_{\mathbb{R}^3} ...$

Next we come to The Procedure: the step-by-step formalism for setting up such integrals. I have summarized this procedure in its entirety on the back page of this unit. Please

tear off the back page

now. It's our roadmap for all the exercises coming up. The key to The Procedure is this:

parametrizing (i.e., expressing) **everything that varies** in your n-dimensional integral
in terms of <u>n integration parameters</u> (IPs) of your choice.

The IPs are <u>the variables over which you will integrate</u>. I call them **sweeping parameters** because, as you vary them, they *sweep out* the path/surface/volume over which you're integrating. They can be coordinates from your chosen coordinate system, or not. We'll label these parameters u_j, where j runs from 1 to n; if we need individual letters for these u_j we'll take them from the list (u,v,w). The role of the IPs is clear when you see what your multidimensional integrals will finally look like:

- dimensionality n: $\mathbb{R}^1$ <u>path integral</u> $\mathbb{R}^2$ <u>surface integral</u> $\mathbb{R}^3$ <u>volume integral</u>

- the n IPs: 1 IP $= u$ 2 IPs $= u, v$ 3 IPs $= u, v, w$

- what your integral $\int_{u_i}^{u_f} G(u)\, du$ $\int_{v_i}^{v_f} \int_{u_i}^{u_f} G(u,v)\, du\, dv$ $\int_{w_i}^{w_f} \int_{v_i}^{v_f} \int_{u_i}^{u_f} G(u,v,w)\, du\, dv\, dw$
   will look like:

Section 11.1: Step 1 → Parametrize $\mathbb{R}^n$

Step 1 is to write down a <u>mathematical description of the integration region $\mathbb{R}^n$</u> = the path / surface / volume over which you will integrate. How do we do that? → Please read Step 1 from The Procedure page now. What you do is express all the coordinates r_i of your coordinate system as functions of 1 parameter (for a line) or 2 parameters (for a surface) or 3 parameters (for a volume). In later steps, these parameters will serve as your <u>IPs</u> = your integration parameters … or to use my favorite term, your *sweeping parameters*.
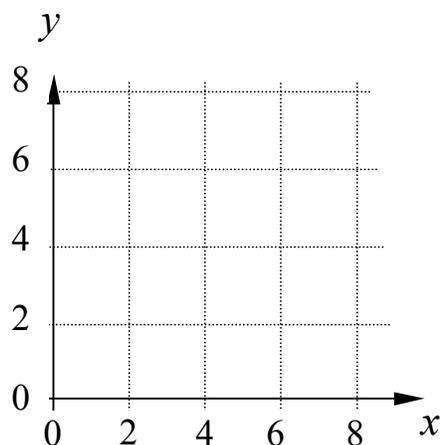
First, I'll give you a parametrization of a region $\mathbb{R}$ and you figure out what it is!

(a) This example is in 2D space. The integration parameter (IP) is u. The parametrization is:

      Coordinates: $x(u) = 1 + 3u$, $y(u) = u^2$

      Bounds: $u: 0 \rightarrow 2$

Sketch this object!

(b) Here is another object in 2D space. This time, we are using one of the actual coordinates of the Cartesian system as our sweeping parameter — generally an *excellent* thing to do if you can.

      Coordinates: $y(x) = 6 - x$

      Bounds: $x: 6 \rightarrow 2$

Plot this object on the same graph above.

(c) One more in 2D space, but in polar coordinates (s,ϕ) and removing the "hint words":

$$s(u) = 6\sqrt{u}$$

$$\phi(u) = \frac{\pi}{2}$$

$$u: 0 \rightarrow 1$$

Plot this curve on the graph above. If you're getting something complicated you've made an error … the expressions may look strange but they describe a very simple path!

(d) Onto 3D. What is this object? (*a* and *b* are constants) Sketch, then describe it in words.

$$s(\phi) = a$$
$$z(\phi) = b\phi / 2\pi$$
$$\phi: \ 0 \rightarrow 2\pi$$

(e) And this?

$$z(x,y) = ax$$
$$x: 0 \rightarrow b$$
$$y: 0 \rightarrow b$$

(f) Your turn! Parametrize a circle of radius *a* that lies in the *xy* plane and is centered on the origin.

(g) Did you just parametrize a filled circle (an area) or an open circle (a path)? Whichever one you did, now write in the *other* one next to it. ☺ (Only a tiny modification is required.)

(h) Parametrize a thin hemispherical shell (i.e. a surface) of radius *a* that is centered on the origin but confined to the region *z* > 0. For practice, please do this in both spherical and cylindrical coordinates (i.e. provide two parametrizations).

(i) Last example: a volume. Parametrize a thick-walled cylindrical tube that has length *L*, inner radius *a*, and outer radius *b*. Pick whatever coordinate system and IPs you like. This is a solid, so guess what: *all three coordinates will be IPs*, no matter what coordinate system you pick. The mathematical description of a volume consists entirely of specifying the bounds on the IPs.

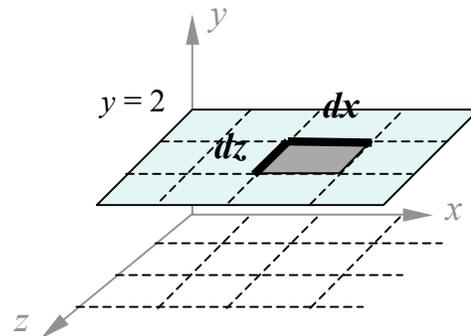**Section 11.2: Step 2 → Parametrize the Differential $d\mathbb{R}^n$**

Next, we must <u>express $d\mathbb{R}^n$ in terms of our chosen IPs</u>. $d\mathbb{R}^n$ is the differential element of path ($d\vec{l}$), area ($d\vec{A}$), or volume (dV) in our multi-D integral. Please read Step 2 on the tear-off page now. Two methods are described. The first one is pure geometric thinking and is all you need for the majority of undergrad problems. How does Method 1 work? → <u>Draw! a! sketch!</u>, then use it to answer the geometric question that defines $d\mathbb{R}^n$:

> If I increase each of my n integration parameters u_j by a little bit du_j,
> *how much length | area | volume do I sweep out?* → **That's $d\mathbb{R}^n$.**

For path or surface integrals, $d\mathbb{R}^n$ is a *vector* differential ($d\vec{l}$ or $d\vec{A}$). Many problems only require the magnitude of $d\mathbb{R}^n$ ($dl \equiv |d\vec{l}|$ or $dA \equiv |d\vec{A}|$), which simplifies things; if you do need the direction, express it using the unit vectors $\hat{r}_i$ of your coordinate system. Every region is different, forget about cooked formulas → your sketch and your knowledge of geometry are the tools you need to find $d\mathbb{R}^n$ for each situation. I will give you two worked examples first.

<u>Example 1</u>: A rectangular plastic plate lies in the plane $y = 2$, with opposite corners at the points (x, y, z) = (0, 2, 0) and (a, 2, b). The plate is in a region of ionized gas and picks up a non-uniform surface charge density of $\sigma = K\, xz^2/y$, where K is a constant with units of C/m^4. Find the total charge on the plate.

This problem is solved by the area integral $Q = \int\limits_{\text{plate}} \sigma \, dA$.

- Step 1: Parametrize the plate.
    A flat plate parallel to the xz-plane? Too easy:
        $y(x,z) = 2$
        $x: 0 \rightarrow a$
        $z: 0 \rightarrow b$

- Step 2: Parametrize the differential dA.
    We are sweeping over our plate with IPs x and z. When we move a tiny bit dx and a tiny bit dz along the plate, at the same time, *how much area do we sweep out?* → the tiny rectangular area $dx\, dz$. That's our differential of area: $dA = dx\, dz$. FYI: if the problem required the vector differential $d\vec{A}$, it must point perpendicular to the surface, so we would write $d\vec{A} = dx\, dz\, \hat{y}$. (The $-\hat{y}$ direction is also fine as this is an open surface.)

- Step 3: Construct the integral.
    The integrand is $\sigma\, dA$, where $\sigma = K\, xz^2/y$. We must express that *entirely* in terms of the IP's (x and z) and constants … so we have to get rid of y. Our parametrization of the plate (Step 1) provides us with the coordinate function we need: $y(x,z) = 2$.
   Thus $Q = \int\limits_{0}^{a}\int\limits_{0}^{b} \frac{Kxz^2}{2}\, dx\, dz = \frac{K}{2} \int\limits_{0}^{a} x\, dx \int\limits_{0}^{b} z^2\, dz = \frac{K}{2} \left.\frac{x^2}{2}\right|_{0}^{a} \left.\frac{z^3}{3}\right|_{0}^{b} = \frac{K}{12} a^2 b^3$.

The next example involves spherical coordinates. To figure out differentials of path/area/volume you have to visualize <u>how much distance you sweep out</u> when you <u>increase each of your IPs by a little bit du_i</u> … well, if your IPs are coordinates from one the three familiar coordinate systems, you have already figured this out! This exact information is **<u>encoded in the line element $d\vec{l}$</u>** of each coordinate system. From Unit 10:

$$d\vec{l} = \sum_i dl_i\,\hat{r}_i \qquad \begin{array}{l}\text{where } dl_i \text{ is the } \textit{distance you traverse} \text{ when you}\\ \textit{change coordinate i} \text{ from } r_i \text{ to } r_i + dr_i.\end{array}$$

The dl_i's – the components of the line element – are *exactly* what you need to build path/area/volume differentials $d\mathbb{R}^n$. For problems with high symmetry, $d\mathbb{R}^n$ is just a product of n dl_i's. For reference:

> Cartesian: $d\vec{l} = (dx)\hat{x} + (dy)\hat{y} + (dz)\hat{z}$
>
> Spherical: $d\vec{l} = (dr)\,\hat{r} + (r\,d\theta)\,\hat{\theta} + (r\sin\theta\,d\phi)\,\hat{\phi}$
>
> Cylindrical: $d\vec{l} = (ds)\,\hat{s} + (s\,d\phi)\,\hat{\phi} + (dz)\,\hat{z}$
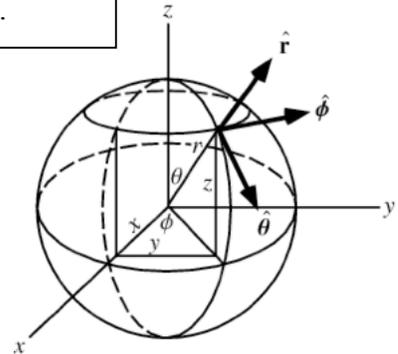
… and what a waste of space that was. These expressions are such *utterly obvious* consequences of the picture that we are just wasting paper writing them down again. ☺

<u>Example 2</u>: Find the surface area of a hemisphere of radius a.

• Step 1: Parametrize a hemispherical surface centered on the origin. You did this already:
> $r(\theta,\phi) = a$
> $\theta: 0 \to \pi/2$
> $\phi: 0 \to 2\pi$

• Step 2: Parametrize the differential dA.
> dA is how much area we sweep out when we move a little bit $d\theta$ and a little bit $d\phi$ on our surface. Those tiny coordinate movements will speep out an area that is so tiny we can *approximate it as a rectangle*; the area is then the product of the lengths of the two sides. Consult the spherical line element to translate the coordinate shifts $d\theta$ and $d\phi$ into physical displacements dl_θ and dl_ϕ:
> $$dA = (dl_\theta)(dl_\phi) = (r\,d\theta)(r\sin\theta\,d\phi) = r^2\sin\theta\,d\theta\,d\phi.$$

• Step 3: Construct the integral.
> The surface area is $\int dA = \int_0^{2\pi}\int_0^{\pi/2} r^2\sin\theta\,d\theta\,d\phi$. <u>Vital check</u> before you integrate:
> *do we have <u>everything</u> expressed in terms of our <u>sweeping parameters</u> and <u>constants</u>?*
> Not quite: r is not yet specified! Is it a variable? Does it vary with θ or ϕ? What *is* it? Our coordinate functions from step 1 that parametrize the hemisphere answer these questions : $r = a$. A constant, hurray! ☺ Thus, the surface area of the hemisphere is:
> $$\int dA = \int_0^{2\pi}\int_0^{\pi/2} a^2\sin\theta\,d\theta\,d\phi = a^2\int_0^{\pi/2}\sin\theta\,d\theta\int_0^{2\pi} d\phi = a^2(-\cos\theta)\big|_0^{\pi/2}(\phi)\big|_0^{2\pi} = 2\pi a^2$$

Your turn! All the problems in this section involve regions that follow the axes or planes one of the standard coordinate systems, so they can all be done with Method 1. Method 1 is the fastest, and by far the best to build up your physical intuition. By all means, try Method 2 if you like (the formalism route) … you'll see that it just collapses to Method 1 in cases like these.[1]

The crux of setting up multi-D integrals is picking your coordinate system, picking your n sweeping parameters, and figuring out the corresponding differential $d\mathbb{R}^n$. Here's the all-important definition of $d\mathbb{R}^n$ once more:

> If I increase each of my n integration parameters u_j by a little bit du_j,
> *how much length | area | volume do I sweep out?* → **That's $d\mathbb{R}^n$.**

Off you go!

(a) Parametrize the region $\mathbb{R}$ and find the differential $d\vec{l}$ for a <u>circular path</u> of radius a that lies in the xy plane and is centered on the origin. Use spherical coordinates $r_i = \{r, \theta, \phi\}$. The choice of IP is up to you, but it will help if you can pick one of those coordinates: if you can, $d\vec{l}$ for this path is trivial as it is just the corresponding component of the spherical line element (i.e., it's a problem you've already solved!). So off you go → find $d\vec{l}$ for a circle in the xy plane.

(b) Let's do an actual integral with that nice differential. Apply Step 3 to calculate the circumference of the circle via $C = \oint_{circle} dl$. Note that you need the *magnitude $dl = \left|d\vec{l}\right|$* in this integrand, not the vector $d\vec{l}$.

---

[1] When do you *need* Method 2? → You need it when you can't figure out $d\mathbb{R}$ from a sketch. Generally, that happens when your region of integration *doesn't follow the axes of planes of a known coordinate system*. We'll come to such regions in a next week!

(c) Parametrize the region $\mathbb{R}$ and differential dV for a <u>hollow sphere</u> of inner radius a, outer radius b, and centered on the origin. Use spherical coordinates for r_i.

(d) Parametrize the region $\mathbb{R}$ and vector differential $d\vec{A}$ for the <u>surface of a cylinder</u> of radius a and length $2L$. The cylinder is centered on the origin and its axis of symmetry coincides with the z axis. Use cylindrical coordinates for r_i.

(e) Parametrize the region $\mathbb{R}$ and the vector differential $d\vec{A}$ for a <u>filled quarter-circle</u> of radius a. The circle is centered on the origin and lies in the xy-plane; the quarter-circle of interest is the one in the "third quadrant" $x < 0$ and $y < 0$. You pick the coordinate system. ☺

Section 11.3: Real Integrals Giving Constant Results

In this section we will set up some real physical problems that require the entire procedure. Important: don't waste any time trying to do the actual integration. All the integrals in this section are easily doable, I just want you to spend your time on the physics side of things, which is *setting up* the integral for a given physical situation. Once you've set up the integral for each problem, please move on to the next problem (unless you have lots of experience with integration and can do the integrals quickly). Homework is a better place to practice integration.

As usual, keep your eye open for footnotes with helpful hints, especially if you're stuck.

(a) A hollow cylinder of inner radius a, outer radius b, and length $2L$ carries a non-uniform mass density $\rho_m(\vec{r}) = \rho_0 \dfrac{x^2}{a^2}$, where ρ_0 is a constant with units kg/m^3. The cylinder is centered on the origin and its axis of symmetry runs along the z axis. Calculate the total mass of the cylinder.[2]

➜ **Tips & Tricks**: This is a classic case where the function F you're integrating is given in a coordinate system that is *not* the best choice for parametrizing the integration region $\mathbb{R}$. Which coordinate system to choose? → Always choose the best system for the <u>integration region $\mathbb{R}$</u>, not the integrand. If you don't believe me, try the other choice and see what a mess you get!

---

[2] The integral you should get is doable, as long as you know this famous result: $\displaystyle\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \sin^2\theta \, d\theta = \pi$.

(b) Let's get some vectors involved. Consider a small bead that is confined to move (without friction) along a circle of radius R that is centered on the origin and lies in the xy plane. The bead is given a charge q and the whole apparatus is placed in an electric field $\vec{E}(\vec{r}) = -Cy\hat{x}$. This strange electric field is able to push the charged bead around the circle, and in doing so accomplishes an amount of work $W = \oint_{circle} q\vec{E} \cdot d\vec{l}$. Calculate W.

(c) A solid sphere of radius R is centered at the origin and has a total mass M that is distributed uniformly throughout its volume (i.e., its volume mass density is a constant, $\rho_m = \dfrac{M}{V} = \dfrac{3M}{4\pi R^3}$). Calculate the moment of inertia I of the sphere. The formula for the moment of inertia of any object around an axis $\hat{a}$ is $I_{\hat{a}} = \int dm \, |\vec{r} \times \hat{a}|^2$, where dm represents the differential of mass = each little piece of mass in the object. You can use any axis $\hat{a}$ for a sphere, of course, and you'll get the same answer … so use $\hat{a} = \hat{z}$ to make your life easier.[3] Go to it: calculate $I_{\hat{z}}$ of the sphere!

Hints about the $I_{\hat{a}}$ formula: Confused about the $\vec{r}$? See footnote.[4] Confused about dm? See other footnote.[5] Did you need to read the footnotes? Ask your instructor! The issues addressed in these footnotes need to be *utterly obvious* to you. If they're not, please *ask until they are!*
Self-check: $I = 2MR^2/5$ (You may remember this well-known result from mechanics).

---

[3] The reason $\hat{a} = \hat{z}$ is the best choice is that the z axis is the **axis of symmetry** of both the spherical and cylindrical coordinate systems. We've discussed this in lecture. If you're not sure what we mean by axis of symmetry, by all means try a different axis! Try $\hat{a} = \hat{x}$. You'll quickly see that $\hat{a} = \hat{z}$ produces *vastly* simpler expressions.
[4] What does the symbol $\vec{r}$ *always* mean, especially when it appears in a formula like this without comment? It is the generic position vector $\vec{r}$ from the Third Key of vector calculus.
[5] What to do with dm: remember "Playing with Differentials". dm is a little piece of mass within the sphere. We're given the mass per unit volume ρ of the sphere. What should you multiply with ρ to get a little piece of mass dm? $\rightarrow$ Answer: a little piece of volume. $dm = \rho \, dV$.

➧ **Tips & Tricks**: What happens when your multi-D integral gives you a **<u>vector result</u>**?
(So far we've seen only scalar integrands.) Here's the key: integrals that give you a vector result in
3D are really <u>three separate integrals</u>, one each for the *x*-, *y*-, and *z*- components of the answer.
Integrals are just big fancy sums … well, when you add vectors together, you *know* you always
have to add them *by components*. It's exactly the same with vector integrals: you integrate all the *x*-
components, the *y*-components, and the *z*-components *separately*. The next problem illustrates this
issue.

(d) A rectangular box has its edges aligned with the *x*, *y*, and *z* axes of a Cartesian coordinate
system, and is positioned so that its opposite corners are at the points $(x,y,z) = (0,0,0)$ and (a,b,c).
The box carries a non-uniform mass density per unit volume of $\rho_m(\vec{r}) = \rho_0 z^2 / c^2$ where ρ_0 is a
constant with units kg/m^3. Calculate the position $\vec{r}_{CM}$ of the box's center of mass via the standard

formula $\vec{r}_{CM} = \left. \int\limits_{object} \vec{r}\, dm \middle/ \int\limits_{object} dm \right.$ where *dm* are all the little differential pieces of mass of which the

object is composed.

(e) A cylinder of radius R and length 2L is centered on the origin, and its axis of symmetry coincides with the z axis. The cylinder has a non-uniform mass density $\rho_m(\vec{r}) = \rho_0(1 + \cos\phi)$. Calculate the position $\vec{r}_{CM}$ of the cylinder's center of mass.

➡ **Tips & Tricks**: Clearly, cylindrical coordinates is the way to go here, but _Watch! Out!_ Can you use cylindrical _unit vectors_ in an integrand? Remember: curvilinear unit vectors like $\hat{s}$ and $\hat{\phi}$ are _position-dependent!_ Consider this integral, for example: $\int_0^{2\pi} \sin\phi \; d\phi \; \hat{s}$ around a ring of

radius a. It's dangerously tempting to do the following: $\int_0^{2\pi} \sin\phi \; d\phi \; \hat{s} = \hat{s} \int_0^{2\pi} \sin\phi \; d\phi = 0 \; \dots$

but this is completely wrong. You cannot treat $\hat{s}$ the way you would $\hat{x}$ or $\hat{y}$: you cannot pull $\hat{s}$ outside the integral because $\hat{s}$ _is a function_, it _varies_ over the integral. So how _do_ you evaluate that integral? → **Switch to Cartesian unit vectors**. They are the only ones that never vary. In general,

<p align="center">**you cannot leave _any_ variable unit vectors in an integrand**!</p>

You must re-express any position-dependent unit vectors in terms of the fixed Cartesian directions $\hat{x}$, $\hat{y}$, and $\hat{z}$ before you can integrate (i.e., add them together at different points). The correct solution to the integral above is as follows:

$$\int_0^{2\pi} \sin\phi \; d\phi \; \hat{s} = \int_0^{2\pi} \sin\phi \; d\phi \; (\cos\phi \, \hat{x} + \sin\phi \, \hat{y}) = \hat{x} \int_0^{2\pi} \sin\phi \cos\phi \, d\phi + \hat{y} \int_0^{2\pi} \sin^2\phi \; d\phi = \hat{y}\pi \, .$$

Note that we didn't switch to Cartesian _coordinates_ → there's no need whatsoever to replace ϕ with atan(y/x), which would be a real mess. We merely replaced $\hat{s}$ with $\hat{x}\cos\phi + \hat{y}\sin\phi$.

The Procedure for Multidimensional Integration

$$\int_{\mathbb{R}^n} F(r_i) \otimes d\mathbb{R}^n$$

Definition of the essential word "**PARAMETRIZE**" as used in this context:	Express **all quantities that vary over the integral** in terms of your **integration parameters** (**IP**s) & constants.

1. <u>Parametrize the **Region** $\mathbb{R}^n$</u>

   a. Pick your **coordinate system** $r_i = (x,y,z)$ Cartesian, or (r,θ,ϕ) spherical, or (s,ϕ,z) cylind.

   b. Pick your n **integration parameters** u_j – a.k.a. IPs or "**sweeping parameters**" – that will sweep out the region $\mathbb{R}^n$. If possible, use one or more of your chosen coordinates r_i.

   c. Describe the *shape* of $\mathbb{R}$ by expressing your coordinates r_i as **functions**[6] $r_i(u_j)$ of the IPs

   d. Describe the *edges* of $\mathbb{R}$ by providing **bounds** on each integration parameter u_j

2. <u>Parametrize the **Differential** $d\mathbb{R}^n$</u> using your coordinate system's **Line Element** $d\vec{l}$

   Method 1: Visualization

   $d\mathbb{R}^n = d\vec{l} \,|\, d\vec{A} \,|\, dV$ is **how much space** (length | area | volume) **you sweep out when you increase every IP u_j by du_j.** Figure it out with a sketch and/or the line element $d\vec{l}$ of your coordinate system. This method works best when the integration parameters u_j are actual coordinates r_j.

   Method 2: Formalism

   $$d\vec{l}_{\text{path}} = \frac{d\vec{l}}{du}\,du$$

   $$d\vec{l}_u \equiv \frac{\partial \vec{l}}{\partial u}\,du \;\rightarrow\; \begin{array}{l} d\vec{A} = d\vec{l}_u \times d\vec{l}_v \\[6pt] dV = \left(d\vec{l}_u \times d\vec{l}_v\right)\cdot d\vec{l}_w \end{array}$$

3. <u>Construct the **Integral**</u> expressing *everything* in terms of your **IPs** and **constants**

   Use your **coordinate functions** $r_i(u_j)$ from to express *everything that varies* in the integrand *entirely* in terms of the IPs and constants. Watch out especially for spher/cylind unit vectors!

   Your integral must be **doable** = something you can type into Wolfram Integrator[7], and must **make sense** = give a result that depends only on quantities that *survive the integration*. (Example of nonsense: a final result with an IP left in it!) Proper integrals have this form:

   $\mathbb{R}^1$ path integral

   $$\int_{u_i}^{u_f} G(u)\,du$$

   $\mathbb{R}^2$ surface integral

   $$\int_{v_i}^{v_f}\int_{u_i}^{u_f} G(u,v)\,du\,dv$$

   $\mathbb{R}^3$ volume integral

   $$\int_{w_i}^{w_f}\int_{v_i}^{v_f}\int_{u_i}^{u_f} G(u,v,w)\,du\,dv\,dw$$

   For **vector integrals**, you will get one such scalar integral **per component**.

---

[6] What to call these functions $r_i(u_j)$? **Constraint functions** is a good name, as that's what they do: constrain the coordinates to lie on your region $\mathbb{R}$. I like the descriptive **shape functions**, but we'll go with **coordinate functions**.

[7] Free integration available online at http://integrals.wolfram.com (indefinite integrals only). The new, insanely powerful WolframAlpha can do definite integrals too → see http://wolframalpha.com/examples/Calculus.html