Physics 225
Relativity and Math Applications
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Unit 12
Multidimensional Integration II
& Maxwell's Equations

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Unit 12: Multidimensional Integration II & Maxwell's Equations

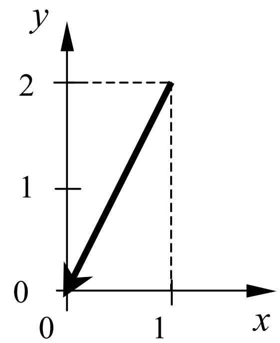
Today we will complete our study of multidimensional integration by covering two types of integrals that are somewhat more complicated than those we practiced last week:

- Funky Regions: cases where the integration region $\mathbb{R}^n$ does not completely follow the natural lines or surfaces of any standard coordinate system.
- Field Integrals: where one must determine a scalar or vector field as a function of a generic *field point* $\vec{r}$ by integrating over many *source points* $\vec{r}_q$.

Our reference sheet for multiD integration in on the back page once again, so go ahead and tear it off if you'd like to refer to it as you work through the exercises. The front side outlines "The Procedure", same as last week. On the back side you'll find a complete collection of our "Tips and Tricks" to address common pitfalls, including those from last week and this week.

Section 12.1: Funky Regions

When our regions of integration do not nicely follow the natural curves or surfaces of our coordinate system, we need **Method 2** from our reference sheet → the one with the hardcore-looking formulas for line/area/volume differentials. Once you work with them, you'll see that these formulas are actually quite obvious applications of two concepts that are now familiar: the line element of a coordinate systems and playing with differentials. Here's an example of Method 2 at work:



Example: Calculate the length $\int dl$ of the line plotted at right.

Step 1: Parametrize the line in Cartesian coordinates:

$$x: 1 \to 0, \qquad y(x) = 2x, \qquad z(x) = 0$$

Here we've chosen the coordinate x as our sweeping parameter.

Step 2: We must get the differential element of path – the line element $d\vec{l} = \hat{x}\,dx + \hat{y}\,dy + \hat{z}\,dz$ – written entirely in terms of our chosen IP: x and its differential dx. The relevant formula is shown in the box below. It is nothing more than *what we just said,* in symbols:

"Express the differential of path ($d\vec{l}$) entirely in terms of your one IP (u and its differential du)".

$$\boxed{d\vec{l}_{\text{path}} = \frac{d\vec{l}}{du}\,du}$$

Applying this sentence-formula to our path we get:

$$d\vec{l} = \frac{d\vec{l}}{dx}dx = \left(\hat{x}\frac{dx}{dx} + \hat{y}\frac{dy}{dx} + \hat{z}\frac{dz}{dx}\right)dx = (\hat{x} + 2\hat{y})\,dx\,.$$

For this problem, we want the magnitude dl which is $\left|d\vec{l}\right| = \sqrt{5}\,dx$.

Step 3: The final integral is $\int_1^0 \sqrt{5}\,dx = -\sqrt{5}$, but we drop the minus as we seek a length.

When exactly must we use the formulas of Method 2 to determine our path/area/volume differentials $d\mathbb{R}^n$? → Whenever we can't figure them out from a sketch ☺ (i.e. Method 1). Last week's problems were all solvable with Method 1, which is the best way to build physical intuition. The situations coming up below have one or both of the following features:

- the functions $r_i(u_j)$ describing the shape of the region of integration are not constants
- the integration parameters u_j are not actual coordinates r_i from your chosen system

(a) Last week you identified the following parametrization as a one-turn helix (a spiral), with radius a and height b:

$$\phi: 0 \rightarrow 2\pi$$
$$s(\phi) = a$$
$$z(\phi) = b\phi / 2\pi$$

Your task today: calculate the total length $\int dl$ of this one-turn spiral.

Time Limit = 5 minutes: If you get stuck, or aren't sure how to proceed, please read through the worked example on the previous page again and try to follow the steps. Keep at it, but if you find yourself hitting the 5 minute mark, please move onward.

Self-Check: How did it go? Compare your solution with this one:
Step 1 is done for you, with a parametrization in cylindrical coordinates and ϕ as the IP. Step 2 requires us to get the cylindrical line element $d\vec{l} = \hat{s}\,ds + \hat{\phi}\,s\,d\phi + \hat{z}\,dz$ for our spiral path written entirely in terms of our integration parameter ϕ. How? Remember "Playing with Differentials" and the trick we learned to obtain derivatives in terms of the variables we want? → Multiply by $d\phi/d\phi$! That's exactly the trick incorporated into our Method 2 formula $d\vec{l}_{\text{path}} = \left(d\vec{l}\,/\,du\right)du$:

$$\begin{aligned}
d\vec{l}_{\text{spiral}} &= \frac{d\vec{l}}{du}du = \left(\frac{ds}{d\phi}\,\hat{s} + s\frac{d\phi}{d\phi}\,\hat{\phi} + \frac{dz}{d\phi}\,\hat{z}\right)d\phi \\
&= \left(\frac{d(a)}{d\phi}\,\hat{s} + a\frac{d\phi}{d\phi}\,\hat{\phi} + \frac{d(b\phi/2\pi)}{d\phi}\,\hat{z}\right)d\phi \\
&= \left(a\;\hat{\phi} + \frac{b}{2\pi}\,\hat{z}\right)d\phi
\end{aligned}$$

- sub in your chosen IP "u" (ϕ) and the line element of your chosen coord system,
- sub in the coordinate functions $r_i(u)$ describing your path,
- and evaluate those derivatives!

All of our coordinate systems are orthogonal, so the magnitude is simple: $dl = d\phi\sqrt{a^2 + b^2/4\pi^2}$.

Step 3 is the integral: $\displaystyle\int dl = \int_0^{2\pi} d\phi\sqrt{a^2 + \frac{b^2}{4\pi^2}} = 2\pi\sqrt{a^2 + \frac{b^2}{4\pi^2}}$.

(b) Here's a trajectory with an IP that has nothing to do with *any* coordinate system:

$$\begin{aligned}
&t\text{: } 0 \to T = 2v_0 \sin\alpha\,/\,g \\
&x(t) = v_0\, t\cos\alpha \\
&y(t) = v_0\, t\sin\alpha - \frac{1}{2}gt^2 \\
&z(t) = 0
\end{aligned}$$

The symbols v_0, α, and g all denote constants, and the sweeping parameter t is *time*. (Do you recognize this trajectory?[1]) Time is a very common choice of IP for describing paths. Your task: set up (but do not calculate!) the integral you would need to find the total length of the trajectory.

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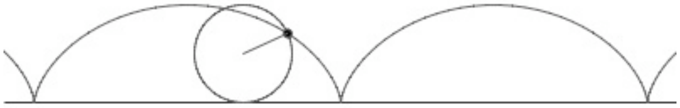
[1] It's the ballistic trajectory of a particle that is launched with speed v_0 at an angle α with the x-axis, then travels under the influence of gravity (which is pointing in the $-y$ direction).

(c) A **cycloid** is the path traced out by a point on a rolling circle. Here is a cycloid parametrized in Cartesian coordinates, but using a sweeping parameter ϕ that also has nothing to do with the coordinate system. (It describes the rotation angle of the circle as it rolls along the x axis.) The symbol R below denotes the radius of the circle and is a constant with units of length.

ϕ: $0 \to 2\pi$
$x(\phi) = R(\phi - \sin\phi)$
$y(\phi) = R(1 - \cos\phi)$
$z(\phi) = 0$



Calculate the total length of one cycle of this cycloid. You should get $8R$. Helpful math formulas:

$$\sin\frac{\phi}{2} = \sqrt{\frac{1-\cos\phi}{2}}$$
$$\cos\frac{\phi}{2} = \sqrt{\frac{1+\cos\phi}{2}}$$

(d) Enough with funky paths, on to funky surfaces! Here's a surface in Cartesian coordinates, where a and b are constant lengths and α and β are constant angles. What is this object?

x: $0 \to a$
y: $0 \to b$
$z(x,y) = x\tan\alpha + y\tan\beta$

(e) To integrate anything over this surface we'll need dA. Our IPs are x and y, so the differential dA represents *how much surface-area we sweep out* when we increase x by dx and y by dy. Hmm … Suppose someone suggests to you that $dA = dx\, dy$ for the above surface (which is a tilted plane, by the way. ☺) Do you agree? If not, is dA for the tilted plane bigger or smaller than $dx\, dy$?

dA is *bigger* than *dx dy!* When you increase x and y, you *increase z too*; geometrically, you move "up" the plane (z direction) as well as "sideways" (x and y) directions, and the result is a swept-out tilted rectangle whose area is bigger than *dx dy*.

$$\boxed{d\vec{A} = d\vec{l}_u \times d\vec{l}_v}$$

$$\boxed{d\vec{l}_u \equiv \frac{\partial \vec{l}}{\partial u}\, du}$$

How do we compute $d\vec{A}$ for surfaces like this? We use the fact that the area vector of a parallelogram with sides $\vec{a}$ and $\vec{b}$ is $\vec{a}\times\vec{b}$. The little pieces of area $d\vec{A}$ in a surface integral are exactly such parallelograms, with sides equal to the spatial displacements we make when we increase our sweeping parameters u and v in little steps du and dv. Let's use the symbols $d\vec{l}_u$ and $d\vec{l}_v$ to denote these displacements. They are just like $d\vec{l}_{path}$:

$d\vec{l}_u$ is *how far you move* along your region of integration $\mathbb{R}$
when you increase your sweeping parameter u by a little bit du

The second boxed formula above shows how to compute that: find the "slope" $\partial\vec{l}\,/\,\partial u$ of your coordinate system's line element with respect to u, then multiply it by du. It is the same operation as we performed for $d\vec{l}_{path}$: basically divide out du, then multiply it back in. ☺ The only difference for a surface is that the coordinates in the line element depend on *two* sweeping parameters (u and v) rather than one (just u). The derivative you get from "dividing out du" is thus a partial derivative rather than a simple 1D derivative.

Let's apply this formalism to our tilted plane. The plane is described in Cartesian coordinates,

$$x\text{: } 0 \to a, \qquad y\text{: } 0 \to b, \qquad z(x,y) = x\tan\alpha + y\tan\beta$$

so we must take the derivatives of the Cartesian line element $d\vec{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$ with respect to our chosen sweeping parameters $u = x$ and $v = y$. Those derivatives are

$$\frac{\partial\vec{l}}{\partial x} = \hat{x}\frac{\partial x}{\partial x} + \hat{y}\frac{\partial y}{\partial x} + \hat{z}\frac{\partial z}{\partial x} = \hat{x} + \hat{z}\tan\alpha \qquad \text{and} \qquad \frac{\partial\vec{l}}{\partial y} = \hat{x}\frac{\partial x}{\partial y} + \hat{y}\frac{\partial y}{\partial y} + \hat{z}\frac{\partial z}{\partial y} = \hat{y} + \hat{z}\tan\beta$$

(f) Finish up the calculation: following our new formulae, take the cross-product of the above "slopes" then multiply it by *dx dy* to get $d\vec{A}$. Finally, calculate the tilted plane's total area.[2]

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[2] Hint: Some integrals require a vector differential like $d\vec{A}$ with direction included, while some require its <u>magnitude</u>, $dA \equiv |\,d\vec{A}\,|$. Which one is needed to calculate a surface area?

Self-check: the tilted plane's surface area is $ab\sqrt{1+\tan^2\alpha+\tan^2\beta}$.

(g) Consider the surface of a cone with half-angle α, expressed in cylindrical coordinates:

u: $0 \rightarrow L$
ϕ: $0 \rightarrow 2\pi$
$s(u,\phi) = u\sin\alpha$
$z(u,\phi) = u\cos\alpha$

First, let's make sure our new formulae are *intuitively* clear. Please make a sketch of the (inverted) cone in the space at right. Then, at a random point on the cone's surface, sketch the little displacements $d\vec{l}_u$ and $d\vec{l}_\phi$ that we make on the surface when we change $u \rightarrow u+du$ and $\phi \rightarrow \phi+d\phi$, and shade in the little patch of area $d\vec{A}$ that is swept out. Hint: the sweeping parameter u represents distance from the origin. More succinctly, u is the spherical coordinate r; I just didn't want to confuse things by mixing variables from two coordinate systems.

(h) Calculate the surface area of the cone.

$$\boxed{d\vec{A} = d\vec{l}_u \times d\vec{l}_v}$$

$$\boxed{d\vec{l}_u \equiv \frac{\partial \vec{l}}{\partial u}\, du}$$

Reference Page: Maxwell's Equations & Helmholtz's Theorem

We've discussed **Maxwell's Equations** in lecture; here they are for reference:

$$\boxed{\begin{array}{cc} \vec{\nabla}\cdot\vec{E}=\dfrac{\rho}{\varepsilon_0} & \vec{\nabla}\cdot\vec{B}=0 \\ \vec{\nabla}\times\vec{E}=-\dfrac{\partial\vec{B}}{\partial t} & \vec{\nabla}\times\vec{B}=\mu_0\vec{J}+\mu_0\varepsilon_0\dfrac{\partial\vec{E}}{\partial t} \end{array}}$$

ρ is charge per unit volume, while $\vec{J}$ is current per unit area-perpendicular-to-flow. In symbols, $\vec{J}(\vec{r})\equiv d\vec{I}\,/\,dA_\perp$. I actually find the inverse, integral form of this definition more clear: the total current I passing through a surface S is $I=\int_S \vec{J}\cdot d\vec{A}$. The reason that Maxwell's four equations are all you need to know about the origin of classical EM fields is the **Helmoltz Theorem**:

Helmholtz Theorem: A vector field is defined uniquely by its divergence and curl, as long as the field is known to fall off as $1/r^2$ or faster as $r\to\infty$.

The complete integral solution of Maxwell's equations in the **static case**, i.e. when nothing is changing with time, is:

$$\boxed{\vec{E}(\vec{r})=\frac{1}{4\pi\varepsilon_0}\int\frac{\vec{r}-\vec{r}_q}{|\vec{r}-\vec{r}_q|^3}\rho(\vec{r}_q)\,dV_q} \qquad \boxed{\vec{B}(\vec{r})=\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}_q)\times\frac{\vec{r}-\vec{r}_q}{|\vec{r}-\vec{r}_q|^3}dV_q}$$

These are the full blown multiD-integral forms of **Coulomb's Law** and the **Biot-Savart Law**. You saw some version of them in Physics 212; now you can use them to calculate *any* static EM field. About the source terms in those integrals, different versions can be substituted:

- $\rho\,dV_q$ (charged volume) → $\sigma\,dA_q$ (charged surface) → $\lambda\,dl_q$ (charged line)
- $\vec{J}\,dV_q$ (current in a solid) → $\vec{K}\,dA_q$ (surface current) → $I\,d\vec{l}_q$ (current in a wire)

Maxwell's equations can also be written in a **potential formulation**, where the fields $\vec{E}(\vec{r})$ and $\vec{B}(\vec{r})$ are written as derivatives of an **electric scalar potential** $V(\vec{r})$ (familiar from Phys 212) and a magnetic vector potential $\vec{A}(\vec{r})$ (which you'll meet in Phys 435). Here is the potential formulation for **electrostatics**:

define: $\boxed{\vec{E}=-\vec{\nabla}V}$ → Maxwell: $\nabla^2 V=-\dfrac{\rho}{\varepsilon_0}$ → solution: $\boxed{V(\vec{r})=\dfrac{1}{4\pi\varepsilon_0}\int\dfrac{\rho(\vec{r}_q)}{\left|\vec{r}-\vec{r}_q\right|}dV_q}$

What a powerful set of multi-dimensional integrals we have here! To use them, we have to pay very careful attention to those symbols $\vec{r}$ and $\vec{r}_q$ …

The Laplacian: A quick note about the symbol ∇^2 that appears in the middle equation . It's called the Laplacian, and not surprisingly, it stands for $\nabla^2 \equiv \vec{\nabla} \cdot \vec{\nabla}$. Its form in Cartesian coordinates is exactly what you'd expect … but beware: the form of the Laplacian in spherical and cylindrical coordinates is <u>*not obvious*</u>, exactly as for the divergence and curl operations. <u>*Look it up*</u> if you need it → it's on your 3D calculus formula sheet.

Section 12.2: Field Integrals

Let's say you are trying to calculate an electric field $\vec{E}(\vec{r})$. To do so, you'll have to add together — integrate — the contributions from all the charges in your system, which will in general be located all over the place.

- The position on which your field $\vec{E}(\vec{r})$ depends is called the **field point** $\vec{r}$.
  It is the point at which you are determining the field. It will be expressed as some triplet of coordinates such as (x,y,z).
- Your integral will sweep over *different* coordinates: those of the **source point** $\vec{r}_q$.
  The source point sweeps over all sources of the field, e.g. the positions of all the charges q in your system. We must carefully distinguish these coordinates, so we flag them with a subscript as in (x_q, y_q, z_q) .

Primes are often used for the source-point coordinates: e.g. (x', y', z'). I personally find it more helpful to denote them with a subscript that indicates what they represent physically, like q for charges. Whatever notation you choose, here's the critical point:

➔ **Tips & Tricks**: If your integral gives you a <u>field</u> as a result (i.e., a function like $V(\vec{r})$ or $\vec{E}(\vec{r})$) you must be careful to distinguish between **field point** and **source point** coordinates:

*The **<u>source-point</u>** coordinates* $\vec{\mathbf{r}}_q = (x_q, y_q, z_q)$ ***<u>vary</u>*** *over the integral, while the **<u>field-point</u>** coordinates* $\vec{\mathbf{r}} = (x,y,z)$ ***<u>do not</u>*** *and can be treated as constants.*

Let's put it to use. You learned in Physics 212 that the electric field of a point charge q located at the origin is kq/r^2 (where $k = 1/4\pi\varepsilon_0$) and points radially outwards from the charge q. Now consider the general case of a point charge located not at the origin but at some **source point** $\vec{r}_q$. In vector notation, the E field of such a point charge is $\vec{E}(\vec{r}) = kq \dfrac{\vec{r} - \vec{r}_q}{\left|\vec{r} - \vec{r}_q\right|^3}$. (See footnote[3] if the cube looks weird.) Coulomb's Law on the previous page is just an integral over this point-charge formula.

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[3] See how that classic $1/r^2$ dependence is still there *despite* the cube in the denominator? We needed to use the trick of "vector / |vector|^3" to encode the direction information, which is absent from the more familiar formula $E = kq / r^2$.

We must be rock solid on the use of $\vec{E}(\vec{r}) = kq\,\dfrac{\vec{r}-\vec{r}_q}{\left|\vec{r}-\vec{r}_q\right|^3}$ before we use it in integrals. We'll start with a example that uses only Cartesian coordinates.

(a) If a point charge q is placed at the source point $\vec{r}_q = c\,\hat{z}$, what is the electric field $\vec{E}$ at the field point $\vec{r} = a\,\hat{y}$? Your answer must be in vector form, of course … and *Draw! A! Sketch!* For your sketch, use different symbols to indicate the source point and field point. My personal favorites are

- • for the source point (because it looks like a physical charge … to me anyway ☺)
- × for the field point (because it reminds me of a treasure map. seriously.)

Consider a large square plate of sides $2R$ that lies in the xy plane, is centered on the origin, and carries a uniform surface charge density σ. We will be treating this as an infinitely large plate by taking the limit $R \to \infty$. Your task will be to calculate the electric field $\vec{E}(\vec{r})$ produced by this plate. But first …

➔ **Tips & Tricks**: Before you embark on a complicated integral, first consider the **symmetries** of the system. What we mean by "a symmetry of the system" is a transformation that leaves the system unchanged. If a system has such a symmetry, then any field it produces or quantity that describes it will *also* be unchanged under that transformation. This allows you to simplify your work in advance! For integrals producing fields, symmetries can restrict the functional dependence of the result. For integrals producing vectors (constant vectors or vector fields), symmetries can restrict the number of components you have to calculate.

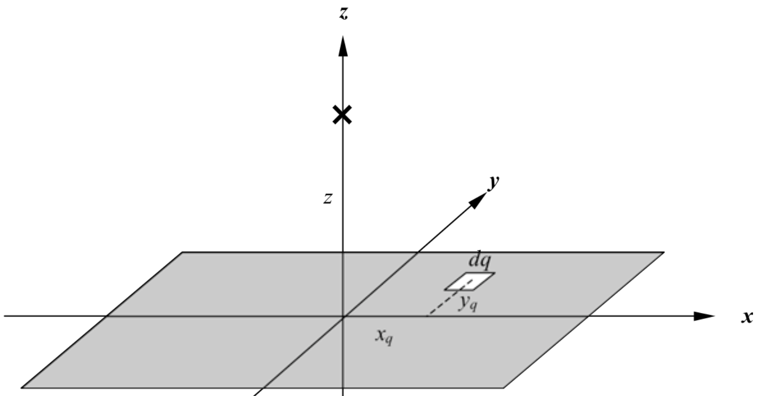
(b) How does this apply to our infinite plate in the xy-plane? → The plate is *symmetric under translations (shifts in position) in x and y.* In Cartesian coordinates, electric fields and potentials generally depend on all three coordinates: $\vec{E}(x,y,z)$ and $V(x,y,z)$. How does the translational symmetry of the infinite plate restrict / simplify the position-dependence of $\vec{E}$ and V?

(c) The plate is also symmetric under the transformation $z \to -z$. How will that simplify our work?

The answers: (b) The field $\vec{E}(\vec{r})$ and potential $V(\vec{r})$ (or anything else!) caused by this plate can only depend on the coordinate z, not x or y. (c) The field and potential will obey the symmetry relations $V(-z) = V(z)$ and $\vec{E}(-z) = -\vec{E}(z)$, so we can restrict our calculations to the $z > 0$ region. If you have any questions about these conclusions, please ask! I also have one more piece of advice that often goes hand-in-hand with symmetry considerations:

➔ **Tips & Tricks**: Cylindrical and spherical coordinates are wonderful for objects whose axis of symmetry lies along the z axis. All coordinate systems also work best for objects that are centered on the origin. These coordinate systems are *dreadful*, however, if the cylinder or sphere over which you're integrating is *not* centered on the origin, or if the cylinder's axis is *not* in the z direction. What to do? If you encounter such a problem, shift and/or rotate the coordinate system so that the object *is* centered and oriented in the most convenient way. (See [4] for a detail we don't need today.)

(d) On with our calculation! First, let's try the potential route → calculate $V(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int \frac{dq}{|\vec{r} - \vec{r}_q|}$ then take its gradient to obtain $\vec{E} = -\vec{\nabla}V$. Since $V(\vec{r})$ can only depend on z, we can pick any x and y values we choose for our field-point $\vec{r}$ → we'll go with $x = 0$ and $y = 0$, as that centers the whole system on the origin. Using Cartesian coordinates, as shown in the figure, set up the integral for $V(\vec{r})$ for field points on the z-axis and for a square plate of charge density σ with corners at $(\pm R, \pm R, 0)$. Don't calculate it, just set it up. In case it's unclear, dq here is just $\sigma\, dA$ → we are sweeping over a surface, hence over little elements of area dA, each with a little bit of charge $\sigma\, dA$. Hints: First, draw $\vec{r}$ and $\vec{r}_q$ on the sketch so you can visualize the ingredients of your integral. Then, realizing that this is your *first field integral,* ***Use! The! Procedure!*** on the back page, step by step.



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[4] If you shift/rotate your coordinate system to match your region of integration — which you should almost always do, that's another of our tips & tricks — you'll have to shift/rotate the *integrand* too. Today's exercises don't require such an operation as they have integrands that follow the symmetries of the problem; just something to keep in mind.

(e) To evaluate your integral, it's much better to use cylindrical coordinates. This also gives us some experience switching coordinate systems. Fortunately, this is an infinite plate, so we can equivalently treat it as a circular disk with radius R instead of a square $2R \times 2R$ plate: we're going to take the limit $R \to \infty$ in the next question, so it doesn't matter what shape you use. Rewrite your integral for $V(z)$ in cylindrical coordinates, i.e., using source-point coordinates s_q and ϕ_q instead of x_q and y_q. (The field point coordinate remains z.) Remember, the area element dA you need will be of a different form in cylindrical coordinates. This time, you can do the integral → go for it!

(f) Now take the gradient to obtain the electric field: $\vec{E} = -\vec{\nabla} V$. Take the limit $R \to \infty$ … your result should match the familiar electric field of an infinite plate from your 212 formula sheet!

Symmetry Again: Check the direction of the field you obtained → From the same symmetry principles you used on the previous page, the electric field of an infinite plate *must* point in the z direction, it *cannot* have any x or y components. Is it completely clear that this is so? If not ask!

(g) And now a tour-de-force: ready for the vector integral? You bet! Set up and integrate the electric field of the large plate using Coulomb's Law: $\vec{E}(\vec{r}) = \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int dq \dfrac{\vec{r}-\vec{r}_q}{|\vec{r}-\vec{r}_q|^3}$. Use all the symmetry simplifications we've discussed: restrict yourself to field-points on the z-axis, and also, realize that you *only have to calculate the z-component* of this field $E_z(0,0,z)$. About that last point: it is very instructive to set up the full integral for all three components … but before you actually *do* any of the integrals, stare at them carefully → you should see at once that only the z-component integral will survive, the others will integrate to zero. You can use either Cartesian or cylindrical coordinates for this task. I suggest cylindrical, if only for practice with this somewhat unfamiliar system. If you use Cartesian, you will need the following integrals (which you could obtain yourself using integration-by-parts and trig substitution):

$$\int \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$$

Section 12.3: Yet Funkier Surfaces

(a) A **parabolic mirror** is a very useful object for focusing light rays. To create the required shape, take the simple equation for a parabola in the xz plane, $z = x^2/a$, and *rotate* that curve around the z axis. This creates a **surface of revolution** (a common construction). Write down the parametrization in cylindrical coordinates for a parabolic mirror that starts at the origin, grows quadratically with s, and extends out to the $s = R$. Suggestion: ϕ and s would make good sweeping parameters.

(b) Determine the area differential $d\vec{A}$ for this mirror. Note: The order in which you do the cross-product determines whether your $d\vec{A}$ will be pointing inward or outward. Have it pointing outward for this example. (We just want the differential here; don't forge ahead and calculate any surface area – unless you want to ☺ – as we have a different use for $d\vec{A}$ coming up on the next page …)

(c) Now the pièce-de-resistance. Running down the z axis is a thin wire carrying a constant line charge density λ. As you know from Physics 212, such a charged wire creates an electric field $\vec{E}(\vec{r}) = \dfrac{\lambda}{2\pi\varepsilon_0 s}\hat{s}$. Your task is to calculate the total electric flux $\Phi = \oint \vec{E} \cdot d\vec{A}$ of this field through the mirror. (You can consider it a closed surface by closing off the top; there's no flux through that top disk anyway.) If all goes well, you should have the most wonderful check of your result: Gauss' Law, $\oint \vec{E} \cdot d\vec{A} = \dfrac{Q_{\text{enclosed}}}{\varepsilon_0}$. ☺

The Procedure for Multidimensional Integration

Definition of the essential word "**PARAMETRIZE**" as used in this context:

> Express **all quantities that vary over the integral** in terms of your **integration parameters** (**IPs**) & constants.

1. Parametrize the **Region $\mathbb{R}^n$**

- **a.** Pick your **coordinate system** $\boldsymbol{r_i} = (x,y,z)$ Cartesian, or (r,θ,ϕ) spherical, or (s,ϕ,z) cylind.
- **b.** Pick your n **integration parameters** $\boldsymbol{u_j}$ — a.k.a. IPs or **sweeping parameters** — that will sweep out the region $\mathbb{R}^n$. If possible, use one or more of your chosen coordinates r_i.
- **c.** Describe the *shape* of $\mathbb{R}$ by expressing your coordinates r_i as **functions**[5] $\boldsymbol{r_i(u_j)}$ of the IPs
- **d.** Describe the *edges* of $\mathbb{R}$ by providing **bounds** on each integration parameter u_j

2. Parametrize the **Differential $d\mathbb{R}^n$** using your coordinate system's **Line Element** $d\vec{l}$

Method 1: Visualization

$d\mathbb{R}^n = d\vec{l}\,|d\vec{A}|dV$ is **how much space** (length | area | volume) **you sweep out when you increase every IP u_j by du_j.** Figure it out with a sketch and/or the line element $d\vec{l}$ of your coordinate system. This method works best when the integration parameters u_j are actual coordinates r_j.

Method 2: Formalism

$$d\vec{l}_{\text{path}} = \frac{d\vec{l}}{du}\,du$$

$$d\vec{l}_u \equiv \frac{\partial\vec{l}}{\partial u}\,du \rightarrow \quad d\vec{A} = d\vec{l}_u \times d\vec{l}_v \qquad dV = \left(d\vec{l}_u \times d\vec{l}_v\right)\cdot d\vec{l}_w$$

3. Construct the **Integral** expressing *everything* in terms of your **IPs** and **constants**

Use your **coordinate functions $\boldsymbol{r_i(u_j)}$** from to express *everything that varies* in the integrand *entirely* in terms of the IPs & constants. Watch out especially for spher/cylind unit vectors!

Your integral must be **doable** = something you can type into Wolfram Integrator[6], and must **make sense** = give a result that depends only on quantities that *survive the integration*. (Example of nonsense: a final result with an IP left in it!) Proper integrals have this form:

$\mathbb{R}^1$ path integral	$\mathbb{R}^2$ surface integral	$\mathbb{R}^3$ volume integral
$\int_{u_i}^{u_f} G(u)\,du$	$\int_{v_i}^{v_f}\int_{u_i}^{u_f} G(u,v)\,du\,dv$	$\int_{w_i}^{w_f}\int_{v_i}^{v_f}\int_{u_i}^{u_f} G(u,v,w)\,du\,dv\,dw$

For **vector integrals**, you will get one such scalar integral **per component**.

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[5] What to call these functions $r_i(u_j)$? **Constraint functions** is a good name, as that's what they do: constrain the coordinates to lie on your region $\mathbb{R}$. I like the descriptive **shape functions**, but we'll go with **coordinate functions**.

[6] Free integration available online at http://integrals.wolfram.com (indefinite integrals only). The new, insanely powerful WolframAlpha can do definite integrals too → see http://wolframalpha.com/examples/Calculus.html

Tips and Tricks for Multi-D Integration

➔ Choose the coordinate system r_i that best matches the integration region $\mathbb{R}$, not the integrand.

➔ If your integral gives a vector result, you must split it into 3 separate integrals, one for each component. (Why? Vectors sum by components, and integrals are just that: sums.)

➔ Beware of non-Cartesian unit vectors in your integrand! If $\hat{r}$, $\hat{s}$, $\hat{\theta}$, or $\hat{\phi}$ appear in your integrand and are associated with coordinates over which you're integrating, you *cannot* pull them out of the integral because they're *not constant* → transform them to fixed, Cartesian unit vectors before you integrate. The one exception is in field integrals (next point): if $\hat{r}$, $\hat{s}$, $\hat{\theta}$, or $\hat{\phi}$ are associated with the *field-point* coordinates rather than the source-point coordinates, they *are* constant over the integral and can be left alone.

➔ To change the direction of a path integral, change the bounds, not $d\vec{l}$. Do not mess with the direction of $d\vec{l}_{\text{path}}$: it is tied to your coordinate system and your parametrization of the path by the strict formula $d\vec{l}_{\text{path}} = (d\vec{l}\,/\,du)\,du$. In contrast, you are completely free to choose the order of the bounds on your IP, *u*. Final point: if you do *both*, you'll have done *nothing* → try it and see!

➔ If your integral gives you a field as a result (i.e., a function like $V(\vec{r})$ or $\vec{E}(\vec{r})$) you must be careful to distinguish between **field point** and **source point** coordinates:

> *The source-point coordinates* $\vec{\boldsymbol{r}}_q$ *vary over the integral, while*
> *the field-point coordinates* $\vec{\boldsymbol{r}}$ *do not and can be treated as constants.*

Be sure to label them differently: use a subscript or prime to identify the source point coordinates in your expressions, and use different symbols on your sketches such as

• for the source point (because it looks like a physical charge … to me anyway ☺)
× for the field point (because it reminds me of a treasure map. seriously.)

➔ Always consider the **symmetries** of the system, namely transformations that leave the system unchanged. If a system has such symmetries, then any field it produces or quantity that describes it will *also* be unchanged under those transformations. This allows you to simplify your work in advance! For integrals producing fields, symmetries can restrict the functional dependence of the result. For integrals producing vectors (constant vectors or vector fields), symmetries can restrict the number of components you have to calculate.

➔ Be sure to shift and/or rotate your coordinate system in order to match its symmetries to those of your region of integration. Note that the *z-axis* is the axis of symmetry for both the cylindrical and spherical coordinate systems, while the *origin* is the point of symmetry for all coordinate systems. Detail to keep in mind: if you shift/rotate your coordinate system to get a nice description of $\mathbb{R}$, you must transform your *integrand too*, in the same way.