

Physics 225
Relativity and Math Applications
Fall 2012

Unit 13
The Zen of i

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Unit 13: Complex Numbers and the Zen of i

Introduction

Nature is filled with things that **oscillate**. Pendulums oscillate, the moon oscillates across the plane of the sky, and atomic electrons oscillate round-and-round the nuclei to which they are bound. In your study of electromagnetism in Physics 212, you encountered yet more examples: currents and voltages oscillate in AC circuits, and electromagnetic fields oscillate around each other to form light.

At their core, these oscillations are **sinusoidal** in nature. Things that oscillate in time – like the angle of a pendulum – vary like $\sin(\omega t)$ or $\cos(\omega t)$ or $\sin(\omega t + \phi)$. Things that oscillate in space – like the magnitude of the electric field in a radio wave – vary like $\sin(kz)$ or $\cos(kx)$ or $\sin(ky + \phi)$... and so on, and so on. So far, we have used only the trigonometric functions \sin and \cos to describe this type of behavior. But there is a much better mathematical framework available. As you will see, using \sin and \cos to describe oscillations is rather like trying to calculate a 15% restaurant tip using Roman numerals. ☺

The best mathematical technology for describing things which oscillate is **complex numbers**. These numbers are strange indeed, and they arise from the mystical contemplation of the following truly bizarre question:

WHAT IS THE SQUARE-ROOT OF -1 ?

This question has no rational answer ... or does it? To answer it, we must *expand our minds*, in the manner of the Zen scholars of ancient times ...



Section 13.1: Complex Algebra and Euler’s Theorem

Have you ever heard of a Zen koan? Koans are mind-expanding riddles that Zen masters have used for centuries to teach their students about the limitations of the mind and its preconceptions. The riddles are “non-sensical”, in the sense that they have no solution that the rational mind can grasp. In contemplating these riddles, the mind realizes that it is limited and becomes quiet, opening the student to the possibility of greater truth. Here are a couple of famous Zen koans:

- What is the sound of one hand clapping?
- What is your original face before you were born?



Here’s another one from Richard Feynman (who used it to torment philosophy students during his grad-school days ☺):

- Show me the inside of a brick.

And now here is a famous mathematical koan:

What is the square-root of -1 ?

That question has no mathematical answer within the world of real numbers. And so, just like the Zen students, we must expand our minds and our mathematics into another dimension!



Since the square-root of -1 has no solution within the world of real numbers, we assign this bizarre object its own symbol:

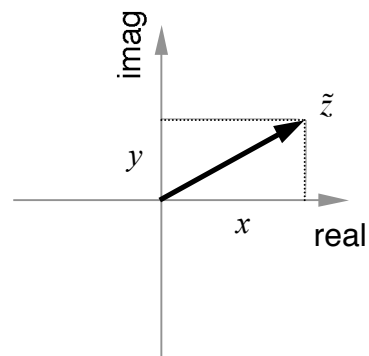
$$i \equiv \sqrt{-1}$$

Using i , we can build an entirely new set of numbers called **imaginary numbers** by multiplying ordinary real numbers with i , e.g. $6i$ or $32.7i$, or $4\pi i/3$. Finally, we define **complex numbers** as the sum of a real number and an imaginary number. It is traditional to use a “tilde” to indicate a complex number:

$$\tilde{z} = x + iy$$

The real part (x) and imaginary part (y) of a complex number are independent in the sense that you can never add two real numbers to get an imaginary number (and vice versa). The addition of complex numbers goes like this:

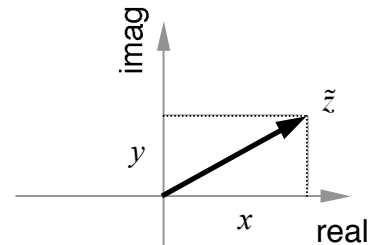
$$\begin{aligned} \tilde{z}_1 &= x_1 + iy_1 \\ \tilde{z}_2 &= x_2 + iy_2 \end{aligned} \quad \Rightarrow \quad \tilde{z}_1 + \tilde{z}_2 = (x_1 + x_2) + i(y_1 + y_2)$$



Well that’s exactly how you add vectors: component-by-component, since the x and y components of a vector are independent. Because of this similarity, we arrive at the **graphical representation** of complex numbers shown above: we can draw a complex number $\tilde{z} = x + iy$ as a 2D vector, with the **real axis** running horizontally and the **imaginary axis** running vertically. The vector \leftrightarrow complex number analogy is not exact, but it is *very* useful, and we’ll refer to it again and again.

(a) Let's do some math with complex numbers. Given two complex numbers $\tilde{z}_1 = x_1 + iy_1$ and $\tilde{z}_2 = x_2 + iy_2$, please calculate the following two quantities: the difference $\tilde{z}_1 - \tilde{z}_2$ and the product $\tilde{z}_1 \tilde{z}_2$. Your answers will also be complex numbers, with real and imaginary parts formed from the ingredients $x_1, y_1, x_2,$ and y_2 . Be sure to manipulate your answer for the product $\tilde{z}_1 \tilde{z}_2$ until it is in the standard form "blah_{real} + i blah_{imag}", where the blahs are both real numbers.

(b) The **complex conjugate** \tilde{z}^* of a complex number is defined as follows: $\tilde{z}^* \equiv x - iy$ i.e., you simply flip the sign of the imaginary part. Why is this a useful operation? To find out, please calculate the product $\tilde{z}^* \tilde{z}$ in terms of x and y . Is it a real number, an imaginary number, or a complex number? And what is the graphical significance of $\tilde{z}^* \tilde{z}$ in terms of the graphical representation of \tilde{z} as a 2-component vector?



This graphical significance explains why the construction $\sqrt{\tilde{z}^* \tilde{z}}$ is called the **magnitude** of \tilde{z} , and gets its own symbol:

$|\tilde{z}| \equiv \sqrt{\tilde{z}^* \tilde{z}}$

(c) Now we're ready to *divide* two complex numbers.¹ Your task is to express the ratio $\tilde{z}_1 / \tilde{z}_2$ in the form $x + iy$ (i.e., find its real and imaginary parts). Here's the best technique to use whenever you have a complex number in a denominator:

Multiply the numerator and denominator by the complex-conjugate of the denominator.

This technique is exceedingly useful, please commit it to memory! Now off you go: calculate $\tilde{z}_1 / \tilde{z}_2$

¹ ... and so we encounter already the breakdown of the analogy between complex numbers and vectors. You cannot divide two vectors, but you can certainly divide two complex numbers. You can't multiply two vectors either. You can take their dot product or their cross-product, but the expression $\vec{a} \vec{b}$ is totally meaningless.

(d) Given this technique, what is $1/i$? Be sure to express your answer in the standard form $x + iy$.

We are now ready to encounter the great mystery of **Euler's Theorem**:

$$e^{i\theta} = \cos\theta + i \sin\theta$$

This *mind-blowing* relation is the magic key that unlocks the usefulness of complex numbers. Without Euler's theorem, complex numbers would be mere curiosities. But how on earth can we understand this bizarre theorem? *Where does it come from?*

Believe it or not, Euler's Theorem comes from Taylor's theorem, which states that any function $f(x)$ can be expressed as an infinite sum via its derivatives²: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$. This polynomial is the **Taylor expansion** of $f(x)$ around the point x_0 (also called a **Taylor series**). If we expand around $x_0 = 0$, we get the simplified form $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$. Here's what it gives you for the three functions that appear in Euler's Theorem:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \dots \quad \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

(e) Evaluate the left and right sides of Euler's theorem using these Taylor series. *See what happens?*

² The notation $f^{(n)}$ signifies the n^{th} derivative of f : i.e., $f^{(0)}(x) = f(x)$, $f^{(1)}(x) = f'(x)$, $f^{(2)}(x) = f''(x)$, etc.



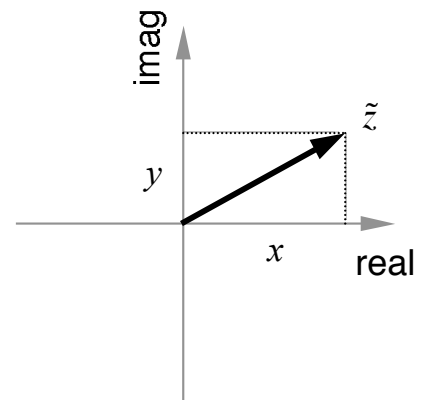
The form $x + iy$ we've been using can be termed the **Cartesian representation** of a complex number \tilde{z} . Euler's Theorem leads us to a second way of representing complex numbers: the **polar representation**. It looks like this: $\tilde{z} = r e^{i\theta}$.

$$\tilde{z} = x + iy = r e^{i\theta}$$

The quantities r and θ — which are both real numbers, just like x and y — are called the **magnitude** and **phase** of \tilde{z} respectively. As you will see in a few seconds, Euler's Theorem shows us that any complex number can be represented in either way. That's what's indicated by the boxed formula above.

(f) Using Euler's Theorem and the definition $\tilde{z} = x + iy = r e^{i\theta}$ of our two representations, determine the transformation equations that give you the magnitude r and phase θ of a complex number from its real part x and imaginary part y . Also find the inverse pair of equations, for obtaining x and y from r and θ . Important convention: **r is always positive**. This is just a convention of the polar representation $\tilde{z} = r e^{i\theta}$ but it is a *meaningful* convention: it ensures that all the phase information about the complex number appears in the phase angle θ , while r is purely a magnitude, with no sign. The convention will make more sense when you get to the next question.

(g) What are the significances of r and θ in our graphical representation of complex numbers? Just look at your transformation equations above, then look at the plot ... r and θ both *mean something* on that plot. Indicate the meaning of these quantities on the figure.



(h) What is the polar representation of i itself?³

³ Need a hint? This question is so simple it's confusing, I know. Think about it like this: " i " is just a complex number with a real part (x) of 0 and an imaginary part (y) of 1. Given that insight, your transformation equations tell you its magnitude (r) and phase (θ). Once you have those two values, you have the polar representation you seek: $i = r e^{i\theta}$. Even better than using the transformation equations to find r and θ is to use the plot: the graphical association you made in part (g) is the best way to translate back and forth between the Cartesian ($x + iy$) and polar ($r e^{i\theta}$) representations of complex numbers!

Checkpoint: the answer to (h) is $i = e^{i\pi/2}$. Makes sense? If not, please go back and read the footnote.

- (i) What is the polar representation of -3 ?
(Draw the plot! Draw the plot!)



- (j) What is the complex-conjugate \tilde{z}^* of $\tilde{z} = r e^{i\theta}$ in the polar representation?

Here are the **most important skills** you need to become a Zen master of complex algebra:



- switching back and forth between polar and Cartesian representations \rightarrow *use the plot!*
- knowing which representation to use when
- and don't forget: if you have a complex number in a denominator, multiply by \tilde{z}^*/\tilde{z}^*

The second bullet is the only one left to address. As you saw earlier, the $x + iy$ representation is ideal for the adding and subtracting complex numbers. The $r e^{i\theta}$ representation, however, is *much* better for multiplication, division, and taking powers. The next examples will make this clear.

- (k) Calculate the polar representations of the product $\tilde{z}_1 \tilde{z}_2$ and ratio $\tilde{z}_1 / \tilde{z}_2$ of $\tilde{z}_1 = r_1 e^{i\theta_1}$ and $\tilde{z}_2 = r_2 e^{i\theta_2}$.

- (l) Given $\tilde{z} = r e^{i\theta}$, what is $\tilde{z}^* \tilde{z}$? Does this agree with the significance of $\tilde{z}^* \tilde{z}$ you determined earlier?

- (m) The polar representation is also ideal for calculating powers. For example, what is \sqrt{i} ?
(How Zen is that!) Express your answer in both the polar $r e^{i\theta}$ and Cartesian $x + iy$ representations.

(n) We now begin to see the great convenience of complex numbers \rightarrow they are an *extremely* elegant and easy-to-use way of representing the trigonometric functions \sin and \cos ! To illustrate: do you remember the formulas for calculating $\sin(a+b)$ and $\cos(a+b)$? If so, write them down here:

(o) No need to remember those formulas anymore! Consider the complex expression $e^{i(a+b)}$. The usefulness of complex numbers comes from the fact that *exponentials are much easier to manipulate than trig functions!* For example, $e^{i(a+b)} = e^{ia} e^{ib}$. Using Euler's Theorem, express both the left- and right-hand sides of this relation in the (x,y) representation \rightarrow in doing so, you will have trivially *derived* those miserable formulas for $\sin(a+b)$ and $\cos(a+b)$!

(p) More Zen practice. What is $1/(1+i)$? Express your answer in both the Cartesian and polar forms.

(q) What is $\sqrt{1+\sqrt{3}i} - \sqrt{\frac{3}{2}}$?

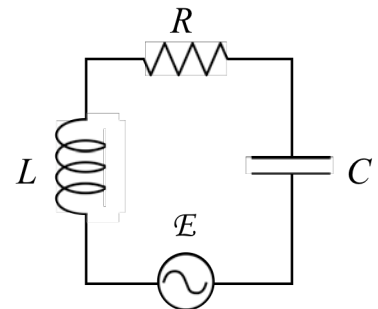


- Sidewalk quote from Physics Day, 2001: “I’m sorry, you have dialed an imaginary number, please rotate your phone by 90° and dial again.”
- One day as Manjusri stood outside the gate, the Buddha called to him, “Manjusri, Manjusri, why do you not enter?” Manjusri replied, “I do not see myself as outside. Why enter?”



Section 13.2: AC Circuit Analysis with Complex Numbers

Complex numbers are an extremely convenient way of representing trig functions. With complex numbers, we can analyze easily problems that would be horribly-difficult using the “primitive” machinery of sin and cos alone. A common example is the analysis of AC circuits, where all currents and voltages are oscillating sinusoidally. In Physics 212, we don’t have the technology of complex numbers at our disposal and so we can only analyze *one single* AC circuit: a generator, resistor, inductor, and capacitor in series. Let’s start by transforming our understanding of that simple circuit to the realm of complex numbers ... then we will see that the analysis of much more complicated circuits becomes simple!



In an AC circuit driven by a generator of frequency ω , all currents and voltages in the problem oscillate at that same frequency ω . In the simple RLC circuit depicted above, all of the circuit elements are in series and so there is only one current: $I(t) = I_{\max} \cos(\omega t)$. To introduce our new mathematical technology, we rewrite this oscillating current as a complex number:

$$\tilde{I}(t) = I_{\max} e^{i\omega t}$$

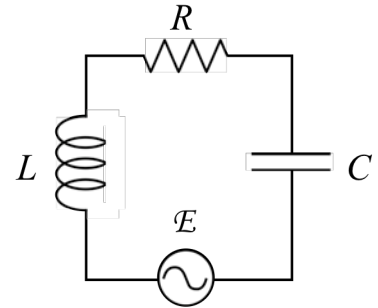
Now here is the key to describing physical quantities in terms of complex numbers:

***The real part of \tilde{I} represents the physical observable I ...
the imaginary part is just mathematical baggage.***

This concept is very similar to the use of the electric potential $V \rightarrow$ the potential is a useful mathematical construction, but its *physical significance* lies only in its *derivatives*: it is the field $\vec{E} = -\vec{\nabla}V$ which is the physical observable, not the potential.

(a) Starting with the complex-current $\tilde{I}(t) = I_{\max} e^{i\omega t}$, calculate the complex-voltages $\tilde{V}(t)$ across the resistor, inductor, and capacitor using the following elementary relations:

- Calculate $\tilde{V}_R(t) = \tilde{I}(t)R$
- Calculate $\tilde{V}_L(t) = L \frac{d\tilde{I}(t)}{dt}$
- Calculate $\tilde{V}_C(t) = \frac{\tilde{Q}(t)}{C} = \frac{1}{C} \int \tilde{I}(t) dt$



Your results should be expressed in terms of the circuit parameters R , L , C , the generator-frequency ω , the peak current I_{\max} , and time of course.

(b) As in Physics 212, the current-voltage relations you derived above can be conveniently expressed with “ $V=IR$ ”-type relations where we introduce the **reactances** X_L and X_C to describe the “effective resistance” of an inductor and capacitor in an AC circuit:

$$\tilde{V}_R(t) = \tilde{I}(t)R, \quad \tilde{V}_L(t) = \tilde{I}(t)\tilde{X}_L, \quad \tilde{V}_C(t) = \tilde{I}(t)\tilde{X}_C$$

... but now the reactances are *complex*. Write down expressions for the complex-reactances \tilde{X}_L and \tilde{X}_C

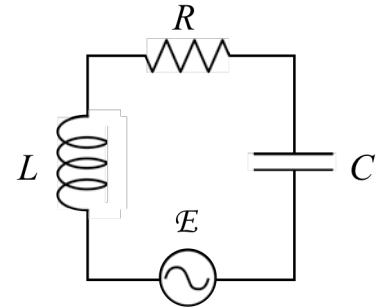


This is the right way to analyze AC circuits!

$$\begin{aligned} \tilde{V}_R(t) &= \tilde{I}(t)R \\ \tilde{V}_L(t) &= \tilde{I}(t)\tilde{X}_L \\ \tilde{V}_C(t) &= \tilde{I}(t)\tilde{X}_C \end{aligned}$$

where

$$\begin{aligned} \tilde{X}_L &= iX_L = i\omega L \\ \tilde{X}_C &= -iX_C = \frac{1}{i\omega C} \end{aligned}$$



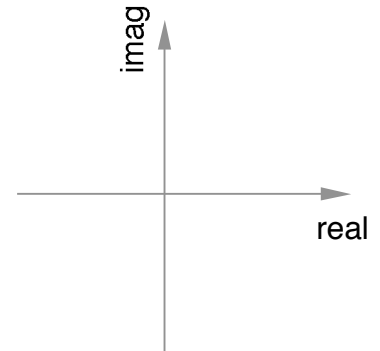
Something wonderful has happened here. In Physics 212, our “ $V=IR$ ”-type formulas apply *only* to the *peak values* of the quantities involved, e.g. $V_{L,\max} = I_{\max} X_L$. On top of that, we have to carry around separate “ELI the ICEman” relations about the *relative phases* of V and I across the different elements:

$$\begin{aligned} V_{R,\max} &= I_{\max} R \\ V_{L,\max} &= I_{\max} X_L \\ V_{C,\max} &= I_{\max} X_C \end{aligned}$$

Ohm’s Law: voltage is in phase with current across a resistor
 ELI: voltage (Emf) across an inductor (L) leads current (I)
 ICE: current (I) across a capacitor (C) leads voltage (Emf)

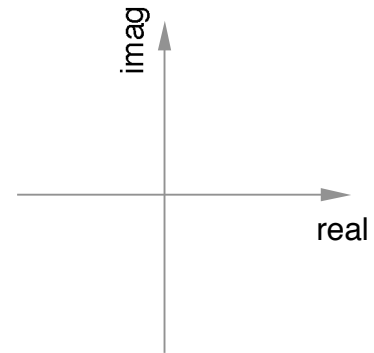
But with complex numbers, **both the amplitude and phase information** is encoded in the reactances!

(c) To demonstrate, write down complex expressions for $\tilde{I}(t)$, $\tilde{V}_R(t)$, $\tilde{V}_L(t)$, and $\tilde{V}_C(t)$, in terms of I_{\max} (use “ I_m ” for short), ω , t , and the real values R , X_L , and X_C . Then find their values at time $t = 0$ and draw the four complex numbers $\tilde{I}(0)$, $\tilde{V}_R(0)$, $\tilde{V}_L(0)$, $\tilde{V}_C(0)$ on the diagram using our vector representation.



(d) Remember that the *physical values* of our current and voltages are always understood to be the *real parts* of our complex numbers. So, at time $t = 0$, which of the voltages are zero? Which are at their peak values?

(e) Repeat the exercise for time $t = \pi/(2\omega)$: calculate the complex numbers $\tilde{I}(\pi/2\omega)$, $\tilde{V}_R(\pi/2\omega)$, $\tilde{V}_L(\pi/2\omega)$, and $\tilde{V}_C(\pi/2\omega)$ and draw them on the diagram. Now which quantities are zero?



The term **impedance** (symbol \tilde{Z}) is used to describe the net resistance of a combination of resistors, capacitors, and inductors. Naturally, \tilde{Z} is a complex number in this language. The beautiful thing is that to calculate \tilde{Z} for any combination of circuit elements, all you have to do is:

Add the complex reactances together just as if they were normal resistances!

(f) For our series R-L-C combination, we have three elements in series, and so $\tilde{Z} = R + \tilde{X}_L + \tilde{X}_C$.

Use your Zen-skills with complex numbers to find the real and imaginary parts of \tilde{Z} in terms of the real numbers R , X_L , and X_C . (This is trivial ...)

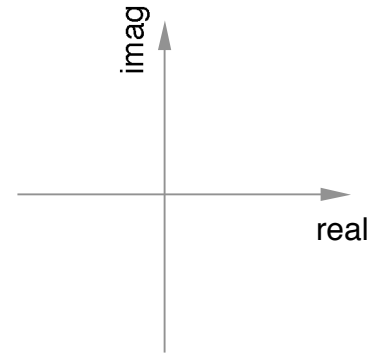
Important tip: Always work impedance calculations using R , X_L , X_C instead of R , ωL , $1/\omega C$. One reason is that R , X_L , X_C all have the **same units**: Ohms. This makes unit-checking infinitely easier than if you substitute in ωL or $1/\omega C$ too early! The other reason is that the convenient symbols R , X_L , X_C *hugely* simplify any algebra you have to do. So: if you really need to calculate an impedance in terms of ω , L , and C , get it in terms of X_L and X_C first, then substitute in $X_L = \omega L$ and $X_C = 1/\omega C$ at the very end.

(g) The R-L-C impedance \tilde{Z} can also be written in the (r, θ) polar representation \rightarrow as $\tilde{Z} = Z e^{i\phi}$.

Determine the magnitude Z and angle ϕ of the impedance, again in terms of the real numbers R , X_L , X_C . Your formulas should look very familiar!



(h) The impedance \tilde{Z} is the total reactance of a *combination* of circuit elements. The current and voltage across the combination are thus related by $\tilde{V}(t) = \tilde{I}(t)\tilde{Z}$. Our R-L-C-series combination is being driven by a generator of voltage $\tilde{\mathcal{E}}(t)$. Multiply $\tilde{I}(t) = I_{\max} e^{i\omega t}$ by $\tilde{Z} = Z e^{i\phi}$ to obtain a complex expression for $\tilde{\mathcal{E}}(t)$. Finally, draw the current $\tilde{I}(0)$ and generator-EMF $\tilde{\mathcal{E}}(0)$ at time $t = 0$ on the diagram.



You've rediscovered the **phasor diagram** of Physics 212. ☺

Let's summarize the impedance \tilde{Z} with some boxed formulas:

$$\tilde{\mathcal{E}}(t) = \tilde{I}(t) \tilde{Z} \quad \text{where} \quad \tilde{Z} \equiv Z e^{i\phi}$$

That's the last of our four " $V=IR$ " relations. To complete our AC-circuit toolbox, we must ensure that we know how to *apply* our complex-number relations to real, physical situations. As you already know:

The physical currents and voltages are given by the real parts of \tilde{I} , \tilde{V} , and $\tilde{\mathcal{E}}$

Our development of complex circuit analysis began with this principle. We turned real quantities written with sines and cosines into complex quantities written with exponentials in order to simplify our mathematics; we just have to remember to *drop the imaginary parts* when we apply our results to the physical world of oscilloscopes and multimeters. It sounds trivial, but there's a bit of a subtlety ...

(i) You have an AC generator, a circuit, and an oscilloscope. The generator produces an EMF of $\tilde{\mathcal{E}}(t) = 60 e^{i(8\pi t + \pi/2)}$ V. What is the actual voltage $\mathcal{E}(t)$ measured by the scope when you attach it to the generator? What is the peak voltage \mathcal{E}_{\max} produced by the generator?

(j) Your circuit has impedance $\tilde{Z} = 3 + 4i$ Ohms at the frequency of your generator. What is the peak current I_{\max} produced in the circuit when you attach it to the generator?

I'm 100% sure you got an I_{\max} of either 20 Amps or 12 Amps. Which one is right? To help you decide:

(k) You now attach a different circuit to the 60 V generator. This circuit has impedance $\tilde{Z} = 4i$. What is I_{\max} now?

(l) You stare at the $\tilde{Z} = 4i$ circuit and see that it consists of only one component. What sort of component is it: a resistor, a capacitor, or an inductor?

Let's see how you did! The answer to (i) is straightforward: the scope displays the real, physical part of $\tilde{\mathcal{E}}(t)$, which is $\mathcal{E}(t) = 60 \cos(8\pi t + \pi/2)$ V = $-60 \sin(8\pi t)$ V. The peak voltage is 60 V. For part (j), you have to find a current, so you divide voltage over impedance. The answer is $I_{\max} = 12$ A. Did you get 20 A instead? If so, you made a very common mistake: you *dropped the imaginary part of \tilde{Z}* .

NEVER drop the imaginary part of a complex impedance \tilde{Z} or reactance \tilde{X} .

Here's why:

- Voltage and current are **physical observables** that we are choosing to *represent* as complex numbers for mathematical convenience. The imaginary parts of such quantities are mathematical baggage: they have no physical meaning, but are of great help in performing calculations.
- Impedances \tilde{Z} and reactances \tilde{X}_L , \tilde{X}_C are *not* physical observables, they are constructed quantities that are **complex by definition**. They are ratios of the *complex-representation-of-a-voltage* (\tilde{V} or $\tilde{\mathcal{E}}$) over the *complex-representation-of-a-current* (\tilde{I}). The imaginary parts of these inherently complex quantities are essential: \tilde{Z} (or \tilde{X}) encodes both the magnitude ratio and the phase difference between the voltage and current in question, and it uses *both* $\text{Re}(\tilde{Z})$ and $\text{Im}(\tilde{Z})$ to do so. You can *never* drop the imaginary part of \tilde{Z} , \tilde{X}_L , or \tilde{X}_C .

Parts (k) and (l) illustrate the point dramatically. That one-component circuit with $\tilde{Z} = 4i$ is an inductor (answer to part (l)) since its impedance is purely imaginary and positive. If you dropped its imaginary part, you'd be left with zero impedance! That can't be: that would produce an infinite current, and we know that doesn't happen when you connect an AC generator to an inductor.

To calculate a physical current as in (j,k), you must calculate the full complex current first via $\tilde{I} = \tilde{\mathcal{E}} / \tilde{Z}$, *then* take its real part. In symbols, physical $I = \text{Re}(\tilde{\mathcal{E}} / \tilde{Z})$; it is **not** $\text{Re}(\tilde{\mathcal{E}}) / \text{Re}(\tilde{Z})$. Our Zen skills tell us that $\tilde{\mathcal{E}} / \tilde{Z}$ – a division – is best done with \tilde{Z} in polar form, so convert \tilde{Z} to $Ze^{i\phi}$ before proceeding.

Revisions needed? If you didn't get the answers (j) $I_{\max} = 12$ A and (k) $I_{\max} = 15$ A, please go back and revise your work using this procedure. Calculating currents from voltages and vice-versa is the *entire point* of circuit analysis, so ask your instructor if you need any assistance! The top of the next page also provides some guidance.

Since every current-voltage formula we have is of the same “ $\tilde{V} = \tilde{I} \tilde{X}$ ” form, we always get the same relations between the peak-values and phase-angles of the physical voltage and current in question:

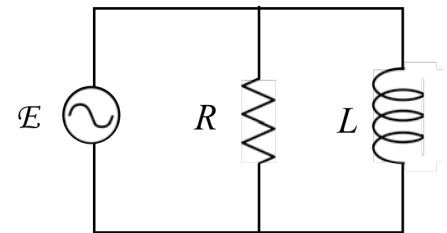
Any current-voltage relation of the form $\tilde{V}(t) = \tilde{I}(t) \tilde{X}$ implies:

- $V_{\max} = I_{\max} |\tilde{X}| = I_{\max} \sqrt{\text{Re}(\tilde{X})^2 + \text{Im}(\tilde{X})^2}$
- V leads I by phase $\phi = \arctan\left(\frac{\text{Im}(\tilde{X})}{\text{Re}(\tilde{X})}\right)$

Nothing in this box is worth memorizing. It all follows directly from applying complex algebra to $\tilde{V} = \tilde{I} \tilde{X}$, then taking the real parts of \tilde{V} and \tilde{I} at the *end* to determine what you will measure in the lab. Instead, the box provides a useful check of your work while all this complex business is still new. ☺

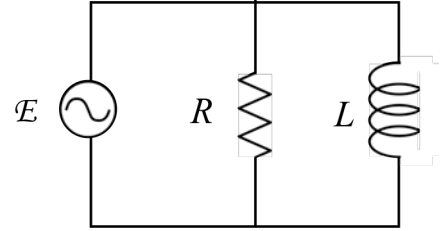
Section 13.3: A New AC Circuit at Last!

(a) We now have the tools to analyze AC circuits *other* than series-R-L-C. For example, let’s connect our generator to a resistor and an inductor in *parallel*. Calculate the complex impedance \tilde{Z} seen by the generator in terms of the real numbers R and X_L . (Remember: just add the resistance R and complex-reactance \tilde{X}_L exactly as you would for two ordinary resistors in parallel → that is the extreme coolness of circuit analysis with complex numbers!) Manipulate your expression for \tilde{Z} until you get it in the form $\text{Re}(\tilde{Z}) + i\text{Im}(\tilde{Z})$.



Checkpoint: the impedance is $\tilde{Z} = \frac{RX_L}{R^2 + X_L^2}(X_L + iR)$.

If that's not what you got, please go back and find the error.



(b) As we discussed, the *usefulness* of impedance is that it relates current to voltage, via $\tilde{\mathcal{E}} = \tilde{I} \tilde{Z}$. Multiplication and division are best done in polar form, so convert \tilde{Z} to $Z e^{i\phi}$ form (i.e., find the magnitude Z and phase ϕ of the impedance in terms of R and X_L).

(c) Does your expression make sense? Check it with a few limiting cases. What happens to Z and ϕ

A. in the limit $R \rightarrow \infty$?

B. in the limit $X_L \rightarrow \infty$?

C. in the limit $R \rightarrow 0$?

Hint: calculate the effect that these extreme values will have on the current $I(t)$ delivered by the generator. Do your results *make sense*? To find out, **draw equivalent circuits** where you replace the capacitor and inductor with either a wire or an open break. (See footnote⁴ for a reminder of how to do that).

⁴ A brief Phys 212 refresher: **Inductors** resist change. At very high frequencies, when things are changing very rapidly, $X_L = \omega L$ becomes arbitrarily large ... and so an inductor behaves like an *open break* of infinite resistance in the limit $\omega \rightarrow \infty$. At very low frequencies, $X_L = \omega L \rightarrow 0$ so the inductor behaves like a resistance-free *wire*. **Capacitors** have the *opposite* behavior with frequency: $X_C = 1/\omega C$. A capacitor looks like a resistance-free wire at very high frequencies (the plates never have time to “charge up”), and like an open break at low frequencies (the plates *can* charge to full capacity and impede further current).