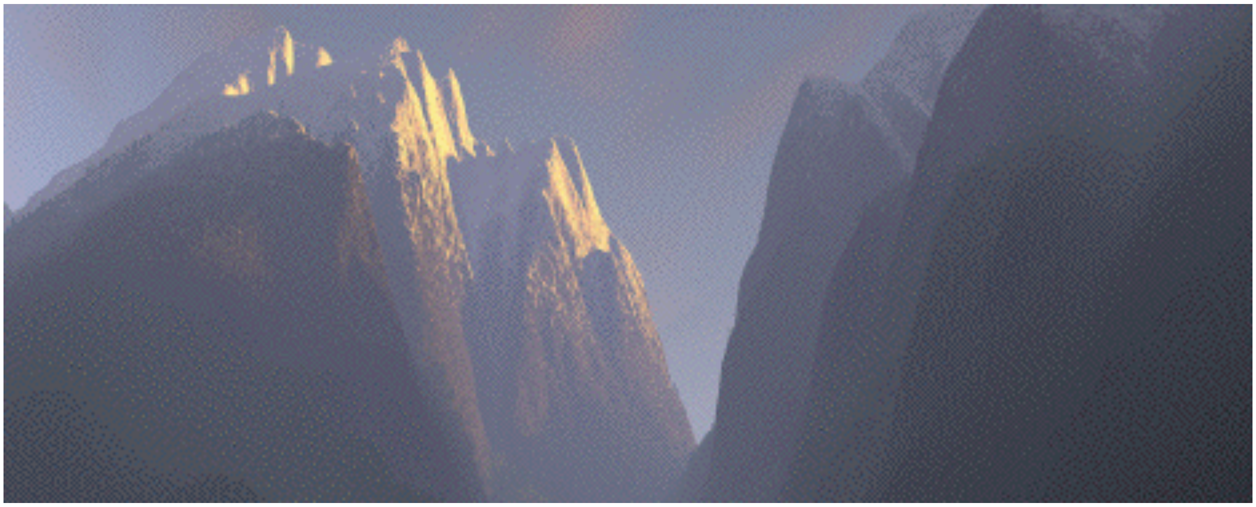

Physics 225

Relativity and Math Applications

Fall 2012

Unit 14

Pitfalls & Mastery

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Unit 14: Pitfalls and Mastery

Section 14.1: The Pitfalls of Power

One of the most common errors people make with complex numbers is in applying this principle: *When an observable quantity is described by a complex number, the real part represents the obvservable's physical value.* This sounds simple, but it requires thought. One of the trickiest cases is electric power.

(a) Consider a completely general AC circuit: the impedance is $\tilde{Z} = Z e^{i\phi}$, the current out of the generator is $\tilde{I}(t) = I_m e^{i\omega t}$, and the generator EMF is $\mathcal{E}(t) = \mathcal{E}_m\, e^{i(\omega t+\phi)}$. What is the **power**, $P(t)$, delivered by the generator as a function of time? We know that $P = I\,V$, so we must calculate "current times generator-EMF". We also want real, physical power, so we must *drop imaginary parts … but where??* Please do these two calculations:

- Calculate $P(t)$ as $\mathrm{Re}\left[\tilde{I}\ \tilde{\mathcal{E}}\right]$
- Calculate $P(t)$ as $\mathrm{Re}\left[\tilde{I}\right]\mathrm{Re}\left[\tilde{\mathcal{E}}\right]$

Once you've calculated both, see if they match. To help you compare them, apply the useful relation $\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) + \cos(a+b)\right]$ to one of them. If the two don't match, *which is correct?* Don't worry if you're not 100% sure; just think carefully, and take your best guess.

(b) The conundrum arises from P being the product of two quantities that we are representing as complex numbers ($\tilde{I}$ and $\tilde{\mathcal{E}}$). But we have calculated such a product before! For example, we've calculated EMF from $\tilde{I}$ times $\tilde{Z}$ many times. Which of these is correct?

- Calculating physical EMF $\mathcal{E}(t)$ as $\mathrm{Re}\left[\tilde{I}\ \tilde{Z}\right]$
- Calculating physical EMF $\mathcal{E}(t)$ as $\mathrm{Re}\left[\tilde{I}\right]\mathrm{Re}\left[\tilde{Z}\right]$

Note: the correct procedure here may or may not be the same as for power … think carefully …

The Answers: $P(t) = \text{Re}\big[\tilde{I}\big]\,\text{Re}\big[\tilde{\mathcal{E}}\big]$ … but $\mathcal{E}(t) = \text{Re}\big[\tilde{I}\;\tilde{Z}\big]$. Why?!? Here's why:

- Power, current, and EMF are **physical observables** that we are choosing to *represent* as complex numbers for mathematical convenience. (What convenience? → The convenience of working with exponentials rather than trig functions, that's all.) The imaginary parts of such physical observables are mathematical baggage: they have no physical meaning, but are of great help in performing calculations. Thus: $P(t) = \text{Re}\big[\tilde{I}\big]\,\text{Re}\big[\tilde{\mathcal{E}}\big]$ is correct because physical-power is physical-current times physical-voltage.

- Impedances $\tilde{Z}$ and reactances $\tilde{X}_L$, $\tilde{X}_C$ are *not* physical observables, they are constructed quantites that are **complex by definition**. They are defined as a ratio of the *complex-representation-of-a-voltage* ($\tilde{V}$ or $\tilde{\mathcal{E}}$) over the *complex-representation-of-a-current* ($\tilde{I}$). Both the imaginary and real parts of these quantities are essential, as together they encode both the magnitude ratio and the phase difference between the voltage and current in question. It is *never* reasonable to drop the imaginary part of $\tilde{Z}$, $\tilde{X}_L$, or $\tilde{X}_C$. (See footnote[1] for a way to remember it.) Thus: $\mathcal{E}(t) = \text{Re}\big[\tilde{I}\;\tilde{Z}\big]$ is correct because $\tilde{\mathcal{E}} = \tilde{I}\;\tilde{Z}$ is literally the definition of $\tilde{Z}$. To find the real part of the left-hand-side, you have to calculate the full right-hand-side *then* take the real part, there are no shortcuts.

Generally, we are less interested in the time-dependence of power than in the **average power** delivered over a cycle. The cycle-average $\langle P\rangle$ of any quantity $P(t)$ that oscillates with angular frequency ω is:

$$\boxed{\langle P\rangle \equiv \frac{1}{T}\int_0^T P(t)\,dt} \qquad \text{where the oscillation period is} \qquad \boxed{T = \frac{2\pi}{\omega}}$$

(c) Calculate $\langle P\rangle$ for the result you got with the correct method, $P(t) = \text{Re}\big[\tilde{I}\big]\,\text{Re}\big[\tilde{\mathcal{E}}\big]$. See footnote[2].

(d) Calculate $\langle P\rangle$ for the incorrect version, $P(t) = \text{Re}\big[\tilde{I}\tilde{\mathcal{E}}\big]$, from the previous page. Problem, eh? ☺

[1] A good way to both understand and remember that you *never* drop the imaginary part of an impedance is to consider dropping the imaginary part of $\tilde{X}_L$ or $\tilde{X}_C$ → what would you get? *Zero!!* Those quantities are purely imaginary, not because they're meaningless, but because they encode phase shifts of 90° between current and voltage.

[2] Notice how the phase shift between the EMF and current plays a role even in the cycle-averaged power? That "cos(ϕ)" is often called the **power factor**.

Let's leave power and return to current-voltage analysis for some more practice relating physical measurements and complex numbers.

(e) You are given a black box whose contents are unknown to you. You attach a generator and measure the EMF and current it delivers. Your scope records $\mathcal{E}(t) = 20\ \sin(\omega t)$ V and $I(t) = -4\cos(\omega t)$ A. What is the impedance $\tilde{Z}$ of this circuit?

(f) When you open the box, what do you see? A resistor, a capacitor, and inductor, or some combination of some/all of them?

(g) One last double-check before we leave this subject. In office hours, I have observed people using all of the following relations to obtain the peak generator voltage, $\mathcal{E}_{\max}$, from the peak current, $I_{\max}$, in a circuit with impedance $\tilde{Z}$. (This is a very common stumbling block!) Which, if any, is correct? If you are even the slightest bit uncertain, please Ask! Your! Instructor!

- $\mathcal{E}_{\max} = I_{\max} |\tilde{Z}|$
- $\mathcal{E}_{\max} = I_{\max}\tilde{Z}$
- $\mathcal{E}_{\max} = I_{\max}\,\mathrm{Re}(\tilde{Z})$
- $\mathcal{E}_{\max} = I_{\max}\tilde{Z}^*\tilde{Z}$

Section 14.2: GGS to the Rescue!

Remember the Gauss-Green-Stokes theorems? They are in the box below. I hope that they are actually becoming obvious to you, thanks to your study of the physical meaning of grad, div, and curl, as the GGS theorems effectively *quantify* this physical meaning. Study the integrals on the right-hand slides, and you'll see that they exactly encode the concepts of "uphill slope" for gradient, "torque" for curl (via work around a closed loop), and "outward flux" for divergence.

$\mathbb{R}^1$: Gradient Theorem	$\mathbb{R}^2$: Stokes' Theorem	$\mathbb{R}^3$: Gauss' Theorem
$\int_{\vec{a}}^{\vec{b}} \vec{\nabla} f \cdot d\vec{l} = f(\vec{b}) - f(\vec{a})$	$\int_{\text{Surf}} (\vec{\nabla} \times \vec{E}) \cdot d\vec{A} = \oint_{\partial\text{Surf}} \vec{E} \cdot d\vec{l}$	$\int_{\text{Vol}} (\vec{\nabla} \cdot \vec{E})\, dV = \oint_{\partial\text{Vol}} \vec{E} \cdot d\vec{A}$

One of the many applications of GGS is as an **integration trick**: a technique to simplify 3D integrals. This trick is only usable in some problems, and it doesn't always help … but it is sometimes a *life saver!* The idea is this: if your integral has the form of one side of a GGS theorem (or can be manipulated until it does), the other side of the theorem might provide a much easier integral to calculate. In both of the following problems, GGS provides a *vastly* easier solution, if you can figure out how to use it. ☺

(a) Consider a cubic region of space, whose sides are parallel to the *x*, *y*, and *z* axes and whose opposing corners are at the points $(x,y,z) = (0, 0, 0)$ and (L, L, L). This region of space is occupied by an electric field of the form $\vec{E}(\vec{r}) = C\, s^3\, \hat{s}$ where *C* is a constant with appropriate units. Calculate the flux $\Phi = \oint_{cube} \vec{E} \cdot d\vec{A}$ through the cube. Try to figure out the GGS trick on your own … but a hint is in the footnote[3]. And a self-check: the correct answer is $8CL^5 / 3$.

[3] You need the divergence theorem (GGS #3) for this one … which is pretty obvious given the form of the integral! ☺

(b) Before we get to the second example, do you recall the **Irrotational Field Theorem** that we proved in lecture? It describes the class of vector fields that have no curl, i.e. fields that *don't "twist"*. The theorem follows from the GGS theorems plus one calculus identity, $\vec{\nabla}\times(\vec{\nabla}g)=0$, and it is intimately related to the physical meaning of the curl and the gradient. The theorem states that the four statements below concerning a vector field $\vec{F}(\vec{r})$ are <u>equivalent</u>: <u>if any one of them is true, then they're all true</u>. See if you can use your intuition about curl and gradient to fill in the blanks below and complete the theorem. To help you, let me remind you of they key physics application we discussed for this theorem: **conservative forces** are irrotational fields.

1. $\vec{\nabla}\times\vec{F}=0$ everywhere
2. $\oint\vec{F}\cdot d\vec{l}=$_______ for any closed loop
3. $\int_{\vec{a}}^{\vec{b}}\vec{F}\cdot d\vec{l}$ is independent of ________, i.e. it depends only on the endpoints $\vec{a}$ and $\vec{b}$.
4. $\vec{F}=$_________ for some scalar field $g(\vec{r})$

Did you figure it out? Flip to the back page and you will find yet another version of our 3D calculus formula sheet, with yet more content. I've added both the irrotational field theorem and its close friend, the **Divergenceless Field Theorem**. We haven't talked about this latter theorem, but have a look: these four statements are also equivalent, and describe the class of fields that *don't "diverge"*:

1. $\vec{\nabla}\cdot\vec{B}=0$ everywhere
2. $\oint\vec{B}\cdot d\vec{A}=0$ for any closed surface
3. $\int_S\vec{B}\cdot d\vec{A}$ depends only on the boundary (∂S) of the surface S
4. $\vec{B}=\vec{\nabla}\times\vec{G}$ for some vector field $\vec{G}(\vec{r})$

(c) Look at those four statements: consider their meaning and and see if their equivalence makes sense to you. Now see if you can answer this question: can you think of a physics application for this theorem? i.e. Can you think of an example in physics of a vector field that is always divergenceless?

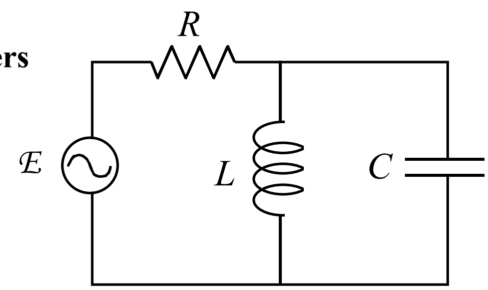
With that discussion under our belts, it's time for our second example of GGS trickery.

(d) A space shuttle pilot flies her shuttle from the equator to the north pole following a meridian (i.e. a line of constant azimuthal angle ϕ). The altitude part of her trajectory is complicated. It follows the curve $r(\theta) = R\,[\,1+\sin(2\theta)\,]$, which means that she starts at the surface of the earth (whose radius is R), then climbs to a very high radius $2R$, then returns to the surface at the north pole. Her shuttle must contend with the force field $\vec{F} = Ks\,\hat{s}$ the whole way. Calculate the amount of work done by the force field on the shuttle over the course of this trip. Try it without a hint first, but if you need one, see the footnote[4]. And a self-check: the correct answer is $-KR^2/2$.

(e) I hope you found the trick! You can do this integral by brute force – it's good practice, in fact! – but it will take you *waaaaay* longer to finish. If the force field experienced by the space shuttle had a different form, however, the trick you just used might not be available. Imagine that you are the instructor and you are preparing the problem above for a final exam. You are a mean and vicious instructor, and you want to make sure the students *cannot* use any GGS trickery! Invent a force field $\vec{F}(\vec{r})$ that would prevent the use of this time-saving technique and force the students to do the integral via brute force. (Hint: you can do it by changing one letter in the force field above, but you'll need the irrotational field theorem to figure out what letter to change.)

[4] Two tricks are possible here: either use the gradient theorem (GGS #1), or use the irrotational field theorem and the *path-independence* it implies for this particular integral. Note that you cannot use Stokes' theorem (GGS #2) because this integral is not over a *closed* path.

Section 14.3: Mastery! Kirchoff's Laws with Complex Numbers

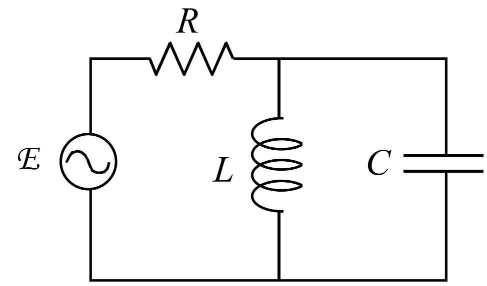


Consider the circuit shown at right → another example of a circuit we cannot solve using Physics 212 technology.

(a) First, find the impedance $\tilde{Z} = Z e^{i\phi}$ of this particular R-L-C combination. Manipulate your expression until you obtain equations for the magnitude Z and phase ϕ of the impedance in terms of the real numbers R, X_L, and X_C.

(b) As you know, the good-old series-R-L-C circuit from Physics 212 has a magic "resonant" frequency $\omega_0 = 1/\sqrt{LC}$ where the impedance is minimized. What is the relation between X_L and X_C at this frequency?

(c) As it happens, our new circuit *also* behaves in a special way at the frequency $\omega_0 = 1/\sqrt{LC}$ → what happens to the impedance $\tilde{Z}$ of the circuit at the special frequency ω_0? And what does that tell us about the current through the resistor, for example? You may be surprised at your result … but can you explain why it makes sense → what is this circuit *doing* at $\omega = \omega_0$? Take your best shot at understand what's going on with this circuit … then continue to the next pages, where you will calculate the behavior of the circuit in detail!



To explore the behavior of this circuit further, we will turn to Kirchoff's Current & Voltage Laws (KCL & KVL). They work just fine for AC circuit analysis with complex numbers:

KCL: $\sum \tilde{I}_{in} = \sum \tilde{I}_{out}$ at each **node**

KVL: $\sum \tilde{V}_i = 0$ around each **loop**

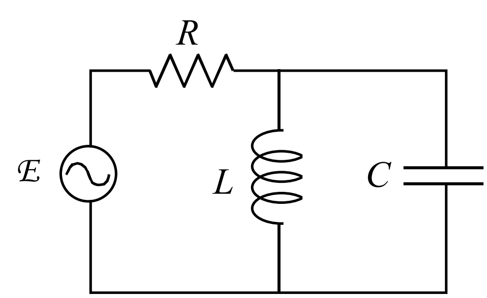
We'll take the generator voltage to be $\tilde{\mathcal{E}}(t) = \mathcal{E}_0 e^{i\omega t}$, then we'll calculate all the (complex) currents.

(d) First, identify the <u>independent branch currents</u> in the circuit. How many are there for this circuit? Assign them symbols and directions (arrows) on the circuit diagram. Normally you number the currents (I_1, I_2, …), but since each one passes through a unique element of our circuit, let's label them $\left(\tilde{I}_{R,} \tilde{I}_{L,} \tilde{I}_C\right)$

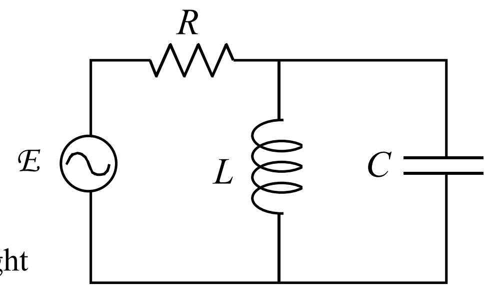
(e) Remember: if there are *N* nodes in a circuit, there will be *N*–1 *independent* KCL equations relating the branch currents. Write down those *N*–1 independent KCL relations (ok ok, there's only one ☺). Just remember, your currents are *complex*, so give them "~" symbols to remind yourself of this.

(f) To solve for the three branch currents in this problem we need three equations. One came from KCL … the other two have to come from KVL. Select two loops in the circuit and write down the corresponding KVL relations. Your equations should involve the branch currents $\tilde{I}_i$, the generator-EMF $\tilde{\mathcal{E}}$, the real values R, X_L, X_C … and of course the Zen-number *i*. Just write them down, no solving yet.

(g) Now it's time to *solve* your three Kirchoff equations for the three branch currents. To simplify the algebra, let's consider *only* the special case when $\omega = \omega_0 \rightarrow$ you may thus just use the symbol X to refer to the equal-reactances $X_C = X_L$. Rewrite your three Kirchoff equations with this simplification.



If all went well, your 3 Kirchoff equations should look like this:



KCL: $\tilde{I}_R = \tilde{I}_L + \tilde{I}_C$

KVL-1: $\tilde{\mathcal{E}} - \tilde{I}_R R - \tilde{I}_L (iX) = 0$

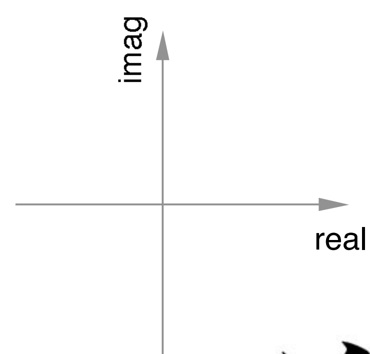
KVL-2: $\tilde{I}_L (iX) - \tilde{I}_C (-iX) = 0$

Note: The specifics might not be exactly the same, since you might have chosen different directions for the currents (my I_R and I_C go to the right at the top, and I_L goes down). You might also have chosen a different KVL loop: the outer one that goes through the battery, *R*, and *C*. (There is always one more loop than you need, and one more node than you need.) But I made the most common choices, so our equations probably match.

(h) Now solve your equations to <u>determine the three complex branch currents</u> $\tilde{I}_i(t)$ in terms of the complex generator-EMF $\tilde{\mathcal{E}}(t)$, *R*, *X*, and/or numerical constants.

(i) For each of your complex currents: (1) substitute in the time-dependence, which comes from the generator: $\tilde{\mathcal{E}}(t) = \mathcal{E}_{\max} e^{i\omega t}$; (2) write down the <u>real, physical currents</u> $I_R(t)$, $I_L(t)$, and $I_C(t)$ that you would measure on a scope; and (3) determine the <u>peak values</u> of these three currents.

(j) To help you visualize what is going on, <u>draw</u> the three complex currents and the generator-EMF on the given diagram at some time *t* of your choosing. (*t* = 0 might be a good choice ☺)



(k) Finally, calculate the (complex) voltages $\tilde{V}_R(t)$, $\tilde{V}_C(t)$, and $\tilde{V}_L(t)$ across the three circuit elements. Add those to the diagram to complete your understanding of the behavior of this circuit!