Least-Action Sandbox

System = mass in uniform gravity, 1D vertical motion

Endpoints = (y,t) from (0,–τ) → (0,+τ)

∴ Lagrangian $L(y,\dot{y},t)\,\frac{2}{m} = \dot{y}^2 - 2gy$

Path Class #1 : <u>constant speed</u> going up & coming down → vary **H** = max height

- paths $\boxed{y(t) = H\left(1 - \frac{|t|}{\tau}\right)} \rightarrow \dot{y} = \pm\frac{H}{\tau}$

- Lagrangian $L(t)\,\frac{2}{m} = \left(\frac{H}{\tau}\right)^2 - 2gH\left(1 - \frac{|t|}{\tau}\right) \rightarrow L(t)\,\frac{2}{m(g\tau)^2} = \left(\frac{H}{g\tau^2}\right)^2 - 2\frac{H}{g\tau^2}\left(1 - \frac{|t|}{\tau}\right)$

define $b \equiv \frac{H}{g\tau^2}$ and vary that ... true path has $H = \frac{g\tau^2}{2}$ $\therefore\ b_{\text{true}} = \frac{1}{2}$

$L(t)\,\frac{2}{m(g\tau)^2} \equiv \boxed{\bar{L}(t) = b^2 - 2b\left(1 - \frac{|t|}{\tau}\right)}$ … vary b and see what happens !

```
plot b^2-2b(1-Abs[t]) for b={0.1,0.25,0.5,0.75,1} from t=-1 to 1
```

- Action $\boxed{\bar{S} = \int_{-\tau}^{+\tau} \bar{L}\,dt = 2\tau\, b(b-1)}$

```
plot b(b-1) from b=0 to 1.5
```

- Least Action $\frac{1}{2\tau}\frac{d\bar{S}}{db} = 2b - 1$ → minimum action at $b = b_{\text{true}} = \frac{1}{2}$! ☺

Path Class #2 : <u>constant acceleration</u> → vary **a** = acceleration

- paths $y(t) = v_0(t+\tau) - \frac{1}{2}a(t+\tau)^2$ and $y(\pm\tau) = 0$ $\therefore\ v_0 = a\tau$ → $\boxed{y(t) = \frac{a}{2}\left(\tau^2 - t^2\right)}$

- Lagrangian $L(t)\,\frac{2}{m} = a^2t^2 + ga\left(t^2 - \tau^2\right) \rightarrow L(t)\,\frac{2}{m(g\tau)^2} = \frac{a}{g}\left[\frac{t^2}{\tau^2}\left(\frac{a}{g} + 1\right) - 1\right]$

define $b \equiv \frac{a}{g}$ and vary that ... true path has $b_{\text{true}} = 1$ of course!

$L(t)\,\frac{2}{m(g\tau)^2} \equiv \boxed{\bar{L}(t) = b\left[\frac{t^2}{\tau^2}(b+1) - 1\right]}$ … vary b and see what happens !

```
plot b(t^2(b+1)-1) for b={0.4,0.7,1,1.3,1.6} from t=-1 to 1
```

- Action $\boxed{\bar{S} = \int_{-\tau}^{+\tau} \bar{L}\,dt = \frac{2\tau\, b}{3}(b-2)}$ .... `plot b(b-2)`

- Least Action $\frac{3}{2\tau}\frac{d\bar{S}}{db} = 2b - 2$ → minimum action at $b = b_{\text{true}} = 1$ ☺