Phys 325 Discussion 11 – Welcome to Lagrangian Mechanics

Procedure for Lagrangian Mechanics: In last week’s lectures, we presented the elements of the Lagrangian approach to mechanics and worked some examples. This week we will prove that the approach is valid, but the proof will be much more meaningful to you if you have worked with the procedure first. Below, we will go through each step of the Lagrangian procedure for solving a mechanics problem using a simple example. This week’s homework also presents these steps, so if you’ve started the homework already, you don’t need to read the paragraphs describing each step.

Problem 1: Step-by-Step

The Example Problem: A bead of mass \( m \) is threaded on a circular wire ring of radius \( R \). The ring is placed in the \( xy \)-plane, centered on the origin, and not allowed to move. Uniform gravity \( g \) points in the \(-y\) direction, and there is no friction between the bead and the wire ring. Our goal is to determine the motion of the bead.

- **Step 1:** Figure out the number of degrees of freedom (DOF) of the system. We will call it \( n \). It is the number of coordinates of the system that can be varied independently. To illustrate, a system composed of \( N \) point-like particles would have \( 3N \) degrees of freedom, since each particle can move in three independent spatial directions. However, if the particles are subject to constraint forces (normal forces, tensions, the binding forces that keep the \( \approx 10^{23} \) atoms of a rigid object in a fixed shape, etc), restrictions are imposed on how the particles can move. This reduces the number of degrees of freedom to \( n = 3N - n_c \), where \( n_c \) is the number of constraint equations imposing restrictions on the coordinates’ values. In summary, \( n \) is the minimum number of coordinates you need to completely describe the system’s configuration, given its constraints.

  (a) How many degrees of freedom, \( n \), does our example problem have?

- **Step 2:** Select the \( n \) generalized coordinates \( \{ q_1, q_2, \ldots, q_n \} \) that you will use to describe the state of your system. These \( q_i \) can be positions, angles, combinations thereof, etc. They can be anything as long as they
  - completely describe the system’s configuration at any given moment, and
  - are independent of each other, meaning that you can change one without changing any of the others.

(b) What generalized coordinate(s) could you choose for this problem?

- **Step 3:** Calculate the Lagrangian \( L(q_i, \dot{q}_i, t) = T - U \) of the system. Here \( T \) and \( U \) are the system’s total kinetic and potential energy respectively, written entirely in terms of your chosen generalized coordinates \( q_i \) and/or the independent variable \( t = \text{time} \). Important distinction: \( U \) is the potential energy under the influence of all supplied forces; the constraint forces (if there are any) do not appear in the Lagrangian at all. As we will prove, the Lagrangian prescription itself takes care of the constraint forces, which is one of the great virtues of Lagrangian mechanics!

\[ L = \frac{1}{2} m R^2 \dot{\phi}^2 - mgR\sin\phi \]

Hints & Checkpoints 1

1 (a) 1 (b) If you are picking a coordinate from one of our 3 standard systems, the azimuthal angle \( \phi \) is by far the best choice. The bead’s \( x, y \), or \( s \) coordinates may seem like reasonable alternatives, but they are a bit dangerous as they don’t completely describe the bead’s position; knowing the bead’s \( x \)-coordinate, for example, only restricts it to one of two positions on the ring. If you are only interested in the motion of the bead on one half of the ring (e.g. if you only care about its motion near equilibrium), then one of these alternatives would be fine, but not very pleasant to work with.

(c) \( L = \frac{1}{2} m R^2 \dot{\phi}^2 - mgR\sin\phi \)  
(d) \( R \dot{\phi} = -g \cos\phi \)

(e) Reminder 1: You’ll need to make a Taylor approximation for small angles, as usual, but \( \phi \) is not small in this situation; rather its deviations from \( \phi_0 \) are small. Reminder 2: Change variables! Rewrite your equation of motion in terms of the variable \( \varepsilon = \phi - \phi_0 \), which is small near equilibrium.

Reminder 3: For a near-equilibrium analysis, \( \varepsilon, \dot{\varepsilon}, \text{and} \ddot{\varepsilon} \) can all be made arbitrarily small; find the lowest order of \( \varepsilon, \dot{\varepsilon}, \text{and/or} \ddot{\varepsilon} \) that appears in your EOM — that’s the “lowest non-vanishing order” — then drop all terms of higher order. The final result is \( \omega = \sqrt{g / r} \). 

(f) \( L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy \), constraint equation is \( x^2 + y^2 = R^2 \) (g) EOM: \( \ddot{x} = 0, \ddot{y} = -g \)
(c) The Lagrangian must be expressed in the form $L(q_i, \dot{q}_i, t)$: it must be written entirely in terms of your chosen generalized coordinates, their derivatives, time, and/or constants. *No other variables may appear!* Why? Because the $q_i$’s provide a complete description of the system, so any other variables would depend on them; if you leave such a non-$q_i$ variable in your Lagrangian, you will have a hidden dependence and this will cause errors. (This is exactly like setting up a multi-dimensional integral: you choose a set of integration parameters, then you make absolutely sure that all variables in your integrand are expressed in terms of them.)

→ Write down the Lagrangian for the bead in terms of $\phi$, $\dot{\phi}$, $t$, and/or constants.

Suggestion: The generalized coordinates you choose are of paramount importance, so it’s good practice to circle them right at the start of your work. Do it: write down $\phi$ and circle it! As you will discover, you usually have to introduce other variables when calculating your Lagrangian, so it is easy to lose track of your choice of $q_i$’s.

● Step 4: Apply the Euler-Lagrange equations to your Lagrangian to get the system’s $n$ equations of motion:

$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right)$$

for each generalized coordinate $q_i$

(d) Write down the bead’s one equation of motion. It will be a second-order differential equation, just like Newton’s second law ($\bar{F}_x = m \ddot{x}$), that can be solved for $\phi(t)$.

● Step 5: Solve the equations of motion to determine the system’s behavior $q_1(t), q_2(t), ..., q_n(t)$.

(e) The differential equation you obtained cannot be solved analytically for $\phi(t)$. We can solve it in an approximate case, however: find the frequency $\omega$ of small oscillations of the bead around the stable-equilibrium position $\phi_0 = -\pi/2$. If you have forgotten the procedure for small-oscillation analysis, three reminders are provided in the checkpoint.

(f) That’s how Lagrangian mechanics works! You go through the same 5 steps every time. Now let’s address a common question: what is the reason for those first two steps? Why did we make such a big deal about figuring out the number of DOF and choosing exactly $n$ generalized coordinates? Answer: the entire formalism depends on it. You will see this when we go through the proof, but nothing builds intuition like experimentation! Let’s try violating this rule and see what happens. ✨ Let’s use $x$ and $y$ as our coordinates. We know that they’re not independent of each other because of the ring, but why not apply that constraint at the end of the procedure instead of at the start? Let’s try it!

→ Forget about the ring and write down the Lagrangian for the bead in terms of $x$, $\dot{x}$, $y$, $\dot{y}$, $t$, and/or constants.

It is a really easy task … this rebellious plan is going well so far! ✨

→ Write down the constraint equation that relates $x$ and $y$ to each other because of the ring’s shape.

Our rebellious plan is to apply this constraint at the end, after we’ve determined the equations of motion.

(g) Since you have two coordinates, you will get two Euler-Lagrange equations of motion. Write them down.

(h) Now apply the constraint equation to replace all the $y$’s with $x$’s … and you will find that it’s impossible. *The equations of motion are already hopelessly wrong.* That, friends, is the reason for the first two steps. ✨
Problem 2: Atwood’s Machine

You remember Atwood’s Machine from Homework 6? It consists of two masses tied to the ends of a massless string of length \( l \), with the string running over a massless pulley whose axle is fixed in place.

(a) Use the Lagrangian prescription to find the accelerations \( \ddot{x} \) and \( \ddot{y} \) of \( m_1 \) and \( m_2 \).

(b) Was that really easier than analyzing this device using Newton’s force-based procedure? You decide! Our Lagrangian procedure is new, so we should investigate exactly how our it compares with our old practice of calculating \( \ddot{x} \) using \( F = ma \). But remember, you have tensions and normal forces — constraint forces — to deal with in a force analysis. You must draw free-body diagrams showing all the forces acting on each object (the two masses and the pulley, though for this particular problem you will only need to analyze the forces on each mass). Use this force-based approach to calculate \( \ddot{x} \). There is a 3-in-1 free-body diagram on the next page to help you.

Problem 3: Double Atwood Machine

So the Lagrangian method didn’t particularly speed up our analysis of the Atwood machine, but even for such a simple problem you can already see one of its advantages: by getting rid of all constraint forces, it requires less thought (if not less paper) and so fewer possibilities for errors. Where the Lagrangian method really shines is for constrained systems with more than one degree of freedom. To see that in action, let’s revisit the notorious double Atwood machine, with two pulleys and two strings (both massless). On Homework 6, your force-based analysis of this system required four separate EOMs to solve the system … let’s see how Monsieur Lagrange does with this one! This system has two degrees of freedom; let’s choose the \( x \) and \( y \) distances labelled on the figure as our generalized coordinates. Recall that the upper pulley has its axle fixed in place, but the lower pulley can move since it is attached to the upper pulley’s string.

(a) Use the Lagrangian prescription to calculate the acceleration \( \ddot{x} \) of the mass \( 4m \).

But first, read this note about technique (also discussed in lecture):

**TECHNIQUE:** Dropping constant terms & scale factors from the Lagrangian

Our only use for the Lagrangian is to plug it into the Euler-Lagrange equations and get our system’s equations of motion. Since these equations involve only the derivatives of \( L \), we get our first piece of strategy:

- You can always **discard constant terms** from a Lagrangian

(Since \( L = T - U \), this is equivalent to the familiar fact that adding a constant to \( U \) changes nothing whatsoever.) A second simplification arises if you notice that \( L \) appears on both sides of the Euler-Lagrange equations:

- You can **drop constant multiplicative factors** from \( L \) without affecting the Euler-Lagrange equations

For example, if your have \( L = 3m\dddot{x} - mg \), you can drop the overall factor of \( m \) and just use \( L = 3\dddot{x} - g \).

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2 (a) Hint: This problem has only one DOF. You can pick \( x \) or \( y \) as your generalized coordinate, but *not both* (!) since they are related by the constraint equation \( x + y = l \). Depending on your choice, the Lagrangian is \( L(x,\dot{x},t) = \frac{1}{2}\dot{x}^2 (m_1 + m_2) + gx(m_1 - m_2) \) or \( L(y,\dot{y},t) = \frac{1}{2}\dot{y}^2 (m_1 + m_2) + gy(m_2 - m_1) \). Final answers: \( \dddot{x} = g(m_1 - m_2) / (m_1 + m_2) \) & \( \dddot{y} = -x \) (b) self-checking

3 (a) With constant terms and factors dropped, \( L = 2\dddot{x}^2 + \dddot{y}^2 - \dddot{x}\dddot{y} + gy \rightarrow \dddot{x} = g / 7 \) and \( \dddot{y} = 4g / 7 \) (b) you decide ☺
(b) Was that easier than the force-based analysis? For a reminder of its complexity, the 5-in-1 free body diagram you needed in Homework 6 is shown below.

**Problem 4: Cylinder on a Ramp**

A cylinder of mass $m$, radius $R$, and moment of inertia $I$ (for rotation around its central axis) rolls without slipping straight down an inclined plane which is at an angle $\alpha$ from the horizontal. Use as your generalized coordinate the cylinder’s distance $x$ measured down the plane from its starting point. Use the Lagrangian procedure to determine the cylinder’s acceleration $\ddot{x}$.

Free-body diagrams for single and double Atwood machines:

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\[ L = \frac{\dot{x}^2}{2} \left( m + \frac{I}{R^2} \right) + mgx \sin \alpha \rightarrow \ddot{x} = \frac{g \sin \alpha}{1 + I/mR^2} \]