The Hamiltonian formulation of mechanics is a modified version of Lagrangian mechanics. At its heart, the Lagrange-to-Hamilton transition is a change of variables. Consider a system with \( n \) degrees of freedom, whose configuration at any given moment is described by \( n \) generalized (i.e., independently-variable) coordinates \( q_i \). The complete condition of the system is described a bit differently by Lagrange and Hamilton:

- **In Lagrangian** mechanics: with \( n \) gen. coordinates \( q_i \) and their corresponding velocities \( \dot{q}_i \).
- The Lagrangian at the heart of the prescription is written as \( L(q_i, \dot{q}_i, t) \) and equals \( T - U \).
- **In Hamiltonian** mechanics: with \( n \) gen. coordinates \( q_i \) and their conjugate momenta \( p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \).
- The Hamiltonian at the heart of the prescription is written as \( H(q_i, p_i, t) \) and equals \( p_i \dot{q}_i - L \).

Here is the “space” terminology for the various descriptors of a system’s condition:

- the set of \( n \) coordinates \( (q_i) = \text{configuration space} \) → used by both Lagrange & Hamilton
- the set of \( n \) coordinates + \( n \) velocities \( (q_i, \dot{q}_i) = \text{state space} \) → used by Lagrange
- the set of \( n \) coordinates + \( n \) momenta \( (q_i, p_i) = \text{phase space} \) → used by Hamilton

Now for the substantive difference in the two prescriptions: the **equations of motion** (which is, after all, what a mechanics prescription is all about!)

- **Lagrangian** mechanics uses \( n \) 2nd-order EOMs : \( \frac{\partial L}{\partial q_i} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) \)
- **Hamiltonian** mechanics uses \( 2n \) 1st-order EOMs : \( \dot{q}_i = \frac{\partial H}{\partial p_i} \) and \( \dot{p}_i = -\frac{\partial H}{\partial q_i} \) with \( H = p_i \dot{q}_i - L \)

That’s all there is to the Hamiltonian prescription: it is completely equivalent to Lagrangian mechanics, but it changes the equations of motion from 2nd-order to 1st-order (an advantage) while doubling the number of such equations (a disadvantage). And don’t forget the change-of-variables aspect: everything in Hamilton mechanics is written in terms of coordinates and momenta \( (q_i, p_i) \) while Lagrange uses coordinates and velocities \( (q_i, \dot{q}_i) \).

So let’s try it out!

### Problem 1: Welcome to Hamilton’s Equations

*(a) Consider a particle of mass \( m \) that can move in 2 dimensions, \( x \) and \( y \), and is subjected to a linear restoring force \( \vec{F} = -kx \hat{x} \) that keeps pulling it back toward \( x = 0 \). There is no force in the \( y \) direction. To keep our starter example simple, there are also no constraints on the particle’s motion. So: write down Hamilton’s four equations of motion for this system using \( x \) and \( y \) as your generalized coordinates.*

**How to start with this task??** The first thing to realize is that you need the Lagrangian, in almost all cases. Write down the Lagrangian, then determine the momenta \( p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \), then find the Hamiltonian \( H = p_i \dot{q}_i - L \), then change all velocities \( \dot{q}_i \) to momenta \( p_i \) to get \( H(q_i, p_i, t) \), and finally write down the four equations of motion. Yes, that’s the procedure.

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1. **(a) \( H = (p_x^2 + p_y^2)/2m + kx^2/2 \) → 4 EOMs: \( \dot{x} = p_x/m, \ \dot{y} = p_y/m, \ \dot{p}_x = -kx, \ \dot{p}_y = 0 \)** (b) same
2. **(c) \( x(t) = A\cos(\omega t - \delta) \) where \( \omega = \sqrt{k/m} \), \( y(t) = y_0 + v_0 t \); note that \( A, \delta, y_0, v_0 \) are the 4 free constants to be set by the initial conditions** (d) paths: \( p_x^2/m^2 + x^2\omega^2 = C^2 \), you figure out what they look like ⊗ (e) enjoy ⊗
Familiar Equations: You will quickly recognize the significance of Hamilton’s 2n equations. The first group $\dot{q}_i = \frac{\partial H}{\partial p_i}$ are always the definitions of the momenta in terms of the velocities, and the second group $\dot{p}_i = -\frac{\partial H}{\partial q_i}$ are the actual equations of motion in the form $F = dp/dt$.

(b) Is there an easier way to obtain Hamilton’s equations? In part (a) I said “you need the Lagrangian, in almost all cases”. What are the exceptions? Well, what did you use the Lagrangian for? You used it to obtain two things: the Hamiltonian $H$ and the generalized momenta $p_i$. The only cases when you can bypass the Lagrangian are when you can obtain both $H$ and $p_i$ by other means.

- The Hamiltonian can quite often be figured out much more quickly than by calculating $H = p_i \dot{q}_i - L$ because it is quite often the total energy of the system. Remember how that works? When there are no time-dependent constraints around, $H = T + U \rightarrow$ much easier! (We will have a reminder problem about that below.)
- However, even if you know that $H = T + U$, that will only give you $H(q_i, \dot{q}_i, t)$; you still need to change variables and write it as $H(q_i, p_i, t)$, and that requires figuring out the generalized momenta $p_i$ in terms of $q_i$ and $\dot{q}_i$. In general, this is only possible in problems like this one where there are no constraints at all. Then generalized-momentum is just momentum, which is just $p_i = mv_i$.

Use these simplifications to obtain Hamilton’s 4 equations for this simple unconstrained system using the fastest possible method.

(c) Obtain the general solution $x(t), y(t)$ for Hamilton’s equations of motion. As you will see, the very first thing you do is combine the 2n equations into exactly Lagrange’s equations ... the advantages of Hamilton’s formulation are not found in the find-the-actual-solution part of analyzing a system.

(d) So where are the advantages? Why are we doing this at all? We will see in lecture that it has to do with phase space: the paths of a system through $(q, p)$ space have some interesting properties that are not found in $(q, \dot{q})$ state space. As a modest introduction to paths of systems through phase space, ignore the $y$ part of this problem for the moment (happily, it is completely independent of the $x$-part so ignoring it is easy!) and see if you can plot the shape of your solutions in the space $(x, p_x)$. To do so, use your general solution for $x(t)$ to come up with a path equation as we did in our study of central force problems, combine your time-dependent solutions $x(t)$ and $p_x(t)$ in such a way as to get rid of $t$ and obtain $p_x(x)$. That will give you the shape of the system’s “orbits” = their “paths” through $(x, p_x)$ space. Now plot them! You should have a free parameter to play with (one of the two constants of integration that you always get when solving EOMs will survive); that parameter produces a class of orbits that are all possible paths of the system, so play around and plot a few of them!

(e) Now ignore the $x$ direction and see what paths the system can take through the $(y, p_y)$ slice of phase space. This seems simpler than the $x$ part, but it is a strange exercise.

Why is Phase Space Interesting? If all went well with your plots, you should have found that none of your system’s paths ever cross. Really! This observation is the start of our upcoming proof of Liouville’s theorem, which has extensive applications in statistical and quantum mechanics. The proof is developed in the last sections of Taylor’s Chapter 13 and we will cover it in lecture.
Problem 2: When is the Hamiltonian not the Energy?

We’ve covered the question in the title before, but it bears repeating. The precise answer is: $H = T + U$ when the generalized coordinates are natural coordinates, i.e. when the $3N$ Cartesian coordinates $r_{ai}$ of the system can be written purely in terms of the generalized coordinates, as $r_{ai} = r_{ai}(q^1, \ldots, q^N)$. (Here, $a$ is the particle index, running from 1 to $N$ particles, and $i$ is a Cartesian subscript running through the 3 values $x,y,z$.) These are the cases when this transformation between the Cartesian coordinates and the generalized coordinates is independent of time, and in these cases, the Hamiltonian is the total energy of the system.

To remind yourself how this works, let’s take an example instead of a general proof. Consider a single particle of mass $m$ that is sitting in some potential $U(s,z,\phi)$ . We will use $s$ and $\phi$ as our generalized coordinates and constrain the particle’s $z$ coordinate in two different ways.

(a) Let’s give the particle the following constraint: $z = z(s)$, i.e. the particle is forced to move on some surface that is cylindrically symmetric around the $z$ axis (for example, a bowl-like surface such as $z(s) = as^2$.) Write down the transformation equations that give the particle’s Cartesian coordinates $r_i = x, y, z$ in terms of the generalized coordinates $q_i = s, \phi$.

(b) Are our generalized coordinates natural coordinates?

(c) The definition of the Hamiltonian is $H \equiv p_i \dot{q}_i - L$. Clearly, the condition $H = \text{energy} = T + U$ is exactly equivalent to the condition $p_i \dot{q}_i = 2T$, so that’s what we’re going to test. Write down $2T$ in cylindrical coordinates, then apply the constraint $z = z(s)$. (You can use the specific form $z(s) = as^2$ if that makes you more comfortable.) Now write down the Lagrangian, figure out the two conjugate momenta $p_s$ and $p_\phi$, then see if $p_i \dot{q}_i = 2T$. Is the Hamiltonian the total energy in this case?

(d) Now for a different case. Let’s give the particle a time-dependent constraint: $z = z(t)$ . (You can use $z = v_0t$ as a specific example if you want → that would be the particle sitting on a flat table that is moving upward with constant velocity $v_0$.) Write down the transformation equations that give the particle’s Cartesian coordinates $r_i = x, y, z$ in terms of the generalized coordinates $q_i = s, \phi$.

(e) Are our generalized coordinates natural coordinates in this case?

(f) Finally, see if $p_i \dot{q}_i = 2T$. Is the Hamiltonian the total energy in this case?

Interpretation: Time-dependent constrains like having our table move serve to sneak in additional forces that do not appear in $U$. This is the underlying reason that time-dependent constraints ruin the relation $H = T + U$, though it is not entirely trivial to see.

(g) Is the Hamiltonian conserved in the case $z = v_0t$ ?

Remember: Even if the Hamiltonian is NOT the total energy, it can still be conserved!

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2 (a) $x = s \cos(\phi), y = s \sin(\phi), z = z(s)$ (b) yes! all $r_i$ are functions of only $s$ and $\phi$ (c) yes! $2T = m(\dot{s}^2 + s^2 \dot{\phi}^2 + z'^2 \dot{z}^2)$, $p_s = ms(1 + z'^2)$, and $p_\phi = ms \ddot{\phi}$. (d) $x = s \cos(\phi), y = s \sin(\phi), z = z(t)$ (e) no! $z$ is now a function of time, not just $s$ and $\phi$. (f) no! $2T = m(\dot{s}^2 + s^2 \dot{\phi}^2 + z'^2)$, $p_s = ms \ddot{s}$, and $p_\phi = ms \ddot{\phi}$ → $2T$ is NOT the same as $p_i \dot{q}_i$ precisely because of the time-dependent constraint $z(t)$ (g) yes! $L = m(\dot{s}^2 + s^2 \dot{\phi}^2 + v_0^2) / 2 - U(s,\phi)$ does not depend on time, so the Hamiltonian is conserved, even though it is different from the total energy by the term $mv_0^2 / 2$. 