Phys 325 Discussion 7 – Small Oscillations & Equilibrium

Here is a phrase that pops up all over the place: **Small Oscillations**. It is closely connected to the notions of **equilibrium** and stable-vs-unstable equilibrium. Let's investigate!

- If we say an object is **oscillating**, we mean it moves "back and forth". Put different, if an object oscillates, it keeps returning to the same position. (This statement might be approximate, e.g. if the oscillation is heavily damped by friction, the oscillations will eventually stop.)
- If the object keeps returning to the same position, there must be a **restoring force** that pulls it toward some **equilibrium position** that it either passes through repeatedly (or circles around, in 2D or 3D problems).
- So we have an equilibrium position around which the object is oscillating. Let's introduce some generic coordinate "***x***" to describe the state of our system, and let "***x* = 0**" denote the system's equilibrium position. (Note that "*x*" could be an angle, or if our system has more than one dimension it could be a set of coordinates; the point is that "*x*" denotes the "state of the system".)
- We must have a restoring force that keeps pulling the system *toward* the equilibrium position $x = 0$. The force component F_x must therefore be an **odd function of *x*** :
  - when $x > 0$, F_x must be negative to pull the system back toward $x = 0$
  - when $x < 0$, F_x must be positive to pull the system back toward $x = 0$
- Now we come to the "**small**" part of "small oscillations". This means we are restricting ourselves to motions where the coordinate x remains **close to the equilibrium position** x=0. In other words, x is always small.
- If x is always small, the restoring force $F_x(x)$ can be fruitfully **approximated** with a **Taylor series** around x=0:

$$F_x(x) \approx F_x\big|_{x=0} + x\frac{dF_x}{dx}\bigg|_{x=0} + \frac{x^2}{2!}\frac{d^2F_x}{dx^2}\bigg|_{x=0} + \text{negligible}$$

  But we also know that $F_x(x)$ is an *odd* function of x (since it's a *restoring* force), so the constant and quadratic terms above have to be zero and the coefficient of the linear term has to be negative. Note: another reason the constant term has to be zero at x=0 is because x=0 is an equilibrium point, i.e. zero force there! Result:

$$F_x(x) \approx -kx \qquad \text{where the constant } k \equiv -\frac{dF_x}{dx}\bigg|_{x=0} \text{ is positive}$$

- The message : if we stick to small enough x, our restoring force must have an approximately **linear** dependence on x. A linear force with a negative constant? We've seen that before ➔ that's the good-old spring force = Hooke's Law! If you inject that into $F_x = m\ddot{x}$, you get this super-familiar equation of motion:

$$\ddot{x} = -x\,(k/m).$$

  We all know the general solution to that: $x(t) = A\cos(\omega t) + B\sin(\omega t)$ or $x(t) = C\cos(\omega t + \phi)$, with $\omega = \sqrt{k/m}$. This ω is called the **frequency of small oscillations** of the system.

Many problems in many physics or engineering end with the phrase "what is the frequency of small oscillations?" When you see that, you know that you must **approximate the restoring force for small deviations of the system from equilibrium** using a **leading-order Taylor approximation**. You won't be *told* to make such an approximation, it is *implied* by the phrase "small oscillations". So here are some questions of this type involving rotations. In all of them, uniform gravity is active (and provides the restoring force).

Problem 1 : Small Oscillations for a Compound Pendulum *Checkpoints* [1]

A rod of length d and mass M has one of its ends attached to a fixed pivot that allows it to swing freely under the influence of gravity. The rod has a non-uniform mass distribution that results in a moment of inertia I for rotation around its end (along any axis perpendicular to the rod) and a center-of-mass position that is a distance R from the end that's attached to the pivot. The given quantities are thus d, M, I, R, and of course g.

(a) What is the frequency of small oscillations of the pendulum?

(b) You undoubtedly used the letter "**ω**" for the small-oscillation frequency. And now let's confront a **massive point of confusion** in rotational problems: *Is this ω the angular velocity of anything? i.e., Is it the time-derivative of any angle?* Answer: *no!* You can verify this yourself, but do ask your instructor if it's not 100% clear! This also carries a message: beware of the formula "$L=I\omega$" in small-oscillation problems … best advice is never to write it like that, but always as $L = I\dot{\theta}$ or $L = I\dot{\phi}$. Same for torque: write $\tau = I\ddot{\theta}$ not $\tau = I\dot{\omega}$ Then ω is reserved for the oscillation frequency. If you prefer, you can alternatively use a different letter for the oscillation frequency, e.g. Ω.

Problem 2 : Small Oscillations for a Ball in a Fixed Cylinder *Checkpoints* [2]

A small ball with radius r and uniform mass m rolls without slipping near the bottom of a fixed cylinder of radius R, where $R > r$. Our goal is to find the frequency of small oscillations of the small ball.

(a) The moments of inertia for a ball of radius b and uniform mass M for rotation around its center (= its CM) is $I' = 2Mb^2/5$. You may find it useful in this problem to also know the moment of inertia, $I^{(\text{edge})}$ for rotation of the ball around an edge-point, with the axis of rotation being tangential to the sphere. For practice, use the **parallel-axis theorem**, $I^{(B)} = I^{(B)}_{\text{CM}} + I'$ to find $I^{(\text{edge})}$ for the ball.

(b) Now off you go: find the frequency of small oscillations of the ball. Hints are available here[3] and here[4]. The first one may be worth reading straight away: it explains how you must introduce two angles to solve this problem. The second hint presents detailed strategies for the solution, so see first if you can manage without it.

(c) If you solved the problem without that second hint, bravo!! But the hint has some useful information : it describes multiple methods for solving this problem. For practice, please re-solve the problem using the other techniques. (You can avoid reading the hint by thinking of additional methods yourself. ☺)

---

[1] Q1 (a) $\omega = \sqrt{MgR/I}$ (b) No, it's an angular **frequency**, $\omega = 2\pi f$, not an angular speed!

[2] Q2 (a) $I^{(\text{edge})} = 7Mb^2/5$ Method: Parallel-axis theorem is $I^{(B)} = I^{(B)}_{\text{CM}} + I'$ where $I^{(B)}_{\text{CM}} = M\left|\vec{R}^{(B)} \times \hat{\omega}\right|^2$. For our problem, $\hat{\omega}$ is tangential to the sphere and so is perpendicular to the vector $\vec{R}^{(\text{edge})}$ that points from the edge to the CM=center-of-sphere. Thus, $I^{(\text{edge})}_{\text{CM}} = M\left|\vec{R}^{(\text{edge})} \times \hat{\omega}\right|^2 = MR^{(\text{edge})\,2} = Mb^2$. Adding that to the known $I' = 2Mb^2/5$ we get $I^{(\text{edge})} = Mb^2 + 2Mb^2/5 = 7Mb^2/5$.

(b,c) $\omega = \sqrt{5g/7(R-r)}$

[3] There are two relevant angles: that of the small ball around its CM as it rolls – call that ϕ – and the angle between the line from the center of the cylinder to the ball and the vertical – call that θ. You will need the relation $r\phi = (R-r)\theta$, provable by a *really good sketch* + consideration of *arc length* & the phrase *"rolls without slipping"*.

[4] You can solve this in multiple ways: (1) with two equations : $F = MA$ tangential to the cylinder & $\tau' = I'\ddot{\phi}$ (2) with one torque equation around a well-chosen point that *zaps* the friction and normal forces (3) using conservation of energy, once you've convinced yourself that the normal and rolling-friction forces *do no work*; this method itself even has two versions if you consider that the ball's kinetic energy can be calculated in two different ways. For all methods, you also need $r\phi = (R-r)\theta$ from the previous hint.