Physics 325 – Homework #12 **due in 325 homework box by Fri, 1 pm**

All solutions must clearly show the steps and/or reasoning you used to arrive at your result. You will lose points for poorly written solutions or incorrect reasoning. Answers given without explanation will not be graded: ***"No Work = No Points"****. However you may always use any relation on the 1DMath, 3DMath or exam formula sheets or derived in lecture / discussion. Write your* **NAME** *and* **DISCUSSION SECTION** *on your solutions.*

GENERAL STRATEGY TIPS that you will find useful in various places throughout this homework, which consists of typical Ph.D. qualifying exam problems (just with a bit more guidance than on the "qual"):

- When you are asked to "find the equations of motion" for a system, if one of your coordinates is cyclic, it is perfectly fine to leave the corresponding EOM in the first-order form " $\partial L / \partial \dot{q}_i = \text{constant}$". You do *not* have to take the time-derivative of that expression unless you find some reason to do so later in the problem.

- About the Hamiltonian $H = \dot{q}_i(\partial L / \partial \dot{q}_i) - L$: We know that H is conserved when time does not appear explicitly in your Lagrangian, which is useful. However, H can be quite messy to construct, particularly when you have more than one degree of freedom (note the implicit sum over i in the definition of $H \rightarrow$ messy!). Well, we also know that H is frequently the total energy $T+U$ of the system.[1] With your vast experience, you can pretty much always tell in advance if $T+U$ is conserved or not; if it is, don't bother constructing H, just write down "$T+U$ = constant" directly as one of your equations of motion. Remember, you already have T and U written in terms of your generalized coordinates from constructing the Lagrangian, so this is very easy!

- In problems with more than one degree of freedom ($n > 1$) you have more than one generalized coordinate whose time-dependence you have to determine. The n equations of motion you end up with will often be **coupled differential equations**, meaning that more than one of the functions $q_1(t), q_2(t), \ldots$ that you are trying to find will show up in each equation. In almost every situation, it is impossible to make any progress until you **decouple** these equations, i.e. obtain **separated** EOMs that each involve only one coordinate q_i (and its derivatives). If you have any cyclic coordinates q_i, evaluate their EOMs first and leave them in 1st-order " $\partial L / \partial \dot{q}_i = \text{constant}$" form; you can often use these expressions to get rid of the cyclic coordinates q_i or their derivatives $\dot{q}_i$ in the *other* equations, which helps a great deal in the decoupling process. Also be careful not to make the mistake of decoupling your equations in one part, then going *back* to the coupled version in a later part! (it's easy to do when things get complicated.) Remember: you can hardly ever make any progress toward a solution until your equations of motion are decoupled.

Problem 1 : Springy Pendulum

A pendulum is made from a massless spring (force constant k and unstretched length l_0) that is suspended at one end from a fixed pivot O and has a mass m attached to its other end. The spring can stretch and compress but cannot bend, and the whole system is confined to a single vertical plane.

(a) Write down the Lagrangian for the pendulum, using as generalized coordinates the usual angle ϕ (the angle with respect to the direction of gravity = downward) and the length r of the spring.

(b) Find the two Lagrange equations of the system. Interpret the equations in terms of $\vec{F} = m\vec{a}$ in the case when $\vec{a}$ is written in *polar* coordinates. (Consult our 3D calculus formula sheet to obtain $\vec{a}$ in that form.)

(c) The equations of part (b) cannot be solved analytically in general. However they *can* be solved for small oscillations around the equilibrium position $\{ \phi = 0, r = l \}$. (Note that l here is not the unstretched length l_0 of

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[1] Reminder from lecture and discussion 12: $H=T+U$ when your generalized coordinates are **natural coordinates**, which basically means that they do not involve any time-dependent constraints.

the spring; you must figure out what it is.) Analyze the small-oscillation motion of the system and figure out the small-oscillation frequencies for both the ϕ and r coordinates.

Problem 2 : Spherical Pendulum

The "spherical pendulum" is just a simple pendulum that is free to move in any sideways direction. (By contrast, the unqualified word "pendulum" or the explicit phrase "plane pendulum" implies that motion is confined to some vertical plane.) The bob of a spherical pendulum moves on a spherical surface, centered on the point of support with radius $r = R =$ the length of the pendulum. A convenient choice of coordinates is spherical polar (r, θ, ϕ) with the origin at the point of support and the z axis pointing straight *down* = in the direction of uniform gravity. The angles θ and ϕ make a good choice of generalized coordinates.

(a) Find the Lagrangian and the two Lagrange equations of motion.

(b) Explain what the ϕ equation tells us about the z component of angular momentum, l_z.

(c) For the special case that ϕ = constant, describe what the θ equation tells us. To be precise, what <u>simple system</u> does the spherical pendulum reduce to / behave like in this situation?

(d) Back to the general case! Use the ϕ equation of motion to replace $\dot{\phi}$ by l_z (and other terms, of course) in the θ equation, thereby constructing a <u>separated</u> equation for $\theta(t)$ alone. Use it to determine whether or not an angle θ_0 exists at which θ can remain constant. Why is this motion called a "conical pendulum"?

(e) Show that if $\theta(t) = \theta_0 + \varepsilon(t)$, with ε very small, then θ oscillates around θ_0 in harmonic motion. (You do <u>NOT</u> need to find the frequency of the oscillations.) Briefly describe in words the motion of the pendulum's bob for this situation.

Problem 3 : Table & String

Two equal masses $m_1 = m_2 = m$ are joined by a massless string of length d that passes through a hole in a frictionless horizontal table. The first mass slides on the table while the second hangs below the table and moves up and down in a vertical line.

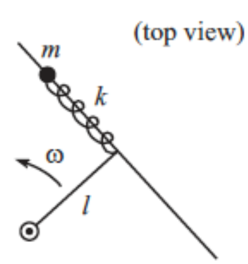
(a) Assuming the string remains taut, write down the Lagrangian for the system in terms of the polar coordinates of the mass on the table.

(b) Find the two Lagrange equations and interpret the ϕ equation in terms of the angular momentum l of the first mass.

(c) Express $\dot{\phi}$ in terms of l and eliminate $\dot{\phi}$ from the radial (s) equation. Now use the s equation to find the value $s = s_0$ at which the first mass can move in a circular path. Interpret your answer in Newtonian terms.

(d) Suppose the first mass is moving in this circular path of radius $s = s_0$ and is given a small radial nudge. Show that the circular orbit is stable with respect to small changes in the radius (i.e. show that the s coordinate oscillates around s_0) and calculate the frequency of small oscillations.

Problem 4 : Spring on a T



A rigid "T" consists of a long rod glued perpendicular to another rod of length l that is pivoted at the origin. The "T" rotates around in a horizontal plane with constant frequency ω. A mass m is free to slide along the long rod and is connected to the intersection of the rods by a spring with spring constant k and relaxed length zero. Find $r(t)$, where r is the position of the mass along the long rod. You should get <u>three</u> solution forms, depending on the relative values of ω^2 and k/m.