Non-Inertial Reference Frames

Inertial reference frames are ones where all vectors are defined as \( \overrightarrow{r} = \overrightarrow{r}_0 \) with \( \overrightarrow{v}_0 = 0 \).

If \( \overrightarrow{F} = m\overrightarrow{\ddot{r}} \), it is also true that \( \overrightarrow{F} = m(\overrightarrow{\ddot{r}} - \overrightarrow{\ddot{r}}_0) \).

If we use a reference frame where \( \overrightarrow{r}_0 \neq 0 \), we can still use \( \overrightarrow{F} = m\overrightarrow{\ddot{r}} \) if we introduce "fictitious forces".

Choose some reference point \( \overrightarrow{r}_0 \) that is accelerating. If \( \overrightarrow{F} = m\overrightarrow{\ddot{r}} \) for some inertial frame, we can define a new \( \overrightarrow{r}' = \overrightarrow{r} - \overrightarrow{r}_0 \implies \overrightarrow{\ddot{r}} = \overrightarrow{\ddot{r}}' + \overrightarrow{\ddot{r}}_0 \).

\[ \implies \overrightarrow{F} = m(\overrightarrow{\ddot{r}}') + m\overrightarrow{\ddot{r}}_0 \]

true forces \( m\overrightarrow{\ddot{r}}_0 \) acceleration of \( \overrightarrow{r}_0 \)

inertial frame relative to \( \overrightarrow{r}_0 \)

\[ \begin{align*}
\text{i.e.:} & \quad m\overrightarrow{\ddot{r}}' = \overrightarrow{F} - m\overrightarrow{\ddot{r}}_0 \\
\text{apparent} & \quad \text{real} \\
\text{force} & \quad \text{force} \\
\text{fictitious force} & \quad \text{force} \\
\overrightarrow{F}_{\text{act}} & = -m\overrightarrow{\ddot{r}}_0 \\
\text{just like gravity points in \( \overrightarrow{r}_0 \) direction} \end{align*} \]

If we attribute \( m\overrightarrow{\ddot{r}}' \) to be due to force \( \overrightarrow{F}' \), then

\[ \overrightarrow{F} = \overrightarrow{F} + \overrightarrow{F}_{\text{act}} \]
At $t=0$ a car accelerates with the door open. Find angular acceleration.

Two methods:

- **Inertial**

The acceleration of the center of mass has two pieces:

$$\frac{F}{m} - \frac{S}{m} = \vec{a}_{cm} = \vec{a} + \ddot{\theta} \left( \frac{\hat{i} \cos \theta + \hat{j} \sin \theta}{2} \right)$$

Now look at torques around center of mass:

$$\tau = \frac{L}{2} \left( -F \sin \theta + S \cos \theta \right) = I \ddot{\theta}$$

$$\tau = -\frac{M L^2 \ddot{\theta}}{12}$$

$$\frac{F}{m} = \vec{a}$$

$$\frac{S}{m} = \ddot{\theta} \left( \frac{\hat{i} \cos \theta + \hat{j} \sin \theta}{2} \right)$$

$$\ddot{\theta} = -\frac{3}{2} \frac{a \sin \theta}{L}$$
(swinging car door)

Now do the same problem in a frame accelerating:

\[ S \rightarrow F \]

\[ \overrightarrow{F} = -m \overrightarrow{a} = -m \overrightarrow{j} \]

This fictitious force is just like \( \overrightarrow{g} \) (uniform),
so we can just treat it as we would gravity.

\[ S \rightarrow \theta \frac{1}{2} \]

Now we can take the torque around the hinge

\[ I \dot{\theta} = -\frac{1}{2} m \sin \theta \cdot a \]

\[ I = \frac{1}{3} m L^2 \] (for a stick around its end)

\[ \frac{1}{3} m L^2 \dot{\theta} = -\frac{1}{2} m \sin \theta \cdot a \]

\[ \Rightarrow \dot{\theta} = -\left( \frac{3}{2} \frac{a}{L} \sin \theta \right) \]

... So much easier...
Another example: A pendulum hanging from the roof of a car that is accelerating.

\[ T \cos \theta = mg, \quad -T \sin \theta = ma_x \]

\[ \Rightarrow \tan \theta = -\left( \frac{a_x}{g} \right) = -\left( \frac{a}{g} \right) \]

\[ T \cos \theta = mg \quad \Rightarrow \quad T \cos \theta = m \cdot \frac{a}{g} \]

- Accelerate left \((\theta > 0)\) \(\phi < 0\)

\[ -T \sin \theta - mg = 0 \]

\[ \Rightarrow \tan \theta = -\left( \frac{a}{g} \right) \]

Same answer, almost trivial.

Now, let's use a helium balloon tied to the floor.

This only makes sense in the accelerating frame, because the buoyancy is in the accelerated frame.

After a short amount of time, the air in the can adjusts itself to adjust to the new \(\vec{g} \): \(\vec{g}_{can} = \vec{g} - a \hat{x}\)

This is a really sneaky question; probably unfair.
What if we are rotating?

Take a rotation in the x-y plane:

\[ \hat{i}' = \cos \theta \hat{i} + \sin \theta \hat{j} \]
\[ \hat{j}' = -\sin \theta \hat{i} + \cos \theta \hat{j} \]
\[ \frac{d\hat{i}'}{dt} = -\sin \theta \dot{\theta} \hat{i} + \cos \theta \dot{\theta} \hat{j} + \sin \theta \hat{k} \]
\[ \Rightarrow \dot{i}' = \dot{\theta} (-\hat{i} \sin \theta + \hat{j} \cos \theta) \]

If we define \( \vec{\omega} = \dot{\theta} \hat{k} \), then we can see that:

\[ \dot{i}' = \vec{\omega} \times \hat{i}' \]

Similarly:

\[ \dot{j}' = -\cos \theta \dot{\theta} \hat{i} - \sin \theta \dot{\theta} \hat{j} = \vec{\omega} \times \hat{j}' \]

\[ \hat{k}' = 0 \] (since this is the rotation axis)

Given this, take any vector \( \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \) and write it in a rotating coordinate system with \( \vec{\omega} = \dot{\theta} \hat{k} \):

\[
\vec{A}' = A_x' \hat{i}' + A_y' \hat{j}' + A_z' \hat{k}'
\]

\[
= \frac{\vec{A}}{2} \left[ \hat{i}' + \hat{j}' + \hat{k}' \right] + \frac{\vec{\omega}}{2} (\vec{\omega} \times \vec{A})
= \frac{\vec{A}}{2} + \vec{\omega} \times \vec{A}
\]

\( \vec{A} \) in rotating coordinates
This is true for the particular case of \( \dot{\omega} = 0 \), but \( \dot{\omega} \times \vec{A} \) is a vector independent of the basis, so in general we have \( \vec{\ddot{A}} = \vec{\ddot{A}}' + \ddot{\omega} \times \vec{A} \).

**Acceleration in rotating frames:**

In inertial coordinates \( \vec{\ddot{A}} = \frac{d}{dt} \left( \frac{d\vec{A}}{dt} \right) \)

\[
\vec{\ddot{A}} = \frac{d}{dt} \left[ \vec{\dot{A}}' \right] + \frac{\ddot{\omega} \times \vec{A}}{\text{uniform rotation}} + \frac{\dot{\omega} \times \vec{\dot{A}}'}{\text{rotating}} + \ddot{\omega} \times \vec{\dot{A}}' = \vec{\ddot{A}}' + \frac{\ddot{\omega} \times \vec{A}}{\text{uniform rotation}} + \frac{\dot{\omega} \times \vec{\dot{A}}'}{\text{rotating}} = \vec{\ddot{A}}' + 2 \dot{\omega} \times \vec{A} + \ddot{\omega} \times (\dot{\omega} \times \vec{A})
\]

In particular, this is true for \( \ddot{\omega} = 0 \)

For rotating frames \( \vec{\ddot{a}} = \vec{a}_{\text{inertial}} = \vec{a}_{\text{rot}} + 2 \dot{\omega} \times \vec{v} + \ddot{\omega} \times (\dot{\omega} \times \vec{r}) \)

If we allow for linear acceleration and time-varying rotation, this becomes \( \vec{a}_{\text{inertial}} = \vec{a}' + 2 \dot{\omega} \times \vec{v} + \ddot{\omega} \times (\dot{\omega} \times \vec{r}) \)

\[
\vec{a}_{\text{linear}} = \vec{a} + \dot{\omega} \times \vec{v} + \ddot{\omega} \times (\dot{\omega} \times \vec{r})
\]
In the accelerating frame, we can still write \( \vec{F} = ma' \), if we add some fictitious forces:

- Coriolis: \( \vec{F}_{\text{cor}} = -2m \omega \times \vec{v}' = +2m \vec{v} \times \omega \)
- Centrifugal: \( \vec{F}_{\text{cent}} = -m \omega \times (\vec{w} \times \vec{v}') \)

Example: Life on a rotating disk with \( \vec{w} = w \hat{k} \), \( \vec{v}' = x' \hat{i} + y' \hat{j} \)

\[
\begin{align*}
\vec{a}' &= \sum \vec{F}_m - 2m \omega \times (x' \hat{i} + y' \hat{j}) \\
&= \sum \vec{F}_m - 2m \omega \times \vec{v}' \\
&= 2m \omega (x \hat{i} - y \hat{j}) - m \omega^2 (x \hat{j} + y \hat{i}) \\
\end{align*}
\]

In absence of external forces:

\[
\begin{cases}
\dot{x} = w^2 x + 2w y \\
\dot{y} = -w^2 y - 2w x
\end{cases}
\]

Start with \( x = 0, y = R, \dot{x} = u, \dot{y} = 0 \)

\[
\begin{align*}
\dot{x} &= 2w u \\
\dot{y} &= -w^2 R \quad \text{is actual and to the right}
\end{align*}
\]
Force-free motion in a rotating frame

\[
\begin{align*}
\dot{x} &= \omega^2 x + 2\omega y \\
\dot{y} &= \omega^2 y - 2\omega x \\
\end{align*}
\]

Trick: [recall magnetic field problem]: \( \vec{F}(t) = x(t) + iy(t) \)

\[
\begin{align*}
\dot{\vec{F}} &= \ddot{x} + i \ddot{y} = \omega^2 x + 2\omega y + i \omega^2 y - 2i\omega x \\
&= \omega^2 (x + iy) + 2\omega (iy - ix) \\
&= \omega^2 (x + iy) + 2\omega (iy + ix) \\
&= \omega^2 \vec{F} - 2i\omega \vec{F} \\
\end{align*}
\]

We've been solving stuff like this for a while.

Try: \( \vec{F} = A e^{st} \) \( \Rightarrow \) \( s^2 = \omega^2 - 2i\omega \)

\( s^2 + 2i\omega - \omega^2 = 0 \) \( \Rightarrow (s + i\omega)^2 = 0 \)

\( s = -i\omega \) is a double root.

Recall critical damping, double root mean 2nd solution is \( t e^{st} \).

\[
\begin{align*}
\vec{F} &= A e^{-it} + B t e^{-it} \quad \text{(adj)} \\
\end{align*}
\]

This seems familiar, but some things are different. \( \omega \) is now a rotation frequency, nothing to do with spins, and \( \vec{F}(t) \) is in general complex, so \( A + B \) will also be complex.

Setting initial conditions: choose \( x(0) = 0, \dot{x}(0) = v, y(0) = -R, \dot{y}(0) = u \)

[Note that we can always choose \( x, y \) axes to make this true!]

\[
\begin{align*}
A &= 0 + i(-R) \quad B = -iR \\
-i\omega A + B &= v + iu \quad B = \omega R + v + iu
\end{align*}
\]
The solution is then

\[
\begin{align*}
    x(t) &= (v + v_R) t \cos(\omega t) + (v t - R) \sin(\omega t) \\
    y(t) &= (v t - R) \cos(\omega t) - (v + v_R) t \sin(\omega t)
\end{align*}
\]

In general, this motion looks complicated. 

A simple case is \( v = -v_R \) (initially moving left with \( v_R - v \)).

In this case \( x(t) = 0 = R \sin(\omega t) \) which just traces out
\( y = -R \cos(\omega t) \) a circle.

This corresponds to an initially stationary object with no forces on it just staying fixed.

In general, there are all straight lines in the inertial frame, but in the rotary frame correspond to things appearing to be thrown outwards and always being pushed to the right.

Simple example:

In the inertial frame,
\[
\begin{align*}
    \vec{v}_B &= \Gamma \vec{v}_A \\
    \vec{v}_A &= \Gamma \vec{v}_A
\end{align*}
\]

If B throws a ball toward A, the true \( \vec{v} = \vec{v}_B + \vec{v}_A \) shows

Since \( |\vec{v}_B| > |\vec{v}_A| \), the ball will miss ahead of A,

appearing to move to the right, as B sees it.
How large is the Coriolis acceleration?

3. Drawing a bathtub
If water is flowing at a rate of \( v \approx 1 \text{ cm/s} \) and the Earth rotates at \( \omega = \frac{2\pi}{24 \text{ hours}} \) (at north pole),

\[
\Rightarrow a_{\text{Coriolis}} \approx 2\omega \cdot v \approx \frac{4\pi}{24 \text{ hours}} \cdot 10^{-2} \text{ m/s} \approx 1.5 \times 10^{-6} \text{ m/s}^2
\]

24.3600 s

So this effect is in general very small.

But, what if you had speeds of \( v \approx 10 \text{ m/s} \) acting for a long time (like several hours)?

In that case, \( a \approx 10^{-3} \text{ m/s}^2 \Rightarrow dv \approx 10 \text{ m/s} \), or if you had a pressure fluctuation that was \( 10^4 \text{ Pa} \) km across,

\( \Rightarrow \) a pressure fluctuation 100 km wide.

\( 10 \text{ m/s} \times 10^4 \text{ s} = 100 \text{ km} \)

\( \sim 3 \text{ hr} \)

This means that a weather pattern 100 km in radius can generate Coriolis flows of 10 m/s.

"Hurricanes" are \( 500 \text{ km} \) + \( \Rightarrow \pm 50 \text{ m/s} \) \( \sim 100 \text{ miles per hour} \)