Velocity Dependent Forces: \( \vec{F} = \vec{F}(\vec{v}) \)

\[ \frac{mdv}{dt} = \vec{F}(\vec{v}) \]

Integrate:

\[ m \int \left( \frac{dv}{F(v)} \right) = \int dt \]

Example:

\[ F = -bv \]

\[ m \int - \frac{dv}{bv} = t \]

\[ ln \left( \frac{v}{v_0} \right) = \frac{-bt}{m} \]

\[ v = v_0 e^{\frac{-bt}{m}} \]

\[ dx = v_0 e^{\frac{-bt}{m}} dt \]

Integrate again:

\[ x - x_0 = \frac{v_0 m}{b} \left( 1 - e^{\frac{-bt}{m}} \right) \]

Does this answer make sense?

Check for limit of \( b \to 0 \):

\[ e^{\frac{-bt}{m}} \to 1 - \frac{bt}{m} \]

\[ (x - x_0) \to \frac{v_0 m}{b} \left( 1 - 1 + \frac{bt}{m} \right) = \frac{v_0 m \cdot 0}{b} \]

i.e. constant uniform motion at \( v = v_0 \)
Another example \( F(v) = -Cv^2 \)

\[
\frac{m\,dv}{dt} = -Cv^2
\]  

\[
\exists \int_{v_0}^{v} -\frac{m}{v} \, dv = \Phi + t
\]

\[
\frac{m}{c} \left( \frac{1}{v} - \frac{1}{v_0} \right) = \Phi + t
\]

\[
\frac{1}{v} = \frac{1}{v_0} + \left( \frac{\Phi}{m} \right)
\]

\[
\exists \quad \frac{dt}{dt} = \frac{1}{\frac{v_0}{c} + \frac{\Phi}{m}}
\]

\[
\exists \quad x-x_0 = \int_{t_0}^{t} \frac{dt}{\frac{v_0}{c} + \frac{\Phi}{m}} = \frac{m}{c} \ln \left( \frac{\frac{v_0}{c} + \frac{\Phi}{m}}{v_0} \right)
\]

\[
x-x_0 = \frac{m}{c} \ln \left( 1 + \frac{Cu_0 \cdot t}{m} \right)
\]

Does this make sense?

Again, take \( C \rightarrow 0 \) \( x-x_0 \rightarrow \frac{m}{c} \left( \frac{Cu_0 \cdot t + \ldots}{m} \right) \)
Which drag law to use? [Side note]

Drag on a moving object is often modeled as \( F(v) \), but I've seen both \( dv \) and \( dv^2 \). Which is right? As always, it turns out to be complicated; "it depends."

"Stokes Drag":

For a small object moving through a non-turbulent fluid ("laminar flow"), the fluid is largely indifferent, so it slightly resists itself to move past the object. There is a boundary layer that is the only actual interaction with area \( \pi R^2 \). The \( \Delta R \) depends on the fluid. In that layer, the fluid "wakes" with the object, and there is a momentum transfer that is \( \Delta \left( \frac{2}{3} \pi R^2 \right) v = \frac{1}{2} \pi R^2 v \). This is Stokes drag: \( \frac{1}{2} = \frac{6 \pi}{R^2} \).

Quadratic:

In the opposite limit, there are fluid elements striking the surface and carrying momentum transfer on an average that is \( \Delta v. \) The number of strikes per second is also \( \Delta v \), so the net force is \( \Delta v^2 \). The cross-section for strikes \( \pi R^2 \) is \( \frac{1}{2} \pi R^2 A \).
Forces in 2-D

\[ \frac{d^2}{dt^2} \mathbf{r} = \frac{d\mathbf{v}}{dt} \] is a vector equation.

E.g. \[ \mathbf{a} = -mg \mathbf{z} - b \mathbf{v} \]

\[ \mathbf{F}_z = -mg - bv_z \]
\[ \mathbf{F}_x = -bv_x \]

\[ \frac{dv_x}{dt} = \frac{-bv_x}{m} \]

\[ \int \mathbf{v}_x = \frac{bt}{m} \]
\[ e \ln (\frac{v_x}{v_0}) = -\frac{bt}{m} \]
\[ v_x = v_0 e^{-\frac{bt}{m}} \]

\[ t \to \infty \]

\[ \frac{dv_z}{dt} = -g - \frac{bv_z}{m} \]

\[ \int \frac{dv_z}{g + \frac{bv_z}{m}} = -t \]

\[ \frac{m}{b} \ln \left( \frac{g + \frac{bv_z}{m}}{g} \right) = -t \]

\[ (1 + \frac{bv_z}{mg}) = e^{-\frac{bt}{m}} \]

\[ \frac{dz}{dt} = \frac{dv_z}{b} \]
\[ \int dv_z = (e^{-\frac{bt}{m}}) \frac{mg}{b} \]

\[ z = z_0 - \left( \frac{mg}{b} \right) t + \frac{m}{b^2} \left( e^{-\frac{bt}{m}} - 1 \right) \]
**Quadratic Drag**

\[
\frac{d\vec{v}}{dt} = -mg - c|\vec{v}| \vec{v}
\]  

\[
\begin{align*}
\frac{d\vec{v}_x}{dt} &= -c \frac{(v_x^2 + v_y^2)^{1/2}}{m} v_x \\
\frac{d\vec{v}_y}{dt} &= -g - c \frac{(v_x^2 + v_y^2)^{1/2}}{m} v_y
\end{align*}
\]

These are messy.

---

**Magnetic Force**

\[
\vec{F} = q \vec{v} \times \vec{B}
\]

\[
\vec{F} = q \vec{v} \times \vec{B}
\]

Take \( \vec{B} = B_0 \hat{k} \)

\[
\begin{align*}
m\vec{v}_x &= q v_y B_0 \\
m\vec{v}_y &= -q v_x B_0 \\
m\vec{v}_z &= 0 \quad \Rightarrow \quad v_z = \text{constant}
\end{align*}
\]

That leaves 2 1st order coupled ODEs.

**Trick:** take \( \frac{d}{dt} \) of \( \vec{v}_x \) \( \Rightarrow \)

\[
\begin{align*}
m\frac{d\vec{v}_x}{dt} &= q B_0 (-q B_0) v_x \\
&= -\frac{q^2 B_0^2}{m^2} v_x
\end{align*}
\]

Then substitute \( v_x = \frac{v_x}{\beta} \)

\[
\beta = \frac{v_x}{\frac{q^2 B_0^2}{m^2}}
\]

\[
\begin{align*}
\beta = \frac{v_x}{\frac{q^2 B_0^2}{m^2}} \\
\frac{v_x}{\beta} &= \frac{v_x}{\frac{q^2 B_0^2}{m^2}} \\
v_x &= \frac{v_x}{\beta} \left( \frac{q^2 B_0^2}{m^2} \right)
\end{align*}
\]
A stunt that solves this more elegantly.

We have two equations: \( \mathbf{v}_x = \omega \mathbf{u}_y \), \( \mathbf{v}_y = -\omega \mathbf{u}_x \)

These are two independent things, \( \mathbf{v}_x \) and \( \mathbf{v}_y \), so we can combine these into one complex object: \( \eta = \mathbf{v}_x + i \mathbf{v}_y \) (or the other way, doesn't matter).

Then \( \mathbf{v}_x = \text{Re}(\eta) \), \( \mathbf{v}_y = \text{Im}(\eta) \), but they live along 2 orthogonal axes.

\[
\eta = \mathbf{v}_x + i \mathbf{v}_y = \omega \mathbf{u}_y - i \omega \mathbf{u}_x = i \omega \left( \frac{\mathbf{u}_y}{\mathbf{i}} - \frac{\mathbf{u}_x}{\mathbf{i}} \right) = i \omega \left( -i \mathbf{u}_y - \mathbf{u}_x \right)
\]

\[
\eta = -i \omega \mathbf{u}
\]

\[
\begin{equation}
\frac{d\mathbf{n}}{dt} = -i \omega (t-t_0) (\mathbf{v}_x, \mathbf{v}_y) = e^{-i \omega (t-t_0)} \text{ where } \mathbf{n}_0 \text{ is a complex constant }
\end{equation}
\]

Write \( \mathbf{n}_0 = A e^{i\theta} \) with \( A, \theta \) real numbers.

\[
\eta = A e^{i\theta} e^{-i \omega (t-t_0)} \text{ take } t_0 = 0 \text{ for simplicity.}
\]

\[
\mathbf{v}_x = \text{Re}[\eta] = A \cos(\omega t - \theta) = A \cos(\theta - \omega t)
\]

\[
\mathbf{v}_y = \text{Re}[\eta] = -A \sin(\omega t - \theta)
\]

\[
\text{just like before!}
\]
Position-dependent forces in multiple D

\[ \mathbf{F} = \mathbf{F}(\mathbf{r}) \]

\[ e^{-k} \mathbf{F} = -k \mathbf{r} = kr \hat{r} \]

\[ \mathbf{F}_x = F_x = -k x \]

\[ \mathbf{F}_y = F_y = -k y \]

\[ \mathbf{F}_z = F_z = -k z \]

→ Still in each direction, phase & amplitude are decoupled.

What about quadratic? \[ F = -kr^2 \frac{\partial}{\partial r} \]

\[ m \frac{d^2 x}{dt^2} = F_x = -k \left( x^2 + y^2 + z^2 \right) \frac{\partial}{\partial x} \]

\[ m \frac{d^2 y}{dt^2} = F_y = -k \left( x^2 + y^2 + z^2 \right) \frac{\partial}{\partial y} \]

\[ m \frac{d^2 z}{dt^2} = F_z = -k \left( x^2 + y^2 + z^2 \right) \frac{\partial}{\partial z} \]

ugh!

We need better coordinates!

We already did cylindrical coordinates.

\[ \mathbf{r} = R \hat{R} + z \hat{z} \]

\[ \mathbf{v} = \dot{R} \hat{R} + \dot{z} \hat{z} + R \dot{\phi} \hat{\phi} \]

\[ \mathbf{a} = \left( \ddot{R} - R \dot{\phi}^2 \right) \hat{R} + \left( \ddot{z} + 2 \dot{R} \dot{\phi} + R \ddot{\phi} \right) \hat{z} + \ddot{\phi} \hat{\phi} \]
**Spherical Coordinates**

\[
x = \hat{r} \sin \theta \cos \phi \quad r
\]
\[
y = r \sin \theta \sin \phi
\]
\[
z = r \cos \theta
\]

\(\hat{r}\) is the normal to the sphere at \((r, \theta, \phi)\)

\(\hat{\theta}\) is tangent to the sphere toward the north pole

\(\hat{\phi}\) is tangent and \(\bot\) to \(\hat{\theta}\), with \(\hat{z} = 0\)

\[
(\hat{r} = -\hat{r})
\]

\[
\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}
\]
\[
\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}
\]
\[
\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}
\]

Ok, now we are all set for any \(\Phi(\vec{r})\) \(\vec{F} = \Phi(\vec{r}) \hat{r}\) (i.e., any central force)

\[
m \frac{d^2 \vec{r}}{dt^2} = \Phi(\vec{r})
\]

Central force \(m \frac{d^2 \vec{r}}{dt^2} = \Phi(\vec{r}) \hat{r}\)

\[
m \frac{d \vec{v}}{dt} = \Phi(\vec{r}) \hat{r}
\]

\[
m \frac{d \vec{a}}{dt} = \Phi(\vec{r}) \hat{r}
\]

\[
m \frac{d \vec{r}}{dt} + \vec{F} = \Phi(\vec{r}) \hat{r}
\]

\[
m \left( \vec{F} + \vec{r} \hat{r} \right) = \Phi(\vec{r}) \hat{r}
\]

\[
m \left( \vec{r} \hat{r} + \vec{r} \hat{r} \right) = \Phi(\vec{r}) \hat{r}
\]

\[
m \left( \vec{r} \hat{r} + \vec{r} \hat{r} + \vec{r} \hat{r} \right) = \Phi(\vec{r}) \hat{r}
\]
Once again, we need to figure out $\dot{r}$ and $\dot{\phi}$, which requires $\dot{\theta}$ and $\ddot{\theta}$. There is probably a clever way to do it, probably an elegant geometric way, or use brute force based on the unit vector definitions:

\[
\dot{r} = (\cos \theta \dot{\theta} \cos \phi - \sin \theta \sin \phi \dot{\phi}) \hat{x} \\
+ (\cos \theta \dot{\theta} \sin \phi + \sin \theta \cos \phi \dot{\phi}) \hat{y} \\
- \sin \theta \dot{\theta} \hat{z}
\]

\[
\dot{\theta} = \begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix} = \begin{bmatrix}
\hat{x} \\
\hat{y} \\
\hat{z}
\end{bmatrix}
\]

\[
\dot{\phi} = -\dot{\phi} \hat{r} + \phi \cos \theta \hat{\phi}
\]

\[
\ddot{r} = -\ddot{r} \hat{r} + \dot{r} \hat{\theta} + \dot{r} \hat{\phi}
\]

Now we can get $\ddot{r} = \ddot{r} = \ddot{\theta} = \ddot{\phi}$

Plugging all this in to find $\dddot{r}$:

\[
\dddot{r} = \left(\dddot{r} - r \dddot{\theta}^2 - r \dddot{\phi}^2 \right) \hat{r} \\
+ \left( \dddot{r} \hat{\theta} + 2 \dot{r} \dddot{\theta} - r \dddot{\phi} \dot{\phi} \dot{\phi} \right) \hat{\theta} \\
+ \left( -2 r \dddot{\theta} \dot{\phi} - \dddot{\phi} \dot{\phi} \dot{\phi} + r \dddot{\phi} \dot{\phi} \dot{\phi} \right) \hat{\phi}
\]

(exhausting) or just look up
Now let's go back to the central force problem briefly. We will revisit in a few lectures:

\[ m \ddot{r} = f(r) \hat{r} \quad \text{when } f(r) \text{ is some central force} \]

\[
\begin{align*}
\dddot{r} - r \ddot{\theta}^2 - r \sin^2 \theta \dot{\phi}^2 &= f(r) \quad (1) \\
\dot{r}\ddot{\theta} + 2\dot{r}\dot{\phi} \sin \theta \cos \theta - r \sin \theta \cos \theta \dot{\theta}^2 &= 0 \quad (2) \\
2 \dot{r}\dot{\phi} \sin \theta + 2r \dot{\phi} \cos \theta + r \sin \theta \dot{\phi}^2 &= 0 \quad (3)
\end{align*}
\]

Notice that if at some time \( \theta = \frac{\pi}{2} \) and \( \dot{\theta} = 0 \) (i.e. in the equatorial plane), then eqn (2) \( \Rightarrow \dot{r} = 0 \) at that moment.

If \( r \neq 0 \), then \( \dot{\theta} \neq 0 \), which means \( \theta \) will stay zero forever more this will be true. But we can always choose this to be true by choosing \( \vec{F} \) such that its directions is aligned with the \( z \)-axis at that moment. For all central force problems there is a plane in which the motion is confined. (Not surprising given conservation of \( I_z \)).

This will be true for any force that has \( \vec{F}_\theta = 0 \) (i.e., as long as nothing is pushing it up or down out of that plane).
Example of non-central force:

A bead is confined to slide along a stick that is rotating uniformly with $\dot{\phi} = \omega$ with $\phi = \pi/2$ (in the equatorial plane).
Find the motion of the bead.

Free body diagram for bead:

Plug in to $m\ddot{r} = \dot{r} \dot{\phi}$ with $\dot{\theta} = \omega$ and $\dot{\phi} = 0$:

1. $\ddot{r} - r \dot{\phi}^2 = 0$  \quad $r \cos(\theta) = 1$, $\dot{\theta} = 0$
2. $2r \dot{\phi} + r \dot{\phi} = \frac{N(\theta)}{m}$

A: $\dot{\phi} = \omega$  \quad $\dot{r} = \omega^2 r$

This equation is solvable in multiple ways:

A) Repeat what was done for $\ddot{x} = -\omega^2 x$

\[ \frac{d}{dx} \left( \frac{1}{2} x^2 \right) = \omega^2 x \]
\[ \frac{d}{dx} \left( \frac{1}{2} \dot{x}^2 \right) = -\omega^2 x \]
\[ \frac{d}{dx} \left( \frac{1}{2} \dot{x}^2 \right) = \omega^2 x \]
\[ \frac{d}{dx} \left( \frac{1}{2} \dot{x}^2 \right) = -\omega^2 x \]
\[ \frac{1}{2} \dot{x}^2 = \omega^2 x \]
\[ \frac{1}{2} \dot{x}^2 = -\omega^2 x \]
\[ \dot{x}^2 = -\omega^2 x \]

After solving:
\[ r(t) = \frac{1}{2} \left( e^{\omega t} + e^{-\omega t} \right) \]
\[ r(t) = \frac{1}{2} \cosh(\omega t) \]

B) Take solution for SHM and plug in $\omega^2 = -\omega^2$

C) Try exp solutions and see if it works (linear, constant coefficient, homogeneous ODE)
shiny bead (continued)

What about \( N(t) \)?

\[
N(t) \approx \frac{2 \pi w}{m} \text{ (just is } N(t) \text{ required to keep } \theta = w) 
\]

Note

\[
\dot{N}(t) - \dot{\theta} = \left( 2 m \cdot \omega \right) \cdot \left( r \dot{w} \right) = m w^2 \frac{d}{dt} (r^2) 
\]

\[
N' \quad \dot{\theta} \quad > 0 
\]

\( \Rightarrow \) wire is doing work on bead

Food for thought:

If rod has length \( L \), what is \( \dot{\theta} \) \( \text{ when it flies off?} \)

\( \dot{\theta} \) \( \text{ must be } \theta \)

Answer:

\[
\dot{\theta} = \frac{\dot{\theta}}{BC_{\text{req}}} 
\]

\[
= \frac{r^2}{r^2} + \left( r \phi \right) \phi 
\]

We know \( r = r_0 \cosh(wt) \)

\( \Rightarrow \) \( L = r_0 \cosh(w \cdot t_L) \)

\( \Rightarrow \) \( t_L = \cosh^{-1} \left( \frac{L}{r_0} \right) \left( \frac{1}{w} \right) \)

\[
\dot{r} = r_0 \left( \sinh(wt) \right) \cdot \omega 
\]

\( \text{ at } \) \( \text{ (looking up)} \)

\( \Rightarrow \) \( \dot{\phi} \)

\( \Rightarrow \) \( \omega r_0 \sqrt{\cosh^2 \left( \frac{L}{r_0} \right) - 1} \)

\( \Rightarrow \) \( \dot{v} = \omega r_0 \sqrt{\cosh^2 \left( \frac{L}{r_0} \right) - 1} \left( \theta + \frac{L \omega}{4} \right) \)

when it flies off.

(need to find \( \phi(t_f) \) and \( \phi(t_f) \)

\( \text{ to get } \) \( \dot{v_x}, \dot{v_y} \)
Another non-central example: Reclining on a spinning hoop

Read constrained to stay on hoop as

\[ \dot{\theta} \text{ spins, } \ddot{\theta} \text{ pointing down.} \]

What is \( \theta(t) \)?

\[ \vec{F} = mg \left( \vec{r} \cos \theta - \vec{r} \sin \theta \right) - N_r \dot{r} + N_\theta \dot{\theta} \]

\[ \dot{r} : \quad mg \cos \theta - N_r = m \left( \ddot{r} - r \ddot{\theta}^2 - r \sin^2 \theta \dot{\theta}^2 \right) \]

\[ \dot{\theta} : \quad -mg \sin \theta = m \left( \ddot{\theta} + 2 \dot{r} \dot{\theta} - r \cos \theta \left( \cos \theta \right) \dot{\theta}^2 \right) \]

\[ \dot{\phi} : \quad \dot{N}_\phi = m \left( 2 \dot{r} \dot{\phi} \sin \theta + 2 \dot{r} \dot{r} \cos \theta + r \dot{\phi} \dot{\theta} \right) \]

By construction, \( \dot{\phi} = 0 \), \( \ddot{\phi} = 0 \), \( \dot{r} = 0 \)
Normal forces are just doing their job, let's focus on \( \dot{\theta} \)

\[ -mg \sin \theta = m \left( \ddot{r} \dot{\theta} + \dot{r} \ddot{\theta} \right) - r \sin \theta \cos \theta \dot{r} \dot{\theta}^2 \]

\[ \Rightarrow \quad \ddot{\theta} = \sin \theta \cos \theta \dot{r}^2 - \frac{g}{R} \sin \theta \]

Analyzing \( \dot{\theta}^2 \):

\[ 2 \dot{\theta} = \sin \theta \cos \theta \dot{r}^2 + \frac{g}{R} \sin \theta \]

\[ \dot{\theta}^2 = \frac{1}{R^2} \left( \sin^2 \theta - \sin^2 \theta_0 \right) + \frac{g}{R} \left( \cos \theta - \cos \theta_0 \right) \]

\[ \Rightarrow \quad \dot{\theta}^2 = \frac{1}{2} \sin^2 \theta + \frac{g}{R} \left( \cos \theta + \cos \theta_0 \right) \]

\[ \theta' = -\frac{\sin \theta}{2 R} - \frac{\theta_0}{R} \]
(spinning hoop cont...) \[\text{if } U(\theta) = -\frac{\Omega^2}{2} \sin^2 \theta - \frac{g}{2} \cos \theta \]

\[
\frac{dU}{d\theta} = -\frac{\Omega^2}{2} \sin \theta \cos \theta + \frac{g}{2} \sin \theta
\]

\[
\frac{d^2 U}{d\theta^2} = -\frac{\Omega^2}{2} \sin^2 \theta - \frac{\Omega^2}{2} \cos^2 \theta + \frac{g}{2} \cos \theta
\]

\[
= \frac{\Omega^2}{2} (1 - 2 \cos^2 \theta) + \frac{g}{2} \cos \theta
\]

Note: \( \frac{dU}{d\theta} = 0 \) at \( \sin \theta = 0 \) and \( \cos \theta = \frac{g}{\Omega^2} \)

but \( \cos \theta \) only exists if \( \theta \leq 1 \)

\[
\Rightarrow \text{solution only if } R_2 > \frac{g}{\Omega^2} \]

(i.e., rotating fast enough.)