Physics 325: Fall 2018

Sample Exam 2. Solutions.

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1 (20 points)

Consider a damped, linear forced simple harmonic oscillator

$$m\ddot{x} + c\dot{x} + kx = F(t). \tag{1}$$

a) Consider first the case where $F(t) = 0$, and solve for the general homogeneous solution.

b) Next, consider the case when $F(t) = \delta(t - \tau)$. Considering quiescent initial conditions, $x(t) = 0$, $\dot{x}(t) = 0$ for all $t < \tau$, solve for $G(t - \tau) = x(t)$ the solution for all t. You may set $\tau = 0$, and then argue that the solution for $\tau \neq 0$ is simply a translated version of the solution for $\tau = 0$.

c) Show that

$$x(t) = \int_{-\infty}^{\infty} G(t-\tau)F(\tau)d\tau \tag{2}$$

is a solution of equation (1) for a general forcing function $F(t)$

a

We find the homogeneous solution by looking for a solution of the form

$$x(t) = \exp(\lambda t), \tag{3}$$

substituting in gives the complementary equation

$$m\lambda^2 + \lambda c + k = 0 \tag{4}$$

introducing

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2m\omega_n} \tag{5}$$

this is

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \tag{6}$$

which has solution

$$\lambda = \frac{-2\zeta\omega \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \tag{7}$$

$$= -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm i\omega_d \tag{8}$$

and thus the general solution is

$$x(t) = ae^{-\zeta\omega_n t + i\omega_d t} + be^{-\zeta\omega_n t - i\omega_d t} = e^{-\zeta\omega_n t}\left(A\cos(\omega_d t) + B\sin(\omega_d t)\right) \tag{9}$$

b

Recall that, the effect of a unit impulse at $t = 0$ is to increase the velocity by an amount

$$v_0 = \frac{1}{m}, \quad x_0 = 0. \tag{10}$$

therefore, we can find the solution by solving for $x(t)$ in free vibration with initial condition $v_0 = 1/m$. From part a, the homogenous solution is

$$x(t) = e^{-\zeta\omega_n t} B \sin(\omega_d t), \tag{11}$$

since $A = 0$. Differentiating,

$$\dot{x}(t = 0) = B\omega_d = 1/m \tag{12}$$

therefore, we have, for $t > 0$

$$x(t) = e^{-\zeta\omega_n t} \frac{\sin(\omega_d t)}{m\omega_d}. \tag{13}$$

Now, time-translation invariance of the answer implies that for a general τ, we have

$$G(t - \tau) = \begin{cases} e^{-\zeta\omega_n(t-\tau)} \frac{\sin(\omega_d(t-\tau))}{m\omega_d}, & t > \tau \\ 0 & t < \tau \end{cases} \tag{14}$$

c

We can insert the solution into the equation of motion

$$\mathcal{L}_t x(t) = \mathcal{L}_t \left[\int_{-\infty}^{\infty} G(t - \tau) F(\tau) d\tau \right] = \int_{-\infty}^{\infty} \mathcal{L}_t G(t - \tau) F(\tau) d\tau \tag{15}$$

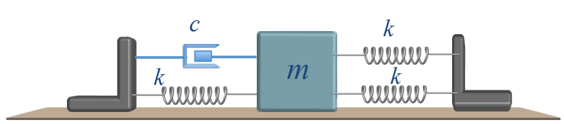
$$= \int_{-\infty}^{\infty} \delta(t - \tau) F(\tau) d\tau = F(t) \tag{16}$$

and thus

$$x(t) = \int_{-\infty}^{\infty} G(t - \tau) F(\tau) d\tau \tag{17}$$

solves the equation of motion.

2 (20 points)



A harmonic oscillator consists of a mass on a spring initially at rest on a horizontal table with a damping force $F_{\text{damp}} = -c dx/dt$. The system has mass $m = 15$ kg, 3 springs of equal spring constant $k = 80$ N/m, and the damping constant $c = 120$ kg/sec .

a) If the damping constant were zero (i.e. $c = 0$), what would be the natural frequency ω_n of the system?

b) Is this an underdamped, overdamped, or critically damped oscillator (with $c = 120$ kg/sec and the other parameters as stated in the problem)?

c) The system is given initial conditions $x(t = 0) = 10$ mm, $v(t = 0) = 0$ mm/sec. Find the resulting damped free vibration $x(t)$ in mm.

d) At what rate does the system dissipate energy as a function of time (in Watts)?

a

In the absence of damping, the natural frequency is found from noting that the effective spring constant is $3k$, so that

$$\omega_n = \sqrt{\frac{3k}{m}} = 4 \text{ rad/sec} \tag{18}$$

b

The nature of damping is determined by the value of the parameter ζ

$$\zeta = \frac{c}{2m\omega_n} = \frac{120}{2 * 15 * 4} = 1 \tag{19}$$

so the system is critically damped.

c

We recall that the critically damped case has repeated roots, so its solution is

$$x(t) = (A + Bt)\exp(-\zeta\omega_n t). \tag{20}$$

Clearly,

$$A = 10 \text{ mm}. \tag{21}$$

Next, we need to solve for B, imposing the second condition, $\dot{x}(0) = 0$

$$\dot{x}(0) = -\zeta\omega_n(A + B0)\exp(-\zeta\omega_n 0) + B\exp(-\zeta\omega_n 0) = 0 \quad \Rightarrow \quad B = 40 \text{ mm/sec} \tag{22}$$

so

$$x(t) = 10\exp(-\zeta\omega_n t) + 40\frac{t}{\text{sec}}\exp(-\zeta\omega_n t) \text{ mm} \tag{23}$$

d

The oscillator dissipates energy at a rate

$$F^{\text{damping}} v = -c\dot{x}^2 \tag{24}$$

we need $\dot{x}$, from part c), this is

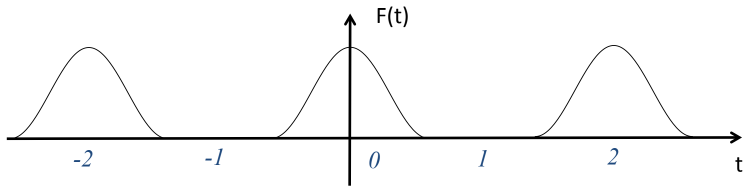
$$\begin{aligned}\dot{x} =& -\zeta\omega_n(A+Bt)\exp(-\zeta\omega_n 0) + B\exp(-\zeta\omega_n t) \\ =& -\zeta\omega_n Bt\exp(-\zeta\omega_n t) = -160\frac{t}{\text{sec}}\exp(-\zeta\omega_n t)\text{mm/sec} \end{aligned} \tag{25}$$

thus

$$F^{\text{damping}}v = -c\dot{x}^2 = -120(0.16)^2\left(\frac{t}{\text{sec}}\right)^2\exp\left(-8\frac{t}{\text{sec}}\right) \quad \text{Watts} \tag{26}$$

$$= -3.072\left(\frac{t}{\text{sec}}\right)^2\exp\left(-8\frac{t}{\text{sec}}\right) \quad \text{Watts} \tag{27}$$

3 (20 points)



A pictured function has Fourier series representation

$$F(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{2n\pi t}{T}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{2n\pi t}{T}\right) \tag{28}$$

You are not asked to calculate the a_0, a_n, or b_n.

a What is the period T?

b Based on the plot of $F(t)$, what can you say about the coefficients a_n and b_n? Do any vanish?

c This force is applied to a simple damped oscillator

$$m\ddot{x} + c\dot{x} + kx = F(t). \tag{29}$$

Find the steady-state response $x_{\text{st-st}}(t)$. Leave your answer in terms of the a_0, a_n, b_n and the system parameters m, k, c, and any secondary variables you may have defined in terms of them (like ζ or β or ω_n or ω_d or G). (make sure you do define any you use.).

d What is the average position of the oscillator?

a

From the plot, we see that the period is 2.

b

The plot is even under $t \to -t$, thus the $b_n = 0$, while $a_n \neq 0$. Further, the time average is non-zero, which implies that $a_0 \neq 0$.

c

We know that the response of a harmonic oscillator to a harmonic driving force $F(t) = F_0 \cos(\omega t)$ is

$$x(t) = F_0 G(\omega) \cos(\omega t - \phi(\omega)) \tag{30}$$

where

$$G(\omega) = \frac{1}{k} \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + 4\zeta^2 \left(\frac{\omega}{\omega_n}\right)^2}} \tag{31}$$

and

$$\phi(\omega) = \tan^{-1}\left[\frac{2\zeta\omega\omega_n}{\omega_n^2 - \omega^2}\right] \tag{32}$$

therefore, using the fact that we can simply sum over the particular solutions

$$x(t) = \frac{a_0}{2k} + \sum_{n=1}^{\infty} a_n G(n\omega) \cos(n\omega t - \phi(n\omega)) \tag{33}$$

where

$$\omega = \pi \tag{34}$$

d

The cosine terms all average to zero, and therefore the average position is

$$\langle x \rangle = \frac{a_0}{2k} \tag{35}$$

4 (20 points)

A ball is thrown at a latitude of 45 degrees north. If it is thrown vertically up (as determined locally) with a velocity v, where does the ball land? Note that in the local frame, the earth's angular velocity vector is given by

$$\vec{\omega} = \omega \hat{p} = (\hat{u} \sin\theta + \hat{n} \cos\theta) \tag{36}$$

a) Neglect the centripetal and elevator forces due to the earth's rotation, and starting from the formula

$$m\ddot{\vec{r}} = \sum \vec{F}^{\text{true}} - 2m\vec{\omega} \times \vec{v} \tag{37}$$

show that the components of the equations of motion for the ball in the local reference frame on the earth are given by

$$\ddot{x} = -2\omega\dot{z}\cos(\theta) + 2\omega\dot{y}\sin(\theta) \tag{38}$$

$$\ddot{y} = -2\omega\dot{x}\sin(\theta) \tag{39}$$

$$\ddot{z} = -g + 2\omega\cos(\theta)\dot{x}. \tag{40}$$

b) Working to first order in the coriolis force, calculate the position of a ball that is thrown vertically at a latitude of 45 degrees north.

a

The only true force that acts is gravity pointing vertically down. So, we have

$$m\ddot{\vec{r}} = -mg\hat{u} - 2m\omega\left(\hat{u}\sin\theta + \hat{n}\cos\theta\right) \times \left(\dot{x}\hat{e} + \dot{y}\hat{n} + \dot{z}\hat{u}\right) \tag{41}$$

Now, since

$$\hat{u} \times \hat{e} = \hat{n}, \quad \hat{n} \times \hat{e} = -\hat{u}, \tag{42}$$

we have

$$\begin{aligned} m(\ddot{x}\hat{e} + \ddot{y}\hat{n} + \ddot{z}\hat{u}) =& - mg\hat{u} - 2m\omega\left(\hat{u}\sin\theta + \hat{n}\cos\theta\right) \times \left(\dot{x}\hat{e} + \dot{y}\hat{n} + \dot{z}\hat{u}\right) && (43)\\ =& - mg\hat{u} - 2m\omega\left(\sin\theta(\dot{x}\hat{n} - \dot{y}\hat{e}) - \cos\theta(\dot{x}\hat{u} - \dot{z}\hat{e})\right) && (44) \end{aligned}$$

and we find

$$\begin{aligned} \ddot{x} =& 2\omega\dot{y}\sin\theta - 2\omega\cos\theta\dot{z} && (45)\\ \ddot{y} =& - 2\omega\sin\theta\dot{x} && (46)\\ \ddot{z} =& - g + 2\omega\cos\theta\dot{x} && (47) \end{aligned}$$

b

Now, we will solve this order by order in the coriolis force, to zeroth order

$$\begin{aligned} \ddot{x} =& 0 && (48)\\ \ddot{y} =& 0 && (49)\\ \ddot{z} =& - g && (50) \end{aligned}$$

and we have

$$z(t) = vt - \frac{1}{2}gt^2 \tag{51}$$

the ball hits the ground again when

$$t = \frac{2v}{g}. \tag{52}$$

We can now put in this solution into the equations to solve for the correction, we see that only the equation for x is corrected, we have

$$\ddot{x}_1 = -2\omega\cos\theta\dot{z}_0 = -2\omega\cos\theta(v - gt) \tag{53}$$

and we can integrate

$$x_1 = x_0 + v_{x_0}t - 2\omega\cos\theta\left(v\frac{t^2}{2} - g\frac{t^3}{6}\right). \tag{54}$$

Our initial conditions are $x_0 = 0$, $v_{x_0} = 0$. With $t = 2v/g$, we have

$$\begin{aligned} x_1 =& - \omega\cos\theta\left(v\frac{4v^2}{g^2} - \frac{g}{3}\frac{8v^3}{g^3}\right) = -\omega\cos\theta\left(\frac{4v^3}{g^2} - \frac{1}{3}\frac{8v^3}{g^2}\right) && (55)\\ =& - \frac{4}{3}\frac{v^3}{g^2}\omega\cos\theta && (56) \end{aligned}$$

with $\theta = \pi/4$, we get

$$x_1 = -\frac{2\sqrt{2}}{3}\frac{v^3}{g^2}\omega \tag{57}$$

5 (20 points)

A bug crawls around on a horizontal turntable rotating with constant angular speed ω. The mass of the bug is m and the coefficient of friction of the bug with the surface of the turntable is μ . Recall that $F_{\text{staticfriction}} \leq |\mu N|$ where $N = mg$ is the normal force. The onset of slippage occurs when $F_{\text{staticfriction}} = |\mu N|$. Gravity points downward. Ignore non-inertial effects due to Earths rotation.

The bug crawls with constant speed v_r relative to the turntable in a radial path.

- How far from the center of the turntable can the bug crawl before starting to slip? (in terms of ω, b, μ and g)

The total ficitious force on the bug is

$$\vec{F}^{fictitious} = -2m\vec{\omega} \times \vec{v} - m\vec{\omega} \times [\vec{\omega} \times \vec{r}] \tag{58}$$

for this problem, we have

$$\vec{\omega} = \omega\hat{k}, \quad \vec{r} = b\hat{i}, \quad \vec{v} = v_r\hat{i}, \tag{59}$$

so

$$\vec{F}^{fictitious} = -2m\omega v_r\hat{k} \times \hat{i} - mb\omega^2\hat{k} \times \left[\hat{k} \times \vec{i}\right] \tag{60}$$

$$= -2m\omega v_r\hat{j} + mr\omega^2\hat{i} \tag{61}$$

Now, the observer in the rotating frame thinks that the forces are in equilibrium (the bug is not accelerating in the rotating frame), thus we have

$$\vec{F}^{\text{friction}} + \vec{F}^{\text{fictitious}} = 0 \tag{62}$$

so that

$$\vec{F}^{\text{friction}} = 2\omega v_r\hat{j} - mb\omega^2\hat{i} \tag{63}$$

these are perpendicular, so we conclude that

$$\mu g = \sqrt{(2\omega v_r)^2 + \omega^4 b^2} \tag{64}$$

we can then solve for b,

$$b = \sqrt{\frac{\mu^2 g^2 - (2\omega v_r)^2}{\omega^4}} \tag{65}$$