Fourier Analysis

Recall: any periodic function (with period T) can be written
\[ F(t) = \sum_{n=1}^{\infty} \left( a_n \cos(n\omega t) + b_n \sin(n\omega t) \right) \quad \text{where} \quad \omega = \frac{2\pi}{T} \]

and \[ a_n = \frac{1}{T} \int_{T} F(t) \cos(n\omega t) \, dt \quad \text{and} \quad b_n = \frac{1}{T} \int_{T} F(t) \sin(n\omega t) \, dt \]

This is handy because we know how to do \[ F(t) = \cos(n\omega t) \]
and \[ F(t) = \sin(n\omega t) = \cos \left( \frac{\pi}{2} - n\omega t \right) = \cos(\frac{\pi}{2} - n\omega t) \]

\[ X_p(t) = \left( a_{0,2} + \sum_{n=1}^{\infty} \frac{a_n}{n} \cos(n\omega t - \phi_n) + \sum_{n=1}^{\infty} \frac{b_n}{n} \cos(n\omega t - \phi_n) \right) \]
\[ \frac{m}{\omega_n^2 - \omega^2} \quad \text{and} \quad \phi_n = \tan \left( \frac{\gamma_{n\omega}}{\sqrt{\omega_n^2 - \omega^2}} \right) \]

Note that this is periodic with some period \( T = \frac{2\pi}{\omega} \), but each term has a slightly different response, so \( X_p(t) \) is not just a simple scaling of \( F(t) \).

For example, if \( F(t) \) is a square wave
\[ F(t) = \frac{1}{\pi} \left[ \sum_{n \text{ odd}}^{\infty} \frac{\sin(n\omega t)}{n} \right] \]
\[ = \frac{4}{\pi} \left\{ \sin(\omega t) + \frac{1}{3} \sin(3\omega t) + \frac{1}{5} \sin(5\omega t) + \ldots \right\} \]

If the periodicity is such that \( \left( \frac{2\pi}{T} \right) = \omega \) is some fraction of the natural frequency \( \omega_n = \omega - n\omega \) with integer \( n \) that happens to be odd (e.g., if \( \omega = \omega_0 \) or \( \omega_0 \) \( \frac{5}{2} \), etc.), then the response could be dominated by the higher harmonics.
\[ x_p(t) = 4 \left\{ \frac{\sin(ut - \phi)}{8 \left( w_o^2 - \omega^2 \right)^2 + \frac{8\omega^2}{2} \omega^2} \right. \\
+ \frac{1}{3} \frac{\sin(ut - \phi_3)}{\left( w_o^2 - 9\omega^2 \right)^2 + \frac{9\omega^2}{2} \omega^2} \omega^{1/2} \\
+ \frac{1}{5} \frac{\sin(ut - \phi_5)}{\left( w_o^2 - 25\omega^2 \right)^2 + \frac{25\omega^2}{2} \omega^2} \omega^{1/2} \\
+ \ldots \right\} \\
\]

\[ \omega_o = 5\omega \]

\[ x_p(t) = 4 \left\{ \frac{\sin(ut - \phi)}{8 \left( 24^2 \omega^4 + \frac{8\omega^2}{2} \right)^{1/2}} + \frac{\sin(3ut - \phi_3)}{3 \left( 16^2 \omega^4 + \frac{9\omega^2}{2} \right)^{1/2}} \right. \\
+ \frac{\sin(5ut - \phi_5)}{5 \left( 25 \omega^4 \right)^{1/2}} + \ldots \right\} \\
\]

for lightly damped, \( \frac{Y}{w_o} \ll 1 \) & probably \( \frac{w}{w_o} \ll 1 \) & finally \( \frac{1}{w_o^2} \ll 1 \)

\[ x_p(t) = 4 \frac{1}{\pi m} W^2 \left\{ \frac{\sin(ut - \phi)}{8 \left( 24^2 \omega^4 + \frac{8\omega^2}{2} \right)^{1/2}} + \frac{\sin(3ut - \phi_3)}{3 \left( 16^2 + \frac{9\omega^2}{2} \omega^2 \right)^{1/2}} + \frac{\sin(5ut - \phi_5)}{25 \left( \frac{\omega^2}{2} \right)^{1/2}} + \ldots \right\} \\
\sim \frac{1}{24} \frac{w^2}{\omega^2} \sim \frac{1}{48} \frac{w^2}{\omega^2} \frac{\omega^2}{\omega^2} \left( \frac{1}{w_o^2} \omega^2 \right) \frac{\omega^2}{\omega^2} \]

\[ \text{can be >> 1 for light damping} \]

Notice that this is only going to happen if \( w < w_o \). If you push every third cycle at the same time, it will still add, if you push every time it goes through zero your pushes cancel \( \theta_{(w = 2w_o)} \).
The Fourier series as written is a bit unwieldy. Let's simplify:

Let's define a new complex coefficient $C_n$ such that

$$F_\ell(t) = \sum_{n=-\infty}^{\infty} C_n \ e^{-i\omega_n t}$$

just like before, but now $n$ now goes from $-\infty$ to $\infty$. If we define $A_n = C_n + C_{-n}$ and $B_n = i(C_n - C_{-n})$, and enforce the condition that $F_\ell(t)$ has no imaginary component, then we recover the original Fourier series: $F_\ell(t)$ is real $\Rightarrow C_n = C_{-n}^*$

As before, the orthogonality of $\sin(n \omega t)$ and $\cos(n \omega t)$ lead to an expression for $C_n = \frac{1}{T} \int F_\ell(t) e^{-i n \omega t} dt$.

In this language, we can again build up $x_p(t)$:

$$x_p(t) = \sum_{n=-\infty}^{\infty} \frac{C_n}{\omega_n} \ e^{-i\phi_n} \ e^{-i\omega_n t}$$

$$\phi_n = -\phi_{-n}, \ C_n = C_{-n}^* \Rightarrow x_p(t) \text{ is real}$$

But $\tan \phi_n = \frac{\omega n}{\omega_n^2 - \omega^2}$, so $\phi_n = \phi_{-n}$

$$e^{-i\omega_n t} = e^{-i\phi_n}$$

But let's look at $C_n$. This is a coefficient that tells us the amplitude of a wave with $\omega_n = m \omega = m / 2 \pi \cdot \frac{1}{T}$. If $T \to \infty$, then $\omega_n$ becomes a continuous field: we could pick any value of $\omega_n$, and there would be some integer $m$ that would work.
Fourier Transforms

1. Define \( \hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \) as some new continuous function of frequency. This is a new function, and once you have it, it is a thing that can be used. What is interesting is that you can invert this.

2. Given \( \hat{f}(\omega) \), \( f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega \)

This means that the function \( \hat{f}(\omega) \) contains all the information about \( f(t) \), since given only \( \hat{f}(\omega) \) you can perfectly reconstruct \( f(t) \).

This is the Fourier transform:

\[
\begin{align*}
\hat{f}(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt \\
f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t} \, d\omega
\end{align*}
\]

Notice that 1 is just a definition, but 2 is actually a non-trivial statement.

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{f}(\omega) e^{i\omega t - \omega t'} \, d\omega \, dt'
\]

This is the Fourier Integral Theorem, and it is true for any functions \( f(t) \) that a physicist will ever come across.

The easiest way that I know of to prove this requires a bit of complex analysis that I don't think you've learned yet, unfortunately.
Solving forced oscillations with the Fourier transform:

Once you have defined this function, you can solve the ODE:

\[
\ddot{x} + \dot{x} + \omega_0^2 x = F(t)
\]

Write \( x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{x}(w) e^{iwt} dw \);

\[
F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{F}(w) e^{iwt} dw
\]

Then:

\[
\tilde{x}(w) = \frac{1}{-w^2 + i\omega + \omega_0^2} \tilde{F}(w)
\]

For this to be true for all \( t \), we need:

\[
\tilde{x}(w) = \frac{\tilde{F}(w)}{-w^2 + i\omega + \omega_0^2}
\]

For example:

\[
F(t) = \frac{F_0}{m} \cos(\omega_0 t) = \frac{F_0}{m} \frac{1}{2} (e^{i\omega_0 t} + e^{-i\omega_0 t})
\]

\[
\tilde{F}(w) = \begin{cases} \frac{F_0}{2m} & w = \pm \omega_0 \\ 0 & \text{otherwise} \end{cases}
\]

\[
\tilde{x}(w) = \begin{cases} \frac{F_0}{2im \omega^2 - \omega_0^2} & w = \pm \omega_0 \\ \frac{F_0}{2im} & w = -\omega_0 \\ 0 & \text{otherwise} \end{cases}
\]
\[ x(t) = \frac{F_y}{2m} \frac{1}{-\omega^2 - j\omega \omega^2} + \frac{F_x}{2m} \frac{1}{-\omega^2 + j\omega \omega^2} \]

...which is what we first found back when we started.

This is a very effective way to solve problems like this, especially if you know the Fourier transform \( \tilde{f}(\omega) \) of \( f(t) \).

You can also do this a version of this in position space: instead of expressing things as a comb of \( e^{j\omega t} \) waves, express things as a comb of individual impulses \( \delta(t) \).
Response to a single sharp impulse

Imagine a strong force acts for a vanishing amount of time, such that all that happens is a change in momentum.

\[ F(t) \]

If we start at \( t = -\infty \) with \( x = 0, v = 0 \), then nothing happens until the impulse arrives. After the impulse, there is again no driving, but \( v = 0 \Rightarrow x(t) = e^{-\frac{t}{\tau}} \left\{ A \cos(w't') + B \sin(w't') \right\} \) with \( t' = t - t'_{\text{impulse}} \), \( w' = \sqrt{w^2 - \frac{2F}{m}} \).

Setting initial conditions

\[ x(t) = \frac{v_0 e^{-\frac{t}{\tau}}}{w'} \cdot \sin(w't') \quad \text{for} \quad t' > 0 \]

and \( x(t') = 0 \) for \( t' < 0 \).

This is a bit disturbing to have something happen like this, but it is an interesting case.

This kind of sharply peaked impulse is modeled using the "Dirac delta function." The property of the delta function is that it is only defined when used within integrals:

\[ f(a) = \int f(t) \delta(t-a) \, dt \]

Conceptually: \( \delta(t-a) \) = \[ \text{Height} \quad \text{in} \lim \, t \to 0 \]

or \( \lim_{a \to 0} \frac{1}{\sqrt{2\pi \sigma^2}} \exp\left(\frac{-(t-a)^2}{2\sigma^2}\right) \). Gaussian w/ integral = 1 but width = 0.
Properties of the Dirac delta,

\[ \delta(t-3) \delta(t-1) \]

E.g. \[ \int_{-\infty}^{\infty} \delta(t-3) \delta(t) \, dt = \delta(3) \]

\[ \int_{2}^{\infty} \delta(t-3) \, dt = \delta(3) \]

\[ \int_{-\infty}^{\infty} \delta(t-3) \delta(t-1) \, dt = 0 = \int_{-\infty}^{\infty} \delta(t-3) \delta(t) \, dt \]

\[ \int_{3}^{6} \delta(t-3) \, dt = \int_{-\infty}^{\infty} \delta(t-3) \, dt = \delta(3) \]

Also, \[ \delta(t-3) = \delta(3-t) \]

But \[ \delta(t^2-a) \neq \delta(t-a) \] for \[ a \neq 0 \]

Define \[ y = t^2 \rightarrow dy = 2t \, dt \]

\[ \int_{-\infty}^{\infty} \delta(y-a) \, f(y) \frac{dy}{2\sqrt{y}} = \frac{f(\sqrt{a})}{2\sqrt{a}} \]

For \[ f(t) = f_0 \delta(t) \]

\[ \mathcal{F}\{f(t)\} = \mathcal{F}\left\{ f_0 \delta(t) \right\} = \mathcal{F}\{f_0\} \delta(w) = \mathcal{F}\{f_0\} \cdot \delta(w) \]

Now we can fix the Normalization. From a few properties:

\[ f(t) = f_0 \cos(\omega t) = \frac{e^{i\omega t} + e^{-i\omega t}}{2} \rightarrow \mathcal{F}\{f(t)\} = \mathcal{F}\left\{ \frac{e^{i\omega t} + e^{-i\omega t}}{2} \right\} = \frac{\delta(w-\omega) + \delta(w+\omega)}{2} \]

However, very useful:

\[ \int_{-\infty}^{\infty} e^{i\omega t} \, dt = 1 \quad \int_{-\infty}^{\infty} e^{-i\omega t} \, dt = 2\pi \delta(w) \]

Can change variables: \[ \int_{-\infty}^{\infty} e^{i\omega t} \, dt = 1 \quad \int_{-\infty}^{\infty} e^{-i\omega t} \, dt = 2\pi \delta(w) \]
Back to the sharp impulse:

\[ F(t) = I \delta(t) \quad \text{where} \quad I = \Delta P = \int f(t) \, dt \]

If \( x = 0 \), \( \dot{x} = 0 \) before impulse

\[ x(t) = \begin{cases} \frac{I}{m} \int e^{-\frac{t}{w'}} \left( \frac{\sin(w't)}{w'} \right) & t > 0 \\ 0 & t < 0 \end{cases} \]

This is sometimes written as

\[ x(t) = \frac{I}{m} \int \left( e^{-\frac{t}{w'}} \frac{\sin(w't)}{w'} \right) H(t) \]

where \( H(t) = \begin{cases} 1 & t > 0 \quad \text{is the Heaviside} \\ 0 & t < 0 \end{cases} \) step function.

Summary: an impulse at time \( t_0 \)

\[ x(t) = \frac{I}{m} \int e^{-\frac{t}{w'}} \left( \frac{\sin(w'(t-t_0))}{w'} \right) H(t-t_0) \]

Because of the linearity of the system, we can write

\[ f(t) = \sum_{n} I_n \delta(t-t_n) \]

\[ x(t) = \sum_{n} \frac{I_n}{m} H(t-t_n) e^{-\frac{t-t_n}{w'}} \frac{\sin(w'(t-t_n))}{w'} \]

(assume \( x(t) = 0 \) at \( t = -\infty \))

But \( I_n = (F_n \Delta t_n) \) in the limit of \( \Delta t_n \to 0 \)

i.e. \( I_n = \int_{-\infty}^{t} f_n \, dt \)
In the continuous limit, we have

\[ f(t) = \int_{-\infty}^{\infty} F(\tau) g(t-\tau) \, d\tau \]

If we write \( x(t) = \sum \frac{e^{-\frac{\pi}{2} (t-t_n)}}{m} \),

with \( G(t-t_n) = e^{-\frac{\pi}{2} (t-t_n)} \sin(w(t-t_n)) H(t-t_n) \),

\[ x(t) \rightarrow \int_{-\infty}^{\infty} \frac{F(\tau)}{m} G(\tau-2) \, d\tau \quad \text{(for } x=0 \text{ at early times)} \]

This is a "convolution": \( x \) is the force as a function of time "convolved" with the response due to an impulse at each time.

We can also write \( x(t) = \int_{-\infty}^{\infty} \frac{F(\tau)}{m} G(t-\tau) \, d\tau \)

by defining \( G(t-\tau) = 2 \rightarrow \)

\( 2 \leftarrow t = T \)

\( 2 \leftarrow t = \tau \)

This is handy, because we know that \( G(t) = 0 \) for \( t < 0 \) and \( G(t) = 0 \) for \( t > T \).

\[ \mathbb{E} \{ x(t) \} = \int_{-\infty}^{\infty} \mathbb{E} \{ G(\tau) F(t-\tau) \} \, d\tau \quad \text{for} \quad x(t) \text{ built up by integrating past} \]

\[ \mathbb{E} \{ x(t) \} = \int_{-\infty}^{\infty} F(\tau) \mathbb{E} \{ G(t-\tau) \} \, d\tau \quad \text{(by causality)} \]
How do we use Green's function?

Example: turn a force on at \( t = 0 \): \( F(t) = F_0 H(t) \)

If we assume that \( x = 0, \frac{dx}{dt} = 0 \) (no disturbance at \( t < 0 \))

\[
x(t) = \int_{-\infty}^{\infty} F(t') G(t-t') \, dt'
\]

\[
x(t) = \int_{0}^{t} \frac{F_0}{m} e^{-\frac{w'(t-t')}{2w}} \frac{\sin[w'(t-t')]}{w'} \, dt'
\]

Step function in force cuts off lower band.

\[
x(t) = \frac{F_0}{mw'} \int_{0}^{t} e^{-\frac{w'(t-t')}{2w}} \sin[w'(t-t')] \, dt'
\]

Define \( z = w'(t-t') \); \( \frac{dz}{dt} = w' \)

\[
x(t) = -\frac{F_0}{mw''} \int_{0}^{w'} e^{-\frac{w'(z)}{2w}} \sin \frac{z}{w'} \, dz
\]

\[
x(t) = -\frac{F_0}{mw''} \left\{ \frac{1}{\sqrt{w''+1}} \right\} \left[ 1 - e^{-\frac{w'}{2w'}} \left( \frac{\cos(w'\frac{t}{2}) + \frac{w}{2} \sin(w'\frac{t}{2})}{w'} \right) \right]
\]

\[
x(t) = -\frac{F_0}{mw''} \left\{ \frac{1}{\sqrt{w''+1}} \right\} \left[ 1 - \frac{e^{-w'/2}}{w'} \left( w' \cos(w'\frac{t}{2}) + \frac{w}{2} \sin(w'\frac{t}{2}) \right) \right]
\]

\[
x(t) = -\frac{F_0}{mw''} \left\{ \frac{1}{\sqrt{w''+1}} \right\} \left[ 1 - e^{-\frac{w'}{2w'}} \left( \frac{\cos(w'z) + \frac{w}{2} \sin(w'z)}{w'} \right) \right]
\]

\[
\boxed{x(t) = \frac{F_0}{mw''} \left[ \frac{\sqrt{w''+1}}{2w'} \cos(w'z) + \frac{w}{2} \sin(w'z) \right] - e^{-\frac{w'}{2w'}} \left( -\frac{\cos(w'z)}{w'} + \frac{w}{2} \sin(w'z) \right)}
\]
However, if we had always had the force on, the solution would generally be just a superposition of a constant force field:

\[ x(t) = \frac{F_0}{m w^2} + e^{-\frac{t}{2w}} \left[ A \cos(w't) + B \sin(w't) \right] \]

\[ \text{\textbullet} \quad F_0 \text{ is equilibrium.} \]

If somehow we knew that \( x(0) = 0 \) and \( x'(0) = 0 \):

\[ x(0) = 0 \Rightarrow \quad 0 = \frac{F_0}{m w^2} + A \Rightarrow A = -\frac{F_0}{m w^2} \]

\[ x'(0) = 0 \Rightarrow \quad 0 = \frac{x}{2} [A] + B w' \Rightarrow B = -\frac{F_0}{m w^2} \frac{1}{2 w'} \]

\[ x(t) = \frac{F_0}{m w^2} \left[ 1 - e^{-\frac{t}{2w}} \left( \cos(w't) + \frac{x}{2w'} \sin(w't) \right) \right] \]

(Green's function)

\[ i.e. \quad \text{the same answer!} \]

However, in this case we managed to set the initial conditions using physics \( x = 0 \) at \( t = -\infty \) and we didn't need anything else.