Useful formulae

Momentum of a particle:

$$\vec{p} = m\frac{d\vec{r}}{dt}$$

Force on a particle:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Center of mass:

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}\,dm = \frac{1}{M}\sum_i m_i\vec{r}_i \qquad (1)$$

Change in momentum of center of mass in presence of external forces F_i:

$$\frac{d\vec{p}_{cm}}{dt} = \sum_i \vec{F}_i \qquad (2)$$

Cylindrical coordinates:

$$x = R\cos\phi \qquad (3)$$
$$y = R\sin\phi \qquad (4)$$
$$z = z \qquad (5)$$
$$R = \sqrt{x^2 + y^2} \qquad (6)$$
$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad (7)$$
$$\hat{\mathbf{R}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \qquad (8)$$
$$\hat{\phi} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \qquad (9)$$
$$\frac{d\hat{\mathbf{R}}}{dt} = \dot{\phi}\hat{\phi} \qquad (10)$$
$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{\mathbf{R}} \qquad (11)$$
$$\vec{v} = \dot{R}\hat{\mathbf{R}} + R\dot{\phi}\hat{\phi} + \dot{z}\hat{\mathbf{z}} \qquad (12)$$
$$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{\mathbf{R}} + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\hat{\phi} + \ddot{z}\hat{\mathbf{z}} \qquad (13)$$
$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial f}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} \qquad (14)$$

Spherical coordinates (3D):

$$x = r\sin\theta\cos\phi \qquad (15)$$
$$y = r\sin\theta\sin\phi \qquad (16)$$
$$z = r\cos\theta \qquad (17)$$
$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad (18)$$
$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \qquad (19)$$
$$\hat{\phi} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \qquad (20)$$
$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi} \qquad (21)$$
$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\mathbf{r}} + \cos\theta\dot{\phi}\hat{\phi} \qquad (22)$$
$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\theta}) \qquad (23)$$
$$\vec{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi} \qquad (24)$$
$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} +$$
$$(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} +$$
$$(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi} \qquad (25)$$
$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) +$$
$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \qquad (26)$$

Force from a potential:

$$\vec{F} = -\vec{\nabla}U \qquad (27)$$
$$U(x) = -\int_{x_\circ}^{x} dx'\,F(x') \qquad (28)$$
$$\frac{1}{2}m(v_2^2 - v_1^2) = \int_{x_1}^{x_2} dx'\,F(x') \qquad (29)$$

Gradient of a scalar field:

$$\vec{\nabla}U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \qquad (30)$$

Curl of a vector field:

$$\vec{\nabla}\times\vec{F} = \hat{\mathbf{i}}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) \qquad (31)$$

Stokes' Theorem, which turns a line integral around a closed path into an integral over the enclosed area, with $d\vec{A}$ a vector pointing in the direction orthogonal to the area (for example, a

path that lies in the $x-y$ plane gets turned into an area integral with $d\vec{A}$ pointing in the z direction, which picks out the z component of the curl):

$$\oint \vec{F} \cdot d\vec{r} = \int (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} \qquad (32)$$

Simple harmonic motion:

$$U(x) = \frac{1}{2}k(x - x_\circ)^2 + constant \qquad (33)$$

$$F(x) = -k(x - x_\circ) \qquad (34)$$

$$\omega \equiv \sqrt{\frac{k}{m}} \qquad (35)$$

If $\ddot{x} = -\omega^2 x$ then $x = A\sin(\omega t + \phi)$ (36)

Taylor expansion:

$$f(x) \sim f(x_\circ) + \frac{df}{dx}(x_\circ)(x - x_\circ)$$
$$+ \frac{1}{2!}\frac{d^2 f}{dx^2}(x_\circ)(x - x_\circ)^2$$
$$+ \frac{1}{3!}\frac{d^3 f}{dx^3}(x_\circ)(x - x_\circ)^3 + ... \qquad (37)$$

Rocket-related useful equation (for exhaust relative velocity $\vec{u}$; careful with signs!):

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt} \qquad (38)$$

Center of mass frame scattering (relative velocity $\vec{V}$):

$$\vec{V} = \vec{v}_1 - \vec{v}_2 \qquad (39)$$
$$\vec{p}_1 = \mu\vec{V} = -\vec{p}_2 \qquad (40)$$
$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (41)$$

Potential energy:

$$\text{Constant force} : U(x) = -Fx + C \qquad (42)$$

$$\text{Spring} : U(x) = \frac{1}{2}kx^2 + C \qquad (43)$$

$$1/r^2 \text{ Gravity} : U(x) = \frac{-GMm}{r} + C \qquad (44)$$

Some useful approximations (for small arguments to first order):

$$\sin\theta \sim \theta - ... \qquad (45)$$
$$\cos\theta \sim 1 - +... \qquad (46)$$
$$\ln(1 + x) \sim x2 + ... \qquad (47)$$
$$e^x \sim 1 + x + ... \qquad (48)$$
$$(1 + x)^\alpha \sim 1 + \alpha x + ... \qquad (49)$$

Central force motion effective potential for reduced mass μ and angular momentum ℓ and potential $U(r)$:

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2} \qquad (50)$$

Orbit in a potential $U = -C/r$:

$$r(\theta) = \frac{(\ell^2/\mu C)}{1 - \sqrt{1 + (2E\ell^2/\mu C^2)}\sin(\theta - \theta_\circ)} \qquad (51)$$

Taking $\theta_\circ = -\pi/2$, and using the following definitions:

$$r = \frac{r_\circ}{1 - \epsilon\cos\theta} \qquad (52)$$

$$\epsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu C^2}} \qquad (53)$$

$$r_\circ \equiv \frac{\ell^2}{\mu C} \qquad (54)$$

For a bound orbit, this is an ellipse, with the length of the longest axis A (all the way across i.e., the thing that would be the diameter, not the radius, for a circle) and the short axis B:

$$A = \frac{2r_\circ}{1 - \epsilon^2} = \frac{C}{-E} \qquad (55)$$

$$B = \frac{2\ell}{\sqrt{-2\mu E}} \qquad (56)$$

The longest and shortest distances from the central object:

$$r_{max} = \frac{r_\circ}{1 - \epsilon} \qquad (57)$$

$$r_{min} = \frac{r_\circ}{1 + \epsilon} \qquad (58)$$

Kepler's third law relating period T to length of long axis A:

$$T^2 = \frac{\pi^2}{2(M+m)G}A^3 \qquad (59)$$

Vis-viva equation for Keplerian orbits:

$$v^2 = GM_{tot}(\frac{2}{r} - \frac{1}{a}) \qquad (60)$$

Poisson's equation:

$$\nabla^2 \Phi = 4\pi G\rho \qquad (61)$$

Fourier series:

$$f(t) = \frac{a_\circ}{2} + \qquad (62)$$

$$\sum_{n=1}^{\infty}[a_n \cos(n\omega t) + b_n \sin(n\omega t)]$$

$$\text{where } \omega \equiv \frac{2\pi}{T} \qquad (63)$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt \qquad (64)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt \qquad (65)$$

Fourier transforms:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \qquad (66)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)e^{+i\omega t}d\omega \qquad (67)$$

Lightly damped harmonic motion (for ϕ and A set by initial conditions):

$$\ddot{x} = -\gamma\dot{x} - \omega_\circ^2 x \qquad (68)$$

$$x(t) = Ae^{-\gamma t/2}\cos(\omega' t + \phi) \qquad (69)$$

$$\omega' \equiv \sqrt{\omega_\circ^2 - \frac{\gamma^2}{4}} \qquad (70)$$

Forced damped harmonic motion: steady state solution (initial conditions are irrelevant after a sufficiently long time):

$$\ddot{x} = -\gamma\dot{x} - \omega_\circ^2 x + \frac{F_\circ}{m}\cos(\omega_F t) \qquad (71)$$

$$x_{ss}(t) = \frac{F_\circ \cos(\omega_F t - \phi)}{m\sqrt{(\omega_\circ^2 - \omega_F^2)^2 + \omega_F^2\gamma^2}} \qquad (72)$$

$$\tan(\phi) = \frac{\omega_F\gamma}{\omega_\circ^2 - \omega_F^2} \qquad (73)$$

Green's function for damped oscillator, where $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$:

$$G(t - \tau) = \frac{\sin[\omega'(t-\tau)]}{m\omega'}e^{-\gamma(t-\tau)/2}H(t-\tau) \qquad (74)$$

$$|\tilde{G}(\omega)| = \frac{1}{m}\frac{1}{\sqrt{(\omega_\circ^2 - \omega^2)^2 + \gamma^2\omega^2}} \qquad (75)$$

$$x(t) = \int_{-\infty}^{t} F(\tau)G(t-\tau)d\tau \qquad (76)$$

Fictitious force: uniform acceleration $\vec{a}$ for mass m:

$$\vec{F}_{fict} = -m\vec{a} \qquad (77)$$

Coriolis force, for system rotating with angular velocity Ω and for a velocity measured in the rotating frame of $\vec{v}_{rot}$:

$$\vec{F}_{cor} = -2m\vec{\Omega} \times \vec{v}_{rot} \qquad (78)$$

Centrifugal force, for system rotating with angular velocity Ω:

$$\vec{F}_{cent} = -m\vec{\Omega} \times (\vec{\Omega} \times \vec{r}) \qquad (79)$$

Relation between velocities measured in rotating and inertial frames:

$$\left(\frac{d\vec{B}}{dt}\right)_{inertial} = \left(\frac{d\vec{B}}{dt}\right)_{rotating} + \Omega \times \vec{B} \qquad (80)$$

Center of mass:

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}dm = \frac{1}{M}\sum_i m_i\vec{r}_i \qquad (81)$$

Change in momentum of center of mass in presence of external forces F_i:

$$\frac{d\vec{p}_{cm}}{dt} = \sum_i \vec{F}_i \qquad (82)$$

Moment of inertia:

$$I_{zz} = \int (x^2 + y^2)dm \qquad (83)$$

Angular momentum of a particle (with angular velocity $\omega = d\phi/dt$):

$$\vec{L} = \vec{r} \times \vec{p} = I\vec{\omega}$$

Torque on a particle:

$$\vec{\tau} = \vec{r} \times \vec{F} = I\frac{d\vec{\omega}}{dt}$$

Some useful moments of inertia (through center of mass for the direction that is the symmetry axis):

$$\text{Solid disk}: \quad I = \tfrac{1}{2}MR^2 \qquad (84)$$
$$\text{Ring}: \quad I = MR^2 \qquad (85)$$
$$\text{Solid sphere}: \quad I = \tfrac{2}{5}MR^2 \qquad (86)$$

Parallel Axis Theorem (moment of inertia for axis parallel to one through center of mass):

$$I = I_{cm} + Ml^2 \qquad (87)$$

Kinetic energy of object with velocity of center of mass v_{cm} and rotating around c.o.m with angular velocity ω:

$$K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\omega^2 \qquad (88)$$

Lagrangian Mechanics:

$$S = \int_{t_1}^{t_2} L\, dt \qquad (89)$$
$$L = T - V \qquad (90)$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha} \qquad (91)$$

In the presence of constraints $C(q_\alpha, t) = 0$, those may be handled as Lagrange multipliers, adding $\lambda C(q_\alpha, t)$ to the Lagrangian and proceeding as before.

A few trig identities:

$$\cos(a + b) = \cos a \cos b - \sin a \sin b \qquad (92)$$
$$\sin(a + b) = \sin a \cos b + \cos a \sin b \qquad (93)$$

Some more calculus:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\vec{v}} \cdot \nabla f \qquad (94)$$
$$\ddot{x} = \frac{1}{2}\frac{d}{dx}\dot{x}^2 \qquad (95)$$

Some integrals:

$$\int \frac{dx}{x + a} = \ln(x + a) + C \qquad (96)$$
$$\int e^x dx = e^x + C \qquad (97)$$
$$\int \ln x\, dx = x \ln x - x + C \qquad (98)$$
$$\int \sin x\, dx = -\cos x + C \qquad (99)$$
$$\int \cos x\, dx = \sin x + C \qquad (100)$$
$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad (101)$$
$$\int \frac{x\, dx}{\sqrt{x^2 + a^2}} = \sqrt{a^2 + x^2} \qquad (102)$$
$$\int \frac{dx}{(a + x)^n} = \frac{(a + x)^{1-n}}{1 - n} + C \qquad (103)$$
$$\int \frac{x\, dx}{(x + a)^3} = -\frac{a + 2x}{2(a + x)^2} \qquad (104)$$
$$\int_{-\infty}^{\infty} f(t)\delta(t - a)dt = f(a) \qquad (105)$$
$$\int_{-\infty}^{\infty} f(t)\delta(a - t)dt = f(a) \qquad (106)$$

Some Fourier transform pairs:

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{iat} dt = 2\pi\delta(\omega - a) \quad (107)$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-at^2} dt = \sqrt{\frac{\pi}{a}} e^{-\omega^2/(4a)} \quad (108)$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-a|t|} dt = \frac{2a}{a^2 + \omega^2} \quad (109)$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} \cos(at^2) dt = \sqrt{\frac{\pi}{a}} \cos\left(\frac{\omega^2}{4a} - \frac{\pi}{4}\right) \quad (110)$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} \sin(at^2) dt = -\sqrt{\frac{\pi}{a}} \sin\left(\frac{\omega^2}{4a} - \frac{\pi}{4}\right) \quad (111)$$