Useful formulae

Momentum of a particle:

$$\vec{p} = m\frac{d\vec{r}}{dt}$$

Force on a particle:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Center of mass:

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}dm = \frac{1}{M}\sum_i m_i\vec{r}_i \qquad (1)$$

Change in momentum of center of mass in presence of external forces F_i:

$$\frac{d\vec{p}_{cm}}{dt} = \sum_i \vec{F}_i \qquad (2)$$

Cylindrical coordinates:

$$x = R\cos\phi \qquad (3)$$
$$y = R\sin\phi \qquad (4)$$
$$z = z \qquad (5)$$
$$R = \sqrt{x^2+y^2} \qquad (6)$$
$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad (7)$$
$$\hat{\mathbf{R}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \qquad (8)$$
$$\hat{\phi} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \qquad (9)$$
$$\frac{d\hat{\mathbf{R}}}{dt} = \dot{\phi}\hat{\phi} \qquad (10)$$
$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{\mathbf{R}} \qquad (11)$$
$$\vec{v} = \dot{R}\hat{\mathbf{R}} + R\dot{\phi}\hat{\phi} + \dot{z}\hat{\mathbf{z}} \qquad (12)$$
$$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{\mathbf{R}} + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\hat{\phi} + \ddot{z}\hat{\mathbf{z}} \qquad (13)$$
$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial f}{\partial R}) + \frac{1}{R^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2} \qquad (14)$$

Spherical coordinates (3D):

$$x = r\sin\theta\cos\phi \qquad (15)$$
$$y = r\sin\theta\sin\phi \qquad (16)$$
$$z = r\cos\theta \qquad (17)$$
$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad (18)$$
$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \qquad (19)$$
$$\hat{\phi} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \qquad (20)$$
$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi} \qquad (21)$$
$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\mathbf{r}} + \cos\theta\dot{\phi}\hat{\phi} \qquad (22)$$
$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\theta}) \qquad (23)$$
$$\vec{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi} \qquad (24)$$
$$\begin{aligned}\vec{a} = {} & (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} + \\ & (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} + \\ & (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi} \qquad (25)\end{aligned}$$
$$\begin{aligned}\nabla^2 f = {} & \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \\ & \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \qquad (26)\end{aligned}$$

Force from a potential:

$$\vec{F} = -\vec{\nabla}U \qquad (27)$$
$$U(x) = -\int_{x_\circ}^{x} dx' F(x') \qquad (28)$$
$$\frac{1}{2}m(v_2^2 - v_1^2) = \int_{x_1}^{x_2} dx' F(x') \qquad (29)$$

Gradient of a scalar field:

$$\vec{\nabla}U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \qquad (30)$$

Curl of a vector field:

$$\vec{\nabla}\times\vec{F} = \hat{\mathbf{i}}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{\mathbf{j}}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{\mathbf{k}}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) \qquad (31)$$

Stokes' Theorem, which turns a line integral around a closed path into an integral over the enclosed area, with $d\vec{A}$ a vector pointing in the direction orthogonal to the area (for example, a path that lies in the $x-y$ plane gets turned into an area integral with $d\vec{A}$ pointing in the z direction, which picks out the z component of the curl):

$$\oint \vec{F} \cdot d\vec{r} = \int (\vec{\nabla} \times \vec{F}) \cdot d\vec{A} \qquad (32)$$

Simple harmonic motion:

$$U(x) = \frac{1}{2}k(x - x_\circ)^2 + constant \qquad (33)$$

$$F(x) = -k(x - x_\circ) \qquad (34)$$

$$\omega \equiv \sqrt{\frac{k}{m}} \qquad (35)$$

$$\text{If } \ddot{x} = -\omega^2 x \text{ then } x = A\sin(\omega t + \phi) \qquad (36)$$

Taylor expansion:

$$f(x) \sim f(x_\circ) + \frac{df}{dx}(x_\circ)(x - x_\circ) + \frac{1}{2!}\frac{d^2 f}{dx^2}(x_\circ)(x - x_\circ)^2 + \frac{1}{3!}\frac{d^3 f}{dx^3}(x_\circ)(x - x_\circ)^3 + \ldots \qquad (37)$$

Rocket-related useful equation (for exhaust relative velocity $\vec{u}$; careful with signs!):

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt} \qquad (38)$$

Center of mass frame scattering (relative velocity $\vec{V}$):

$$\vec{V} = \vec{v}_1 - \vec{v}_2 \qquad (39)$$

$$\vec{p}_1 = \mu\vec{V} = -\vec{p}_2 \qquad (40)$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (41)$$

Potential energy:

$$\text{Constant force} : U(x) = -Fx + C \qquad (42)$$

$$\text{Spring} : U(x) = \frac{1}{2}kx^2 + C \qquad (43)$$

$$1/r^2 \text{ Gravity} : U(x) = \frac{-GMm}{r} + C \qquad (44)$$

Some useful approximations (for small arguments to first order):

$$\sin\theta \sim \theta - \ldots \qquad (45)$$

$$\cos\theta \sim 1 - \ldots \qquad (46)$$

$$\ln(1 + x) \sim x + \ldots \qquad (47)$$

$$e^x \sim 1 + x + \ldots \qquad (48)$$

$$(1 + x)^\alpha \sim 1 + \alpha x + \ldots \qquad (49)$$

Central force motion: for reduced mass μ and angular momentum ℓ,

$$\ell \equiv \mu r^2 \dot{\phi}\,, \qquad (50)$$

using a known potential $U(r)$ we can define an effective potential:

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2} \qquad (51)$$

Orbit in a potential $U = -C/r$:

$$r(\theta) = \frac{(\ell^2/\mu C)}{1 - \sqrt{1 + (2E\ell^2/\mu C^2)}\sin(\theta - \theta_\circ)} \qquad (52)$$

Taking $\theta_\circ = -\pi/2$, and using the following definitions:

$$r = \frac{r_\circ}{1 - \epsilon\cos\theta} \qquad (53)$$

$$\epsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu C^2}} \qquad (54)$$

$$r_\circ \equiv \frac{\ell^2}{\mu C} \qquad (55)$$

For a bound orbit, this is an ellipse, with the length of the longest axis A (the "major axis", all the way across i.e., the thing that would be the diameter, not the radius, for a circle) and the short axis B:

$$A = \frac{2r_\circ}{1 - \epsilon^2} = \frac{C}{-E} \qquad (56)$$

$$B = \frac{2\ell}{\sqrt{-2\mu E}} \qquad (57)$$

$$a \equiv A/2 \qquad (58)$$

$$b \equiv B/2 \qquad (59)$$

The longest and shortest distances from the central object:

$$r_{max} = \frac{r_\circ}{1-\epsilon} \tag{60}$$

$$r_{min} = \frac{r_\circ}{1+\epsilon} \tag{61}$$

Kepler's third law relating period T to length of long axis A:

$$T^2 = \frac{\pi^2}{2(M+m)G}A^3 \tag{62}$$

$$T^2 = \frac{4\pi^2}{(M+m)G}a^3 \tag{63}$$

Vis-viva equation for Keplerian orbits:

$$v^2 = GM_{tot}(\frac{2}{r} - \frac{1}{a}) \tag{64}$$

Poisson's equation:

$$\nabla^2\Phi = 4\pi G\rho \tag{65}$$

A few trig identities:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b \tag{66}$$

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \tag{67}$$

Some more calculus:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\vec{v}} \cdot \nabla f \tag{68}$$

$$\ddot{x} = \frac{1}{2}\frac{d}{dx}\dot{x}^2 \tag{69}$$

Some integrals:

$$\int \frac{dx}{x+a} = \ln(x+a) + C \tag{70}$$

$$\int e^x dx = e^x + C \tag{71}$$

$$\int \ln x dx = x \ln x - x + C \tag{72}$$

$$\int \sin x dx = -\cos x + C \tag{73}$$

$$\int \cos x dx = \sin x + C \tag{74}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \tag{75}$$

$$\int \frac{xdx}{\sqrt{x^2+a^2}} = \sqrt{a^2+x^2} \tag{76}$$

$$\int \frac{x^2dx}{(1+x^2)^{5/2}} = \frac{x^3}{3(1+x^2)^{3/2}} \tag{77}$$

$$\int \frac{dx}{(a+x)^n} = \frac{(a+x)^{1-n}}{1-n} + C \tag{78}$$

$$\int \frac{xdx}{(x+a)^3} = -\frac{a+2x}{2(a+x)^2} + C \tag{79}$$

$$\int \frac{dx}{x^2(x+a)} = \frac{x\ln(a+x) - a - x\ln(x)}{a^2x} + C \tag{80}$$

$$\int \frac{dx}{x(x+a)} = \frac{\ln(x) - \ln(a+x)}{a} + C \tag{81}$$