Useful formulae

Momentum of a particle:

$$\vec{p} = m\frac{d\vec{r}}{dt}$$

Force on a particle:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Center of mass:

$$\vec{r}_{cm} = \frac{1}{M}\int \vec{r}dm = \frac{1}{M}\sum_i m_i\vec{r}_i \qquad (1)$$

Change in momentum of center of mass in presence of external forces F_i:

$$\frac{d\vec{p}_{cm}}{dt} = \sum_i \vec{F}_i \qquad (2)$$

Cylindrical coordinates:

$$x = R\cos\phi \qquad (3)$$
$$y = R\sin\phi \qquad (4)$$
$$z = z \qquad (5)$$
$$R = \sqrt{x^2+y^2} \qquad (6)$$
$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad (7)$$
$$\hat{\mathbf{R}} = \cos\phi\hat{\mathbf{i}} + \sin\phi\hat{\mathbf{j}} \qquad (8)$$
$$\hat{\phi} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \qquad (9)$$
$$\frac{d\hat{\mathbf{R}}}{dt} = \dot{\phi}\hat{\phi} \qquad (10)$$
$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}\hat{\mathbf{R}} \qquad (11)$$
$$\vec{v} = \dot{R}\hat{\mathbf{R}} + R\dot{\phi}\hat{\phi} + \dot{z}\hat{\mathbf{z}} \qquad (12)$$
$$\vec{a} = (\ddot{R} - R\dot{\phi}^2)\hat{\mathbf{R}} + (R\ddot{\phi} + 2\dot{R}\dot{\phi})\hat{\phi} + \ddot{z}\hat{\mathbf{z}} \qquad (13)$$
$$\nabla^2 f = \frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial f}{\partial R}) + \frac{1}{R^2}\frac{\partial^2 f}{\partial\phi^2} + \frac{\partial^2 f}{\partial z^2} \qquad (14)$$

Spherical coordinates (3D):

$$x = r\sin\theta\cos\phi \qquad (15)$$
$$y = r\sin\theta\sin\phi \qquad (16)$$
$$z = r\cos\theta \qquad (17)$$
$$\vec{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \qquad (18)$$
$$\hat{\theta} = \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \qquad (19)$$
$$\hat{\phi} = -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \qquad (20)$$
$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\theta} + \sin\theta\dot{\phi}\hat{\phi} \qquad (21)$$
$$\frac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{\mathbf{r}} + \cos\theta\dot{\phi}\hat{\phi} \qquad (22)$$
$$\frac{d\hat{\phi}}{dt} = -\dot{\phi}(\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\theta}) \qquad (23)$$
$$\vec{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi} \qquad (24)$$
$$\begin{aligned}\vec{a} = {} & (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\hat{\mathbf{r}} + \\ & (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} + \\ & (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi} \qquad (25)\end{aligned}$$
$$\begin{aligned}\nabla^2 f = {} & \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial f}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial f}{\partial\theta}) + \\ & \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2} \qquad (26)\end{aligned}$$

Force from a potential:

$$\vec{F} = -\vec{\nabla}U \qquad (27)$$
$$U(x) = -\int_{x_\circ}^{x} dx' F(x') \qquad (28)$$
$$\frac{1}{2}m(v_2^2 - v_1^2) = \int_{x_1}^{x_2} dx' F(x') \qquad (29)$$

Gradient of a scalar field:

$$\vec{\nabla}U = \frac{\partial U}{\partial x}\hat{\mathbf{i}} + \frac{\partial U}{\partial y}\hat{\mathbf{j}} + \frac{\partial U}{\partial z}\hat{\mathbf{k}} \qquad (30)$$

Curl of a vector field:

$$\vec{\nabla}\times\vec{F} = \hat{\mathbf{i}}(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}) + \hat{\mathbf{j}}(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}) + \hat{\mathbf{k}}(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}) \qquad (31)$$

Stokes' Theorem, which turns a line integral around a closed path into an integral over the enclosed area, with $d\vec{A}$ a vector pointing in the direction orthogonal to the area (for example, a path that lies in the $x-y$ plane gets turned into an area integral with $d\vec{A}$ pointing in the z direction, which picks out the z component of the curl):

$$\oint \vec{F}\cdot d\vec{r} = \int (\vec{\nabla}\times\vec{F})\cdot d\vec{A} \tag{32}$$

Simple harmonic motion:

$$U(x) = \frac{1}{2}k(x-x_\circ)^2 + constant \tag{33}$$
$$F(x) = -k(x-x_\circ) \tag{34}$$
$$\omega \equiv \sqrt{\frac{k}{m}} \tag{35}$$
$$\text{If } \ \ddot{x} = -\omega^2 x \text{ then } \ x = A\sin(\omega t + \phi) \tag{36}$$

Taylor expansion:

$$f(x) \sim f(x_\circ) + \frac{df}{dx}(x_\circ)(x-x_\circ) + \frac{1}{2!}\frac{d^2f}{dx^2}(x_\circ)(x-x_\circ)^2 + \frac{1}{3!}\frac{d^3f}{dx^3}(x_\circ)(x-x_\circ)^3 + \ldots \tag{37}$$

Rocket-related useful equation (for exhaust relative velocity $\vec{u}$; careful with signs!):

$$\frac{d\vec{P}}{dt} = M\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt} \tag{38}$$

Center of mass frame scattering (relative velocity $\vec{V}$):

$$\vec{V} = \vec{v}_1 - \vec{v}_2 \tag{39}$$
$$\vec{p}_1 = \mu\vec{V} = -\vec{p}_2 \tag{40}$$
$$\mu = \frac{m_1 m_2}{m_1 + m_2} \tag{41}$$

Potential energy:

$$\text{Constant force}: U(x) = -Fx + C \tag{42}$$
$$\text{Spring}: U(x) = \frac{1}{2}kx^2 + C \tag{43}$$
$$1/r^2 \text{ Gravity}: U(x) = \frac{-GMm}{r} + C \tag{44}$$

Some useful approximations (for small arguments to first order):

$$\sin\theta \sim \theta - \ldots \tag{45}$$
$$\cos\theta \sim 1 - +\ldots \tag{46}$$
$$\ln(1+x) \sim x2 + \ldots \tag{47}$$
$$e^x \sim 1 + x + \ldots \tag{48}$$
$$(1+x)^\alpha \sim 1 + \alpha x + \ldots \tag{49}$$

Central force motion effective potential for reduced mass μ and angular momentum ℓ and potential $U(r)$:

$$U_{eff}(r) = U(r) + \frac{\ell^2}{2\mu r^2} \tag{50}$$

Orbit in a potential $U = -C/r$:

$$r(\theta) = \frac{(\ell^2/\mu C)}{1 - \sqrt{1 + (2E\ell^2/\mu C^2)}\sin(\theta - \theta_\circ)} \tag{51}$$

Taking $\theta_\circ = -\pi/2$, and using the following definitions:

$$r = \frac{r_\circ}{1 - \epsilon\cos\theta} \tag{52}$$
$$\epsilon \equiv \sqrt{1 + \frac{2E\ell^2}{\mu C^2}} \tag{53}$$
$$r_\circ \equiv \frac{\ell^2}{\mu C} \tag{54}$$

For a bound orbit, this is an ellipse, with the length of the longest axis A (all the way across i.e., the thing that would be the diameter, not the radius, for a circle) and the short axis B:

$$A = \frac{2r_\circ}{1-\epsilon^2} = \frac{C}{-E} \tag{55}$$
$$B = \frac{2\ell}{\sqrt{-2\mu E}} \tag{56}$$

The longest and shortest distances from the central object:

$$r_{max} = \frac{r_\circ}{1-\epsilon} \tag{57}$$
$$r_{min} = \frac{r_\circ}{1+\epsilon} \tag{58}$$

Kepler's third law relating period T to length of long axis A:

$$T^2 = \frac{\pi^2}{2(M+m)G}A^3 \quad (59)$$

Vis-viva equation for Keplerian orbits:

$$v^2 = GM_{tot}(\frac{2}{r} - \frac{1}{a}) \quad (60)$$

Poisson's equation:

$$\nabla^2\Phi = 4\pi G\rho \quad (61)$$

Fourier series:

$$f(t) = \frac{a_\circ}{2} + \sum_{n=1}^{\infty}[a_n\cos(n\omega t) + b_n\sin(n\omega t)] \quad (62)$$

$$\text{where } \omega \equiv \frac{2\pi}{T} \quad (63)$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)dt \quad (64)$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)dt \quad (65)$$

Fourier transforms:

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt \quad (66)$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(\omega)e^{+i\omega t}d\omega \quad (67)$$

Lightly damped harmonic motion (for ϕ and A set by initial conditions):

$$\ddot{x} = -\gamma\dot{x} - \omega_\circ^2 x \quad (68)$$

$$x(t) = Ae^{-\gamma t/2}\cos(\omega' t + \phi) \quad (69)$$

$$\omega' \equiv \sqrt{\omega_\circ^2 - \frac{\gamma^2}{4}} \quad (70)$$

Forced damped harmonic motion: steady state solution (initial conditions are irrelevant after a sufficiently long time):

$$\ddot{x} = -\gamma\dot{x} - \omega_\circ^2 x + \frac{F_\circ}{m}\cos(\omega_F t) \quad (71)$$

$$x_{ss}(t) = \frac{F_\circ\cos(\omega_F t - \phi)}{m\sqrt{(\omega_\circ^2 - \omega_F^2)^2 + \omega_F^2\gamma^2}} \quad (72)$$

$$\tan(\phi) = \frac{\omega_F\gamma}{\omega_\circ^2 - \omega_F^2} \quad (73)$$

Green's function for damped oscillator, where $H(x) = 0$ for $x < 0$ and $H(x) = 1$ for $x > 0$:

$$G(t-\tau) = \frac{\sin[\omega'(t-\tau)]}{m\omega'}e^{-\gamma(t-\tau)/2}H(t-\tau) \quad (74)$$

$$|\tilde{G}(\omega)| = \frac{1}{m}\frac{1}{\sqrt{(\omega_\circ^2 - \omega^2)^2 + \gamma^2\omega^2}} \quad (75)$$

$$x(t) = \int_{-\infty}^{t} F(\tau)G(t-\tau)d\tau \quad (76)$$

Fictitious force: uniform acceleration $\vec{a}$ for mass m:

$$\vec{F}_{fict} = -m\vec{a} \quad (77)$$

Coriolis force, for system rotating with angular velocity Ω and for a velocity measured in the rotating frame of $\vec{v}_{rot}$:

$$\vec{F}_{cor} = -2m\vec{\Omega}\times\vec{v}_{rot} \quad (78)$$

Centrifugal force, for system rotating with angular velocity Ω:

$$\vec{F}_{cent} = -m\vec{\Omega}\times(\vec{\Omega}\times\vec{r}) \quad (79)$$

Relation between velocities measured in rotating and inertial frames:

$$\Big(\frac{d\vec{B}}{dt}\Big)_{inertial} = \Big(\frac{d\vec{B}}{dt}\Big)_{rotating} + \Omega\times\vec{B} \quad (80)$$

A few trig identities:

$$\cos(a+b) = \cos a\cos b - \sin a\sin b \quad (81)$$

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \quad (82)$$

Some more calculus:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \dot{\vec{v}}\cdot\nabla f \quad (83)$$

$$\ddot{x} = \frac{1}{2}\frac{d}{dx}\dot{x}^2 \quad (84)$$

Some integrals:

$$\int \frac{dx}{x+a} = \ln(x+a) + C \tag{85}$$

$$\int e^x dx = e^x + C \tag{86}$$

$$\int \ln x dx = x \ln x - x + C \tag{87}$$

$$\int \sin x dx = -\cos x + C \tag{88}$$

$$\int \cos x dx = \sin x + C \tag{89}$$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \tag{90}$$

$$\int \frac{xdx}{\sqrt{x^2+a^2}} = \sqrt{a^2+x^2} \tag{91}$$

$$\int \frac{dx}{(a+x)^n} = \frac{(a+x)^{1-n}}{1-n} + C \tag{92}$$

$$\int \frac{xdx}{(x+a)^3} = -\frac{a+2x}{2(a+x)^2} \tag{93}$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a) \tag{94}$$

$$\int_{-\infty}^{\infty} f(t)\delta(a-t)dt = f(a) \tag{95}$$

Some Fourier transform pairs:

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{iat} dt = 2\pi\delta(\omega - a) \tag{96}$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-at^2} dt = \sqrt{\frac{/pi}{a}} e^{-\omega^2/(4a)} \tag{97}$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-a|t|} dt = \frac{2a}{a^2+\omega^2} \tag{98}$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} \cos(at^2) dt = \sqrt{\frac{\pi}{a}} \cos(\frac{\omega^2}{4a} - \frac{\pi}{4}) \tag{99}$$

$$\int_{-\infty}^{\infty} e^{-i\omega t} \sin(at^2) dt = -\sqrt{\frac{\pi}{a}} \sin(\frac{\omega^2}{4a} - \frac{\pi}{4}) \tag{100}$$