Lecture 13 Wednesday March 7 , 2012

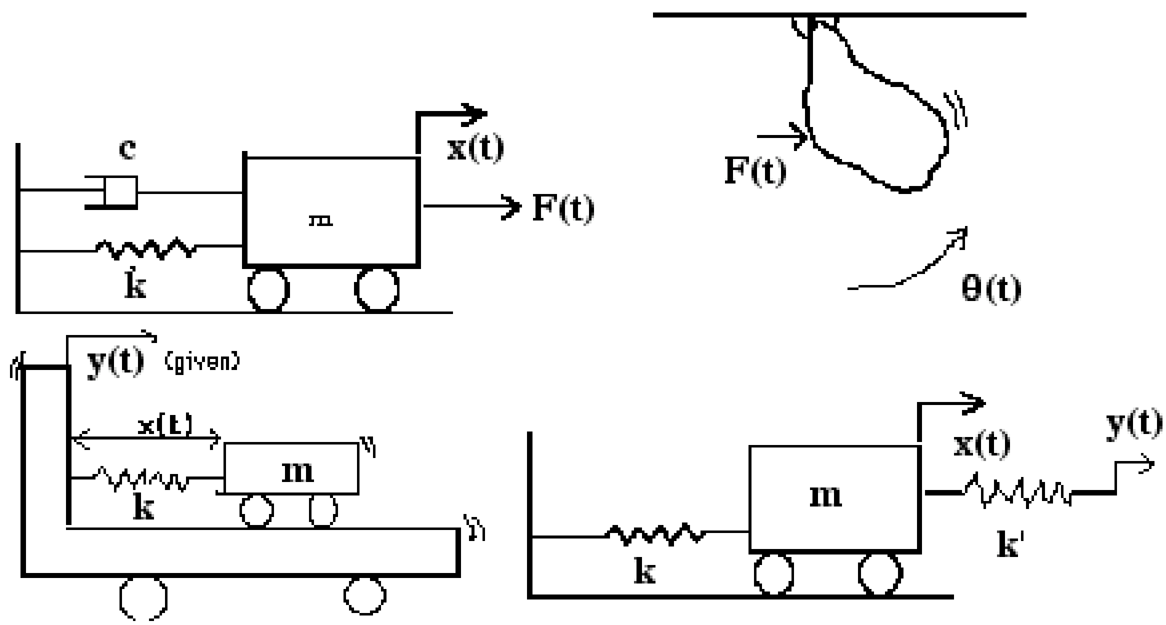
<u>Forced single-degree of freedom oscillators</u>

Generic differential equation:

$$m\{d^2x/dt^2\}+c\{dx/dt\}+kx =F(t)$$

where F is an applied force, or maybe something equivalent…..

examples:



All of the above systems have *EquationsOfMotion* of the above generic form. The last two are particular noteworthy because they are not driven by prescribed forces F(t) but rather by prescribed displacements y(t).

The fourth case has a force on the mass of k' { y-x } - k x acting towards the right. Thus the EOM is

$$md^2x/dt^2 + \{k+k'\}x = k'\, y(t) = F^{effective}(t)$$

which is of standard form. Note that the applied displacement y(t) is the cause of the "**effective force**" k'y(t). Also note the effective stiffness is the sum k + k'.

In the third case where m is on a moving table, m is forced inertially by the acceleration of the table d^2y/dt^2. The force on the mass is kx to the left because x is the amount by which the spring is stretched. The absolute displacement of the mass is, however, x + y; it is its second derivative that gives acceleration. This situation is left for your further development in HW 6

In all four cases, we get an ODE of the same general form. They are all "forced linear single-degree-of-freedom oscillators." Mathematically, these ODE's are classified as linear 2nd order, constant coefficient, but inhomogeneous. The term on the right hand side is the inhomogeneous term because it is independent of the variable x being solved for. It is, in general, an arbitrary function of time.

Solution of the Forced Problem, general comments

The EOM can be written in the more abstract form

$$\mathcal{L}\{x\} = F$$

Where $\mathcal{L}$ is a "**linear operator**". Linear operators, of which "take a second derivative" is an example, map functions of time into new functions of time in a linear manner. They have the property that, for any functions $f_1(t)$ and $f_2(t)$ and constants a and b.

$$\mathcal{L}\{ a f_1(t) + b f_2(t) \} = a \mathcal{L}\{ f_1(t) \} + b \mathcal{L}\{ f_2(t) \}$$

for example $\mathcal{L}$ operating on the sum of two functions gives the sum of $\mathcal{L}$ operating on each function separately.

The differential operator

$$m\{d^2/dt^2\} + c\{d/dt\} + k$$

is a linear operator.

Because of the property of linearity, the differential equation $\mathcal{L}\{x\} = F(t)$ has a solution of the form

$$x(t) = c_1 h_1(t) + c_2 h_2(t) + x_p(t)$$

in terms of three functions $h_1(t)$ and $h_2(t)$ and x_p defined as:

$h_1(t)$ and $h_2(t)$ are solutions of the homogeneous ODE without F(t)

(These solutions were found by the looking for $\exp(\lambda t)$, as discussed above.)

$x_p(t)$ is any *particular* solution of the full ODE $\mathcal{L}\{x\} = F(t)$.

That this linear combination $x(t) = c_1 h_1(t) + c_2 h_2(t) + x_p(t)$ solves the full ODE for any values of the constants c_1 and c_2 can be seen by substitution into $\mathcal{L}\{x\} = F$.

The combination $c_1 h_1(t) + c_2 h_2(t)$ is sometimes called the "*homogeneous*" (because it solves the homogeneous ODE) or "*complementary*" (because it complements the particular) solution.

For reasons to be discussed later, the homogeneous part is sometimes also called the '*transient*' solution and the particular solution $x_p(t)$ is sometimes called the "*steady state solution*"

Using this prescription, we write the solution to our forced, damped, ODE in the form

$$x(t) = \exp(-ct/2m)\,[\,A\cos\omega_d t \quad + \quad B\sin\omega_d t\,] \; + \qquad x_p(t)$$

or (if we prefer)

$$x(t) = \exp(-ct/2m)\;C\cos(\omega_d t - \varphi) \qquad + \qquad x_p(t)$$

The above forms each have two adjustable constants (A and B, or C and φ) which may be used to match the initial conditions. This is therefore not just *a* solution, it is *the general* solution. *Any* solution to the ODE, for any initial condition, may be put in either of the above forms.

We cannot rightly speak of "*the*" particular solution, because there are a double infinity of different solutions, any one of which may be taken as $x_p(t)$. In fact, if you can manage to choose an $x_p(t)$ which is exactly equal to the actual x(t), then the constants A, etc, must be zero. We note therefore, that the constants A, etc. take values that depend on what we've chosen for x_p, as well as the initial conditions.

The problem now reduces to finding any one particular solution to the full ODE with the forcing term. Any approach will do, any trick. We shall consider several different forms for F(t) and find particular solutions to each.

Power series forcing

The easiest is a forcing which is a power series in time. $F = a + b\,t + c\,t^2 + \ldots$
It is easy to see that one solution of the ODE

$$m\{d^2x/dt^2\}+c\{dx/dt\}+kx = F(t)$$

is obtainable by trying $x = \alpha + \beta t + \gamma t^2 + \ldots$ where the latter power series goes precisely as far as the former. Algebra then gives us the coefficients $\alpha, \beta, \gamma \ldots$

We won't spend time on this, as it is not very common. (But see the miniquiz..)

Harmonic Forcing

$$\text{Let } F(t) = F_o\cos(\omega t - \theta),$$

a force which varies sinusoidally in time. We seek a particular solution for this forcing. **N. B:** ω is not necessarily equal to ω_n or ω_d. ω is set by the mechanism that is doing the forcing; it is arbitrary.

This type of forcing is important in practice. There are two main reasons: Often systems are driven by reciprocating machinery which act periodically in time and can therefore be represented in their action by a Fourier series composed of terms like $F_o\cos(\omega t - \theta)$. Furthermore, driving forces can often come from the free vibration of supports; this free vibration is therefore harmonic. The other reason is that any force can be written, by means of a Fourier integral, as a superposition of such simple harmonic forces.

We wish to find any single particular solution to the ODE.

$$m\{d^2x/dt^2\}+c\{dx/dt\}+kx = F_o \cos(\omega t-\theta)$$

The usual trick is to *try* a solution that looks rather like the force, but with unknown amplitude and phase. So try $x_p(t) = D\cos(\omega t-\psi)$ and solve for D and ψ? This would work.

We will instead though, and equivalently, do it in a more abstract and easier way.

Write the force F(t) as

$$F_o \text{Real}\{ \exp(i\omega t-\theta) \} = \text{Real}\{ F_o \exp(-i\theta) \exp(i\omega t) \}$$

and try

$$x_p(t) = \underline{X} \exp(i\omega t)$$

where $\underline{X}$ is a complex number representing the phase and amplitude of x(t). That we will ultimately take only the real part of $\underline{X}$ exp(iωt) is implicit. We now substitute into the EOM:

$$-m\,\omega^2 \underline{X}\,\exp(i\omega t) + i\omega c\,\underline{X}\,\exp(i\omega t) + k\,\underline{X}\,\exp(i\omega t) = F_o \exp(-i\theta)\exp(i\omega t)$$

The common time dependences exp(iωt) drop out, leaving an equation for $\underline{X}$

$$\underline{X} = \underline{F} / [k + i\omega c - m\omega^2]$$

$$= \{ \underline{F} / k \} / [1 - (\omega/\omega_n)^2 + 2i\,\zeta\,\omega/\omega_n]$$

Thus the attempt to find a particular solution has worked. $x_p(t) = \text{Re}(\underline{X} \exp(i\omega t))$.

We define an important *complex* function $\underline{G}$ of ω (which we will be seeing lots of) by

$$\boxed{\underline{G}(\omega) = (1/k) / [1 - (\omega/\omega_n)^2 + 2i\,\zeta\,\omega/\omega_n]}$$

$\underline{G}$ may be broken up into its magnitude and phase by means of the definitions implicit in:
$\underline{\mathbf{G}}(\omega) = |\underline{G}(\omega)| \exp(-i\phi(\omega))$ (not the same ϕ met previously)

These real quantities are

$$\boxed{\begin{aligned} G = |\underline{G}| &= \frac{1/k}{\sqrt{(1-(\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2}} \\ \phi(\omega) = -\arg \underline{G} &= \tan^{-1}[2\zeta\omega\omega_n / (\omega_n^2 - \omega^2)] \end{aligned}}$$

In terms of these quantities we may conclude

$$\boxed{x_p(t) = \text{Re}\{ \underline{X} \exp(i\omega t) \} = F_o\, G \cos(\omega t-\theta-\phi)}$$

We see that

=> the particular response $x_p(t)$ is sinusoidal at the same frequency ω as the forcing (not necessarily the natural frequency ω_n).

=> The response has an amplitude which is different from the forcing by a factor $G(\omega)$ and has a phase delay with respect to the forcing by $\phi(\omega)$.

Also note that even in the presence of damping, this x_p does not decay in time; the force keeps doing work on the system which replaces that lost through dissipation.

We often speak of this x_p as being "the response"; it is actually only the "steady state" response, at times long after the effect of any initial conditions have died out. That is, at times $t \gg 2m/c$. For more on this, see below, where we discuss the imposition of initial conditions. $\{F_o/k\}$ would be the static response to a force F_o. The factor G shows how the response is more or less than this due to the dynamics. Dynamic response also differs from static by the phase delay ϕ.

Harmonic Forcing, Total Solution

What about initial conditions? What about the other part of the solution: $x_{homogeneous}$ or $x_{complementary}$? The full x is of the form:

$$x(t) = \exp(-ct/2m)\,[A \cos \omega_d t \quad + \quad B \sin \omega_d t\,] \quad + \qquad x_p(t)$$

What are the coefficients A and B, etcetera?

First note: It often does not matter. The $x_{complementary}$ dies away in a time period of order 2m/c. Therefore it is often called the "**transient**" solution. The *particular* solution, though, doesn't die away - even if there is damping. It is therefore sometimes called the **"steady state"** solution.

Nevertheless, we are sometimes interested in the behavior at times before the transient has decayed, therefore we need to determine coefficients A & B. To do this, we invoke Initial Conditions:

$$x(0) = x_o = x_p(0) + A$$
$$v_o = dx_p(0)/dt + \omega_d B - cA/2m.$$

which can be solved for A and B. The algebra is straightforward.

Particular Solution for *sum* of Harmonic Forcings?

What if the forcing consists of two distinct terms?

$$\mathcal{L}\{x\} = F_1 + F_2$$

The linearity of the ODE permits to establish that the solution is of the form

$x(t) = c_1\phi_1(t) + c_2\phi_2(t) + x_{p1}(t) + x_{p2}(t)$

where the two particular solutions pertain to the two distinct terms F_1 and F_2.

Thus the case $F(t) = F_1 \cos(\omega_1 t - \theta_1) + F_2 \cos(\omega_2 t - \theta_2)$

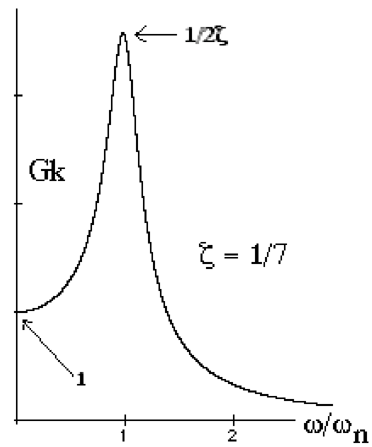
has solution

$$x(t) = \exp(-ct/2m)[\, A \cos(\omega_d t) + B \sin\omega_d t\,] + F_1\, G(\omega_1) \cos(\omega_1 t - \theta_1 - \phi(\omega_1)) + F_2\, G(\omega_2) \cos(\omega_2 t - \theta_2 - \phi(\omega_2))$$

Similarly, it is should be easy to convince yourself that the case $F(t) = F_0 \sin(\omega t)$ has solution

$$x(t) = \exp(-ct/2m)[\, A \cos(\omega_d t) + B \sin\omega_d t\,] + F_o\, G(\omega) \sin(\omega t - \phi(\omega))$$

<u>Resonance</u>



One of the most dramatic features of the function $G(\omega) = |\underline{G}|$ is that it can be very large for frequencies of excitation close to the natural frequency of the system. (see plot.) This means that the response of a system to harmonic forcing near the system's natural frequency has amplitude much greater than the quasi-static response $\{F_o/k\}$. This phenomenon is called "resonance". The forcing is said to be in "resonance" with the natural frequency. The peak is close to $\omega/\omega_n = 1.0$, (but is actually at $\omega = \omega_n \sqrt{(1-2\zeta^2)}$, slightly less than ω_n, and also less than ω_d. The difference depends on the type of damping and is often not even noticed.) We usually ignore this difference, and say that resonance occurs *at* ω_n. At that point G is seen to be $1/(2k\zeta)$. This will be large if ζ is small, and infinite when $\zeta = 0$.

The behavior at $\omega = 0$ may be comprehended also: This is the static limit, corresponding to a constant force. A constant force F_o of course has the obvious static response: $x_p = F_o/k$.

The behavior as $\omega \Rightarrow \infty$ where $|G| \sim 1/\omega^2$, may also be comprehended. Imagine pushing on a large mass with a rapidly time-oscillating force: it won't move much.

<u>questions</u>:

Why should response be especially large when the forcing frequency is close to the natural frequency? Answer(?) The system responds most enthusiastically when told to do what it wanted to anyway, to go at its favorite frequency. More physically, imagine pushing someone on a swing. If you push in a jittery way at too high a frequency, the swing just jiggles. If you push slowly, you get a quasi-static response and the swing moves only a few inches, but if you push at the same instant in each natural cycle, the motions builds up.

What is going on when $\zeta = 0$? How could response be infinite? At small damping, the transient (complementary, homogeneous) part of the full solution is more important. You will find when you do the HW for a very lightly damped system, that it takes some time (of order 2m/c) for the steady state to build up. In the limit that $\zeta=0$, one would find that the full solution looks something like $x \sim t \sin \omega_n t$ and grows without bound.

miniquiz 13 Wednesday March 7, 2012

____
name

An undamped mass – spring system

$$m\, d^2x/dt^2 + k\, x = F(t)$$

is subjected to a steadily increasing force

$$F(t) = a+bt.$$

1) Show that $x_p(t) = (a+b\, t)/k$ is *a* solution of the differential equation

Substitute $x_p(t) = (a+b\, t)/k$ *into* $m\, d^2x/dt^2 + k\, x$ *, get*

$$m\,[\ \ 0\ \ ] + k\,[\ a + b\, t\ ] / k$$

This is clearly equal to a + bt, so [a + bt] / k does satisfy the differential equaion.

2) Find the x(t) that satisfies both "quiescent initial conditions" [i.e., $x_o = 0$, and $v_o = 0$] *and* the differential equation. Do this by writing

$$x(t) = x_c + x_p = A \cos\omega_n t + B \sin \omega_n t + x_p(t)$$

and solving for A and B.

If $x(t) = x_c + x_p = A \cos\omega_n t + B \sin \omega_n t + (a+b\, t)/k$

then $x(t=0)$ *is* $A + a/k$. *If this is zero, then* ➔ *conclude* $A = -a/k$

and dx/dt *at* $t= 0$ *is* $B\, \omega_n + b/k$. *If this is zero, then* ➔ *conclude* $B = -b/k\omega_n$

Thus $x(t) = A \cos\omega_n t + B \sin \omega_n t + (a+b\, t)/k = (a/k)(1-coswnt) + (b/\omega_n k)\, t\omega_n - \sin\omega_n t)$

Lecture 14 Monday March 12 2012

Recap of previous lecture's discussion of steady state response to a harmonic forcing

$$m\{d^2x/dt^2\}+c\{dx/dt\}+kx = F_o \cos(\omega t-\theta)$$

$$x_{steady\text{-}state}(t) = F_o\, G \cos(\omega t-\theta-\phi)$$

where the ω-dependent functions G and ϕ are defined by

$$G = |\underline{G}| = \frac{1/k}{\sqrt{(1-(\omega/\omega_n)^2)^2+(2\zeta\omega/\omega_n)^2}}$$

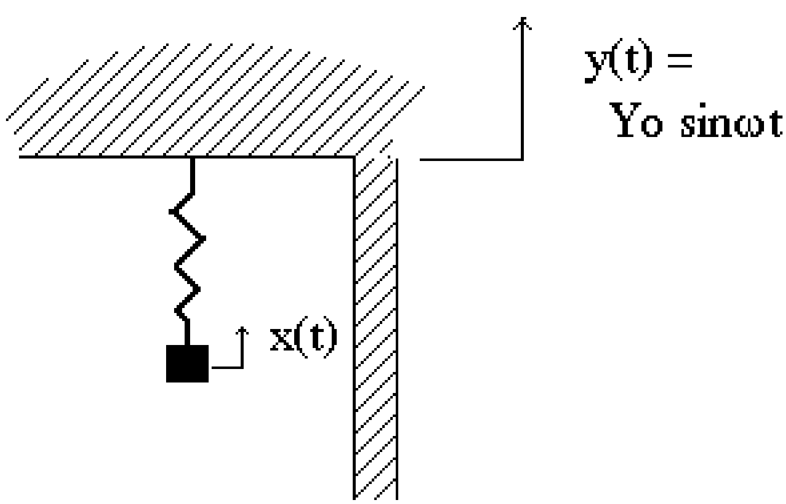
$$\phi(\omega) = -\arg \underline{G} = \tan^{-1}[2\zeta\omega\omega_n/(\omega_n^2-\omega^2)]$$

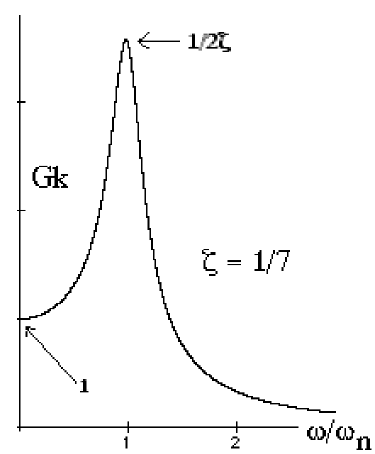
Main conclusions:

=> The steady-state response is sinusoidal at the <u>same frequency</u> ω as the forcing (not necessarily the natural frequency ω_n).

=> The ss response has an amplitude which is different from the forcing by a factor G(ω) and has a phase delay with respect to the forcing by ϕ(ω).

<u>Resonance continued…</u> The response of a mechanical oscillator to a harmonic force is illustrated in several on-line videos. For example, see http://www.youtube.com/watch?v=aZNnwQ8HJHU. This video shows a (vertical) mass-spring system driven by a prescribed harmonic displacement of its support. Schematically, it is





The equation of motion is, (for x defined as the absolute displacement, i.e., relative to the lab, not the ceiling) neglecting the dissipation terms,

$$m\, d^2 x/ dt^2 + k\, x = k\, y(t)$$

The video shows the motion of the mass together with the motion of the vertical rod (equivalent to the ceiling). It shows

1) at low frequency, the ceiling motion and the mass motion are in phase $\phi=0$ (and very nearly equal) . . as if the entire structure moves rigidly, $x = y$.

2) that at a resonant frequency, the mass moves much more than the ceiling does, and does so about $\phi=90$ degrees out of phase from the ceiling. This is resonance.

3) At high frequency, it shows that the mass has little amplitude, and what motion it has is $\phi=180$ degrees out of phase with the ceiling.

<u>Energy Balance and Power flow in the steady state</u>

Power dissipated is $c\, v^2$. Power input by the force is Fv. Both of these vary in time. We first recall

$$F(t) = F_o \cos(\omega t-\theta)$$

and

$$x = G\, F_o \cos\{ \omega t-\theta-\phi\}$$

so

$$v = dx/dt = \quad - G\omega\, F_o \sin\{ \omega t-\theta-\phi\}$$

(By the way, at resonance $\omega=\omega_n$, ϕ is $\pi/2$ and the velocity is exactly in phase with the force. This is another indication of why the amplitude is high at resonance, the force is doing maximal work.)

The power output of the force is the force times the velocity. F v =

$$- F_o \cos(\omega t-\theta)\, (F_o)\ |\underline{G}(\omega)|\ \omega \sin(\omega t-\theta-\phi)$$

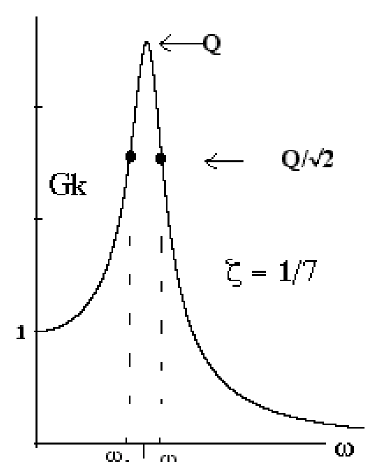
When not at resonance, the two factors are not in precise phase, so there are times during the cycle when the force is doing negative work on the system. And other times it is doing positive work.

The average power output could be obtained by integrating the above expression for Fv over one period and then dividing by the period. (T is $2\pi/\omega$) It is though, more easily obtained by equating the average power input to the average power dissipation (which is the time average of cv^2, being the velocity times the dissipating force.)

This is easy to do because the average of $\sin^2$ is 1/2.

$$\frac{1}{T}\int_{oneperiod} c\{-(F_o)G(\omega)\omega\sin(\omega t-\theta-\phi)\}^2 dt = cF_o^2\omega^2 G^2\ /\ 2$$

we see this is proportional to G^2 and independent of ϕ.



Definition: Quality Factor "Q" = the peak value of G compared to its quasi static value at $\omega = 0$, $G = 1/k$. The maximum amplification possible. See plot. It happens at $\omega \sim \omega_n$. Our theory tells us that $Q = 1/2\zeta$.

The points ω_1 and ω_2 are called the "half power points" because there the force is doing work on average at a rate half as fast as at resonance, because the amplitude of the motion there is down by a factor of $\sqrt{2}$.

Q is not usually measured by the peak value of G, because the low frequency limit where Gk is unity is not usually available to normalize against. Far more commonly, the resonance region is analyzed by measuring its "width", the distance between the half power points. From a scrutiny of the formula for G one can see, after a bit of algebra and if the damping is small, that Q is also equal to the natural frequency divided by the width,

$$\text{width} = \Delta\omega = \omega_2 - \omega_1 \qquad = \omega_n / Q.$$

These values of G can be derived by recalling the expression for $1/(k^2 G^2)$

$$\frac{1}{G^2 k^2} = (1 - (\omega/\omega_n)^2)^2 + (2\zeta\omega/\omega_n)^2$$

Let us analyze it in the vicinity of the resonance. At $\omega = \omega_n(1+\varepsilon)$ for small ε, it is…

$$\frac{1}{G^2 k^2} = (1 - (1+\varepsilon)^2)^2 + (2\zeta(1+\varepsilon))^2 \approx (1 - 1 - 2\varepsilon)^2 + (2\zeta)^2 = 4\varepsilon^2 + 4\zeta^2$$

which has a minimum value of $1/4\zeta^2$, (hence G_{max} is $1/2\zeta$). It takes a value of twice that at $\varepsilon = \pm\zeta = \pm 1/2Q$, these are the half power points, where $G \sim Q/\sqrt{2}$.

Correspondences: large Q - low damping - high peak - narrow peak

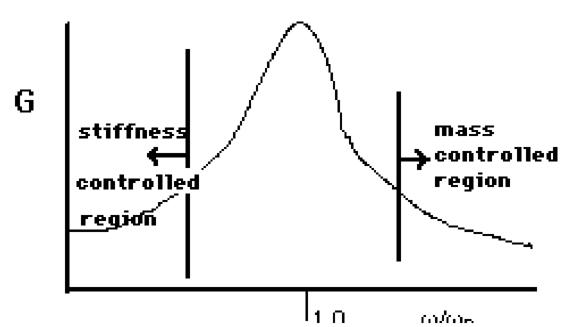
Q is another dimensionless (but *inverse*) measure of damping. It can be interpreted as the number of cycles of diminishing free oscillation (e.g. as illustrated in Lecture 12) required for the amplitude to drop to 4.3% of its original amplitude. Because each cycle drops a free vibrations's amplitude by a factor of $\exp(\delta)$ where δ is the log-decrement, Q cycles drops the amplitude by a factor

$$\exp(Q\delta) \sim \exp(\,(1/2\zeta)\, 2\pi\zeta\,) = \exp(\pi)$$

But $\exp(-\pi) = 0.043 = 4.3\%$

<u>Resonance Continued</u>

Another plot of the amplification factor, with named regions



Why are these regions so named?

Consider the low frequency regime, on the left. Here x_p = (Fo) G cos(ωt-θ-ϕ). In the low frequency limit, G is close to 1/k, and ϕ ~0, so x_p is very nearly F(t)/k. Minor changes in m or c will not change x. We therefore term this limit "stiffness controlled". This can be understood physically by realizing that a slowly varying F(t) (ie, low frequency ω) allows the mass to track the force as if it were in static equilibrium at all times. We find, therefore, that x = F(t)/k if F varies only slowly.

Consider the high frequency (mass controlled) region, where G is given approximately by Gk = $(\omega_n/\omega)^2$ which is very small for frequencies well above the natural frequency. Also ϕ is about π, so

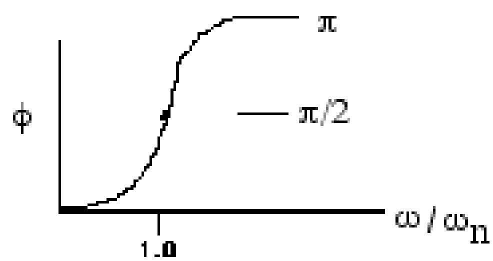
$$x_p = (F_o)\ G\ \cos(\omega t-\theta-\phi) \quad = -F_o/(m\omega^2) * \cos(\omega t-\theta)$$

from which we see that it is the mass m rather than the other system parameters which has the most influence on the response. Indeed, in this regime, we can write $md^2x/dt^2 =\sim F(t)$. This can be understood physically by realizing that a very rapidly varying F (ie, high frequency) will not allow a sluggish mass to follow it.

The phase delay ϕ..

$$\phi(\omega) = -\arg \underline{G} = \tan^{-1}[2\zeta\omega\omega_n / (\omega_n^2 - \omega^2)]$$

(but beware the π ambiguity in arctan)



The region on the left is the stiffness controlled region, where ϕ is approximately zero. This indicates, as the above physical argument indicated, that the displacement and force are in phase in this region, ie that the displacement tracks the force.

The high frequency mass controlled region is on the right and has ϕ approximately equal to π. Thus the displacement is 180° behind the force. Note however, that the acceleration is in phase with the force, as it should be when we recall F(t) ~ ma.

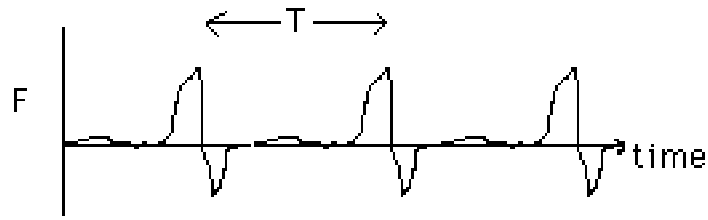
So much for i) power series F(t), and ii) Harmonic forcing. Now iii) Periodic forces F(t)

Periodic Forcing

Let us seek the particular solution for the case of periodic forcing

$$m\ d^2x/dt^2 + c\ dx/dt + k\ x = F(t)$$

in which F is periodic, i.e, for F(t) = F(t+T) for all times t. It is periodic with period T, not necessarily equal to T_n or T_d. T is in fact a parameter set by the mechanism doing the excitation. Such forces are common because systems are often forced by repetitive machinery, eg, an engine may have a bad valve which disturbs its output every time the cylinder with the bad valve fires. Even in a smoothly running engine, the loads will be periodic.



Such forces, by means of a Fourier series decomposition may be written as a sum of an infinite number of simple sines and cosines of time. By means of the principle of superposition, we will then be able to write the particular solution x_p to the above force as a similar infinite sum of particular solutions to each harmonic forcing.

Superposition follows from the linearity of $\mathcal{L}$:

if $\mathcal{L}\{x(t)\} = f(t)$ and $\mathcal{L}\{y(t)\} = g(t)$ for x, y, f, and g being functions of time, then for all numbers a and b

$$\mathcal{L}\{ a\,x(t) + b\,y(t) \} = a\,f(t) + b\,g(t)$$

The particular solution to a sum of forcings is the sum of the particular solution to each forcing separately.

We use the above property in constructing, by inspection, the particular solution to a periodic force. First, though, we have a digression on how to write a periodic function as a sum of sines and cosines:

For periodic forces F(t), with ω defined as $2\pi/T$ = the circular frequency of a harmonic force with the same period. We invoke Baron Fourier's observation that (any) periodic function F(t) may be written as

$$F(t) = a_o/2 + \Sigma\, a_n \cos(n\,\omega t) \qquad \{= F_{even}(t)\}$$
$$+ \Sigma\, b_n \sin(n\,\omega t) \qquad \{= F_{odd}(t)\}$$

where the sums run from n=1 to infinity. That any periodic force F(t) can be so decomposed is not obvious (and widely not believed for a while until it was made more rigorous). With minor caveats regarding discontinuities in F(t), and regarding the speed of convergence, it is, though, true.

Some observations:

$a_o/2$ is the time average of F.
The even part of F has only cosine terms.
The odd part of F has only sine terms.

Terms with large values of n vary rapidly in time and are therefore possibly unimportant (their coefficients must be small) if F(t) itself is not varying rapidly in time.

The above series is explicitly periodic with period T.

The cosine and sine parts of the series correspond respectively to the Even and Odd parts of F(t) (this can be useful for finding the coefficients)

An even function such as F_{even} has the property that

$F_{even}(t) = F_{even}(-t)$

An odd function has the property that

$F_{odd}(t) = - F_{odd}(-t)$

Any function can be de-composed into even and odd parts

$F = F_{even}(t) + F_{odd}(t)$

and these parts may be constructed from

$F_{even}(t) = (1/2) * [F(t) + F(-t)]$

$F_{odd}(t) = (1/2) * [F(t) - F(-t)]$

And most importantly,…… The Fourier coefficients, the a's and b's, may be constructed from the *formulas*:

$$a_m = (2/T) \int F(t)\cos(m\omega t)dt$$
$$b_m = (2/T) \int F(t)\sin(m\omega t)dt$$

where the integral's limits do not matter as long as they extend over one full contiguous period, eg, 0 to T, or -T/2 to + T/2.

The above integral formulas for the a and b may be checked by substituting the Fourier series for F into the integrals. Because of the properties of the integrals of sines and cosines over full periods, most of the resulting integrals give zero and the above expressions for the a's and b's are established as true.

It is worthwhile to demonstrate these *orthogonalities*, as the phenomenon recurs throughout physics and mechanics. Consider a few simple integrals: (for n and m ≠ 0, and for any contiguous integration range T hat runs over a full period 2π/ω)

$$\int_T \cos(n\omega t)\cos(m\omega t)dt = (T/2) \quad (n = m); \qquad \int_T \sin(n\omega t)\sin(m\omega t)dt = (T/2) \quad (n = m)$$
$$\int_T \cos(n\omega t)\cos(m\omega t)dt = 0 \quad (n \neq m); \qquad \int_T \sin(n\omega t)\sin(m\omega t)dt = 0 \quad (n \neq m)$$
$$\int_T \sin(n\omega t)\cos(m\omega t)dt = 0$$

These results may be proved by using trig identities. For example,

$$\int_T \cos(n\omega t)\cos(m\omega t)dt = \int_T \frac{\cos\{(n+m)\omega t\}}{2}dt + \int_T \frac{\cos\{(n-m)\omega t\}}{2}dt$$

The first term is an integral of cosine over n + m cycles; it is therefore zero.

If n≠m, the second term is an integral of cosine over n-m cycles, so it is zero also. If n = m, the cosine is unity, and the second term is the integral of (1/2) over the period T; therefore it is T/2.

The other integrals may be evaluated using similar procedures.

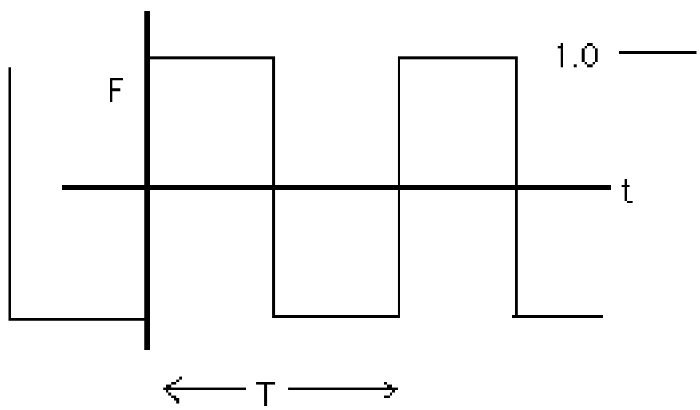
Let us check the formula for a_m:

$$a_m =^? (2/T)\int \cos(m\omega t)F(t)\,dt = (2/T)\int \cos(m\omega t)[\frac{a_0}{2} + \Sigma\, a_n \cos(n\omega t) + \Sigma\, b_n \sin(n\omega t) \quad]dt$$

• If m = 0, we see that the first term in the integration is $a_o/2$ and all the others vanish. QED

• If m ≠ 0, the first term vanishes, as do all the terms in cosine times sine.
We further note that the terms in cosine times cosine vanish except for the mth. QED.

<u>Example: A Square wave</u>



One apply the above integral formulas to calculate the a's and b's, but it is wise to first notice
i) the time average of this F(t) is zero, hence a_o is zero.
ii) the above function F(t) is odd; hence all the a's vanish.

We therefore need only calculate the b's:

$$b_m = (2/T)\int F(t)\sin(m\omega t)\,dt$$

We can chose any contiguous period, but we notice that it may be simplest to choose the range from -T/2 to + T/2.

Therefore:

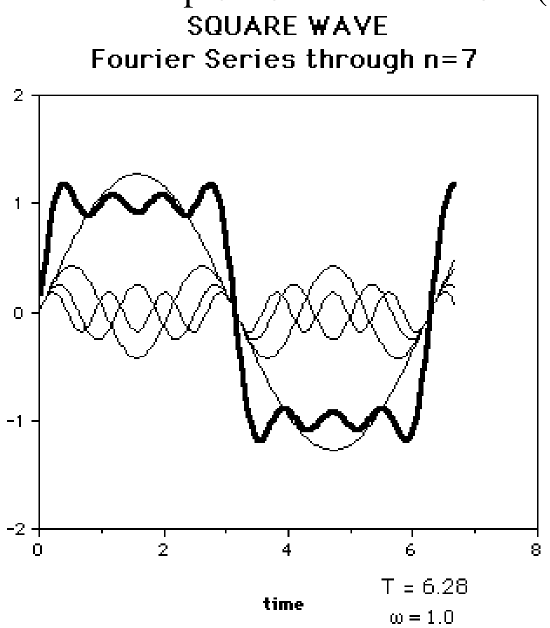
$$b_m = (2/T)\int_{-T/2}^{T/2} F(t)\sin(m\omega t)\,dt = (4/T)\int_0^{T/2}\sin(m\omega t)\,dt = (4/m\omega T)[1-\cos(m\omega T/2)]$$

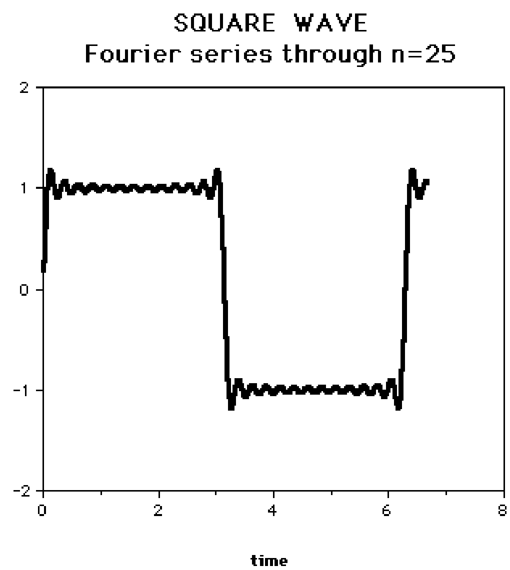
but T = 2π/ω so the cosine is 1 if m is an even number and -1 if m is odd.

b_m = 4/(m π) if m is odd
= 0 if m is even

F(t) = (4/π) [sin ωt + (1/3) sin 3ωt + (1/5) sin(5ωt) + . . .]

Here we see plots of this series for F(t), truncated to a finite number of terms,





The truncated Fourier series has difficulty capturing the very fast transitions. The overshooting followed by, or preceded by, a ringdown is called the "Gibbs" phenomenon and is characteristic of a truncated Fourier series's attempt to describe sharp changes in behavior such as discontinuities.

The error in the truncated Fourier series wiggles at a rate of order the first term neglected. In the 2nd of the above two graphs that term goes like sin(27t) and has a period of 2π/27 = 0.23. That can be seen to be the period of the wiggles in the above graph.

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Evaluate the following definite integrals by inspection

$\int_0^{2\pi} \cos(3t)\,dt =$ *This is zero, as it is the integral of the oscillation over a full period.*

$\int_{\pi}^{3\pi} \sin(17t)\cos(17t)\,dt =$ *This is zero, as could be seen by rewriting the integrand as (1/2) sin34t , which when integrated over a full cycle has to be zero.*

$\int_4^{4+2\pi} \cos(8t)\cos(9t)\,dt =$ T*his is zero as can be seen by rewriting the integrand as*

(1/2) [cos(17t) – cos(t)],

each term of which oscillates so an integration over a full period gives zero.

$\int_{-\pi}^{\pi} \cos^2(19t)\,dt =$ *cosine*^{2} *has an average value of 1/2, so a integration over a full period is the period times 1/2, The answer is therefore* π.

Lecture 15 Wednesday March 14

Response x_p to such a force....

if

$$F(t) = a_o/2 + \Sigma a_n \cos(n\,\omega t) + \Sigma b_n \sin(n\,\omega t)$$

for which the coefficients a and b were found from the original description of F(t) their integral formula.

Then

$$x_p(t) = a_o/2k + \Sigma (a_n) G(n\omega) \cos(n\,\omega t-\phi(n\omega)) + \Sigma (b_n) G(n\omega) \sin(n\,\omega t-\phi(n\omega))$$

where the G and the ϕ are found from

$$G(n\omega) = \frac{1/k}{\sqrt{(1-(n\omega/\omega_n)^2)^2 + (2\zeta n\omega/\omega_n)^2}}$$

$$\phi(n\omega) = -\tan^{-1}[2\zeta n\omega\omega_n/(\omega_n^2 - n^2\omega^2)]$$

This form for x_p is established by appeal to the known form for the particular solutions for each individual harmonic force composing the Fourier series for F, and to the principle of superposition discussed in earlier {*the x_p for a sum of forces is the sum of the individual x_p*}

The above particular response is ugly. What is the essential physics? Let us not that this xp is

i) steady state; that is, it has no component which decays with time.

ii) periodic with the same period as F(t), that is, it can be seen that if t increases by T = $2\pi/\omega$ then x_p is unchanged. By employing trig formulas for the cosine and sine of the sum of two terms, the above could, if you wished, be explicitly rewritten as a Fourier series (cosines and sines without the phases ϕ) . That is rarely a useful thing to do.

iii) looks like the original Fourier series, but with each term amplified by a different factor G depending on the frequency $n\omega$ of that term, and phase delayed by a different amount $\phi(n\omega)$.

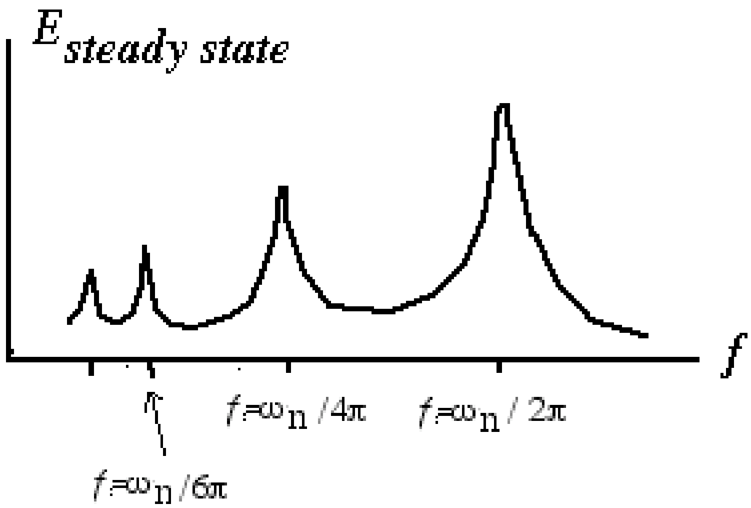
Resonant responses to periodic forcings.

The above expression

$$x_p(t) = a_o/2k + \Sigma (a_n) G(n\omega) \cos(n\,\omega t-\phi(n\omega)) + \Sigma (b_n) G(n\omega) \sin(n\,\omega t-\phi(n\omega))$$

will be resonant if any of the terms are resonant. That is, if $n\omega \sim \omega_{natural}$ for some n, then the G factor for that term will be large and that term can dominate the sum. (assuming the corresponding a and b do not both vanish.) A forcing might be a square wave, but the response will be mostly a single harmonic sinusoid at the subharmonic of the forcing closest to the natural frequency of the mass-spring system.

Schematically we might plot the time-averaged energy of the steady state response versus the frequency $f = \omega/2\pi$ of the periodic forcing. It will show peaks at *subharmonics* of the natural frequency $\omega = \omega_n / n$



This can be slightly counter-intuitive. Why, if the frequency of forcing is half, or a third, of the natural frequency, do we still get resonance? But a little thought establishes its reasonableness. Think of the swing set again; if you push only every other swing, you still get a good response.

Complex Fourier Series

Another version of the Fourier series, the **Complex Fourier Series**. is mathematically more abstract, but simpler too. Consider

$$F(t) = \Sigma \underline{c}_n \exp(in\omega t)$$

where the sum is now over all n from -infinity to +infinity and the $\underline{c}$ are complex.

The reality of F; F = F* implies

$$\underline{c}_{-n} = \underline{c}_n^*$$

By expansion of the exp(inωt)'s using the Euler identities one can make the following identifications between the coefficients of the regular Fourier Series and the complex Fourier Series:

$$a_n = \underline{c}_n + \underline{c}_{-n} \qquad \text{and} \qquad b_n = i\,(\underline{c}_n - \underline{c}_{-n})$$

The integral formula for the Fourier coefficients c_m can be obtained very easily. Multiply both sides of the above complex Fourier series for F by exp(-i m ωt) and integrate over a full contiguous period T.

$$\int F(t)\exp(-im\omega)dt = \int[\,\Sigma\; c_n \exp(in\omega t)\,]\exp(-im\omega)dt$$

On the right hand side all terms will vanish upon integration except for the one term where n happens to equal m. This term will consist of an integral over the period T of c_m. Therefore

$$\underline{c}_m = (1/T)\int F(t)\exp(-im\omega t)dt$$

which , BTW, does obey $\underline{c}_{-n} = \underline{c}_n{}^*$.

Response to a complex Fourier series… Given an F(t) represnted as above as a sum of complex exponentials, we recall that each such term $c_n \exp(in\omega t)$ gives rise to a particular solution $c_n \underline{G}(n\omega)\exp(in\omega t)$ where the complex amplification factor $\underline{G}$ was given by $\underline{G}(\omega)$

$$\underline{G}(\omega) = (1/k) / [\;1 - (\omega/\omega_n)^2 + 2i\,\zeta\,\omega/\omega_n\;]$$

(see lecture 13). Thus the response to the complex Fourier series F(t) is

$$x_p(t) = \Sigma\; \underline{c}_n\; \underline{G}(n\omega)\; \exp(\,in\omega t)$$

The notation for complex Fourier series is more compact than it is for the cosines and sines, at the expense of having to deal with complex quantities.

Query: This x_p looks to be complex? Is it? Answer: No, because the terms in negative n are complex conjugates of the terms in positive n. $c_{-n}= c_n{}^*$ and because $\underline{G}(-\omega) = \underline{G}(\omega)^*$.

So far we've discussed power series forces and their particular solutions, harmonic, and periodic. Another classic is the impulse response, which provides an introduction ot the important topic of Green's functions:

The impulse response G(t)

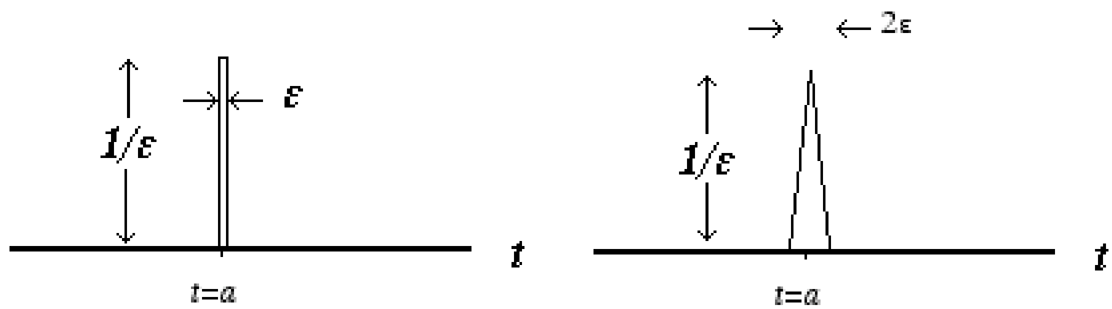
G(t) is specific to a specified initial condition. It is the x(t) that follows quiescent initial conditions in which $x_o = 0$, and $v_o = 0$, and then forcing by a unit "impulse".

$$m\, d^2x/dt^2 + c\, dx/dt + k\, x = \delta(t)$$

where δ(t) is the "Dirac delta-function," This force δ(t) is an idealization of a large force which acts for a very short amount of time.

The 'generalized function' δ(t-a) is zero for all t except t = a, and infinite for t=a. It has the further property that its integral over any region which includes t=a is unity. (Its integral over any other region is zero).

There are several practical conceptualizations of δ(t-a). The simplest are as limits of high and narrow peaks......eg, a rectangle centered on t=a of magnitude 1/ε and width ε in the limit that ε becomes small.



Or a triangle of base 2ε and height 1/ε. Or a Gaussian of narrow width and large height.

The proper definition, from the mathematical theory of *Distributions* or *generalized functions*, is that linear operator with the property, for any function *f* continuous at a,

$$f(a) = \int_{-\infty}^{\infty} f(t)\delta(t-a)dt$$

The above formula may easily be established as true if one's idea of the delta function is the limit of the narrow high rectangle or triangle.

The delta function δ(t) has dimensions of inverse argument (time in this case).. why?

Integrals with delta-functions in them are easy to evaluate.   For example:

$$\int_{-\infty}^{\infty} \delta(t-3)\,\sin(t)\,dt = \sin(3); \qquad \int_{0}^{2} \delta(t-3)\,\sin(t)\,dt = 0;$$

$$\int_{-\infty}^{\infty} \delta(t-4)\,(t^3 - \sqrt{t})\,dt = 62 \qquad \delta(t-3) = \delta(3-t);$$

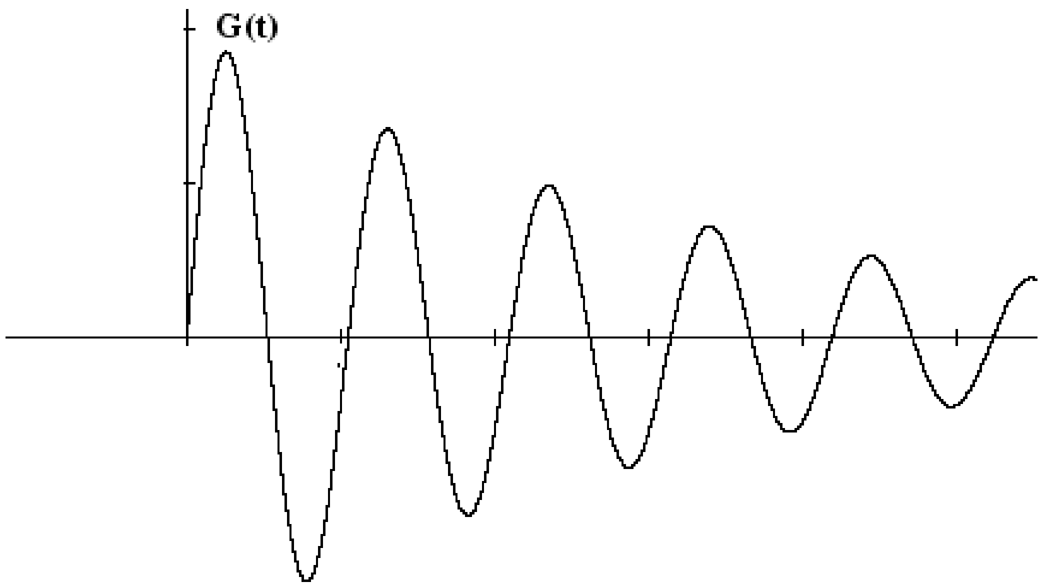
$$\int_{-\infty}^{\infty} \delta(t^3-8)\,\sin(t)\,dt = \int_{-\infty}^{\infty} \delta(u-8)\,\sin(u^{1/3})\frac{du}{3u^{2/3}} = \sin(2)/12$$

A force F(t) which acts for a short time Δt may be idealized as $F(t) = I\,\delta(t)$ where I is the impulse (ie, momentum) which the force applies, being its integral over all time. This idealization is convenient because it does not require a detailed description of how F varies in time, merely its net impulse. It is a good approximation if Δt is much smaller than any time scale in the system being studied, ie , if $\Delta t \ll 1/\omega_n$.

In the idealization of a force as "impulsive", $F(t) = I\,\delta(t)$, we may recognize that the only effect of such a force is to increase the momentum of the mass it acts on by an amount I. The position of the mass is unaffected during the action of this force. If $F(t) = \delta(t)$ acts on an initially quiescent system (and thereby by definition produces the impulse response g(t)) the system is left immediately after the action of the force with position $x_o = 0$ and velocity $v_o = I/m = 1/m$. For times after the force has quit, then we may construct g(t) by solving the initial value problem for the free vibration under initial conditions $x_o = 0$, $v_o = 1/m$. We get:

$$G(t) = \exp(-ct/2m) * \sin(\omega_d t) * (1/m\omega_d) \qquad \{t > 0\}$$

$$G(t) = 0 \qquad \{t < 0\}$$



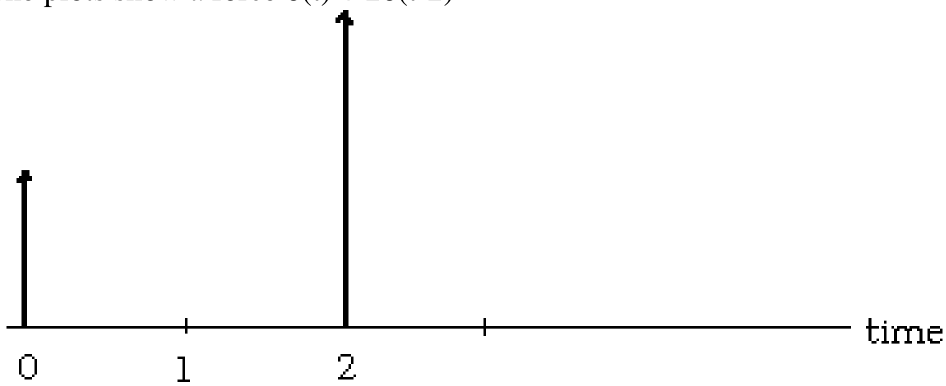
Convolution - Arbitrary forcing.

Consider a force F(t) consisting of two successive impulses of magnitudes I_1 and I_2 which act at times τ_1 and τ_2. A particular solution x_p may be constructed by superposing the impulse responses to each impulse:

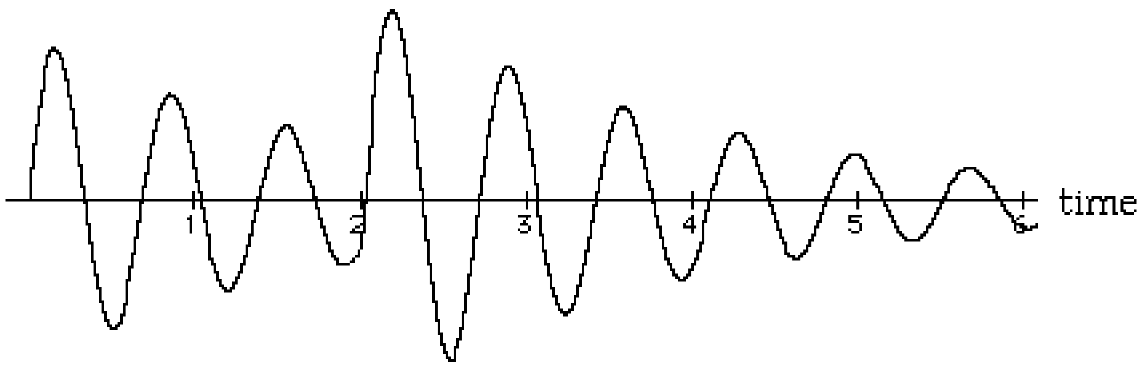
$$F(t) = I_1\,\delta(t-\tau_1) + I_2\,\delta(t-\tau_2)$$

so $$x = I_1\,G(t-\tau_1) + I_2\,G(t-\tau_2)$$

The plots show a force $\delta(t) + 2\delta(t-2)$



and its response $G(t) + 2\,G(t-2)$



At the instants of the impulses (t= 0 and t = 2) , the slope changes discontinuously.

Note that the above particular solution, constructed by superposition of impulse responses G, is a particular solution to

$$m\, d^2x/dt^2 + c\, dx/dt + k\, x = I_1\, \delta(t-\tau_1) + I_2\, \delta(t-\tau_2)$$

But it is also the full solution to this forcing if Initial conditions are quiescent before the force started acting.

==============

Construction of the **convolution** integral which gives the solution for arbitrary forcing:

Consider a forcing, like that considered above, but instead of merely two impulses, a large number:

$$F(t) = \Sigma\, I_m\, \delta(t-\tau_m)$$ where the sum is over m = 1 to whatever.

This can be conceptualized as a series of a large number of hammer taps, each at a different time, each of a different impulse.

The response to such a forcing is given by

$$x(t) = \Sigma\ I_m\ G(t-\tau_m)$$

being a superposition of impulse responses originating at the times τ_m. We note that this x(t) satisfies initial conditions of quiescence for times before the earliest of the τ_m.

Consider now a continuous force F(t) as represented by a set of closely spaced impulses of small magnitude (Now the small hammer taps come continuously and smoothly)

$$F(t) \approx \Sigma\ \{ F(\tau)\, \Delta\tau \}\ \delta(t-\tau)$$

where the impulse acting at time τ has been, wo/log, replaced with $F(\tau)\, \Delta\tau$. The sum is now over all τ.

F is written as a sum (over a large number of discrete times τ separated by amounts of time $\Delta\tau$) of impulses which occur at times $t= \tau$ and have magnitude $F(\tau) * \Delta\tau$ equal to the integral (ie, impulse) of the force over the short interval $\Delta\tau$.

In the limit that $\Delta\tau$ goes to zero we replace it with $d\tau$, and the above sum becomes an integral. We write

$$F(t) = \int_{-\infty}^{\infty} F(\tau)\, \delta(t-\tau)\, d\tau$$

which may be recognized as one of the definitions of the delta function. Recall that an integral with a delta function in it is easy; it is carried out by evaluating the rest of the integrand at the point where the argument of the delta function vanishes. *[caveat: make sure this point is included in the range of the integration. And make sure you are integrating with respect to a variable that occurs directly in the argument of δ ; this may require a change of variables first.]*

The response to a force that is a sum of impulses is merely a superposition of impulse responses.

$$x \approx \Sigma\ \{ F(\tau)\, \Delta\tau \}\ G(t-\tau)$$

Correspondingly, in the limit of vanishing $\Delta\tau$ one finds,

$$x(t) = \int_{\infty}^{\infty} F(\tau)\, G(t-\tau)\, d\tau$$

A change of variables puts this in an alternate form

$$\int_{-\infty}^{\infty} F(t-\tau)\, G(\tau)\, d\tau$$

These are often written as $x = F \otimes G = G \otimes F$ where the $\otimes$ stands for convolution.

The above gives the response x for *arbitrary* forcing. The response to any arbitrary external loading (F) is given by an integral of the loading together with the Green's function. This form for a solution is not always the most convenient, but it is general. The above expression for x satisfies the ODE (by construction. It can also be confirmed by explicit substitution. It satisfies initial conditions of quiescence at times before the force begins to act. If you have different initial conditions you can add the appropriate complementary solution.

The limits on the above integrals are often written somewhat differently. The above limits are exact, but in much of the full range from $-\infty$ to $+\infty$ the integrand is zero, os we can omit that part of the range. Because G vanishes for negative arguments, the range of τ over the interval t to ∞ contributes nothing to the first integral, the integrand vanishing everywhere in that range. Hence one often replaces the upper limit of the first of the above two integrals with t. Similarly in the alternate form of the two integrals, the lower limit may be replaced with zero.

Very commonly we consider forces which begin at time zero and vanish for all previous times; hence the lower limit of the first integral may be replaced, in those circumstances, with zero.

$$x(t) = \int_0^t F(\tau)\, G(t-\tau)\, d\tau$$

Similarly we replace the upper limit in the alternate form of the two integrals with t.

Example of convolution (not presented in class). Consider m = k = 1, c = 0, $\omega_n = 1$, $F = \sin\omega t$.

$$x(t) = \int_0^t \{\sin(\omega t)/m\omega_n\} \sin(t-\tau)\, d\tau.$$

We do this integral by means of the trig identity

$$\sin A \sin B = (1/2)[\cos(A-B) - \cos(A+B)]$$

$$x(t) = (1/2) \int_0^t \{ \cos[\omega\tau+\tau - t] - \cos[\omega\tau-\tau+ t] \}\, d t.$$

$$= (1/2)\left[(1/\omega+1) \sin(\omega\tau+\tau-t) - (1/\omega-1) \sin(\omega\tau-\tau+t) \right] \Big|_{\tau=0}^{\tau=t}$$

$$= -\sin\omega t / (\omega^2-1) + \omega \sin t / (\omega^2-1)$$

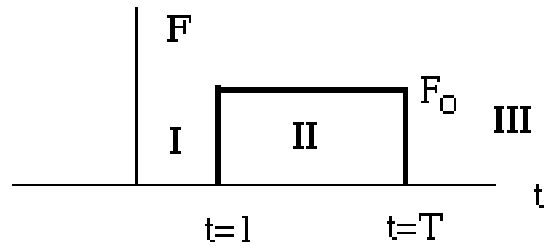
The first of the above two terms comes from the evaluation of the antiderivative at the upper limit; the second term from the lower limit. Note that the first term is an oscillation at the driving frequency and would have been called a few lectures back, the steady state solution.

The second term is an oscillation at the natural frequency $\omega_n = 1$ and would have been called the complementary solution.

Another example:
Let $F = F_o$ for $1<t<T$ and zero otherwise, a square pulse. We write

$$x(t) = \int_{\tau=0}^{\tau=t} F(\tau)\, G(t-\tau)\, d\tau$$

corresponding to quiescent initial conditions at the time of the lower limit.



We3 will take the system to be undamped, so $G(t) = \sin(\omega_n t)/m\omega_n$.

Solution:

Recognize that F is piece-wise analytic, and that therefore we can expect x to take on a different analytic form in the three regions I II and III.

In region I, before t=1, the convolution looks like $\int_0^{t<1} [F(\tau)=0][G(t-\tau)d\tau = 0$

In region II, between t = 1 and t = T, we write

$$\int_0^1 [F(\tau)=0][G(t-\tau)d\tau + \int_1^{t<T} [F(\tau)=F_o][G(t-\tau)d\tau = 0 + F_o \int_1^t G(t-\tau)d\tau$$

$$= \frac{F_o}{m\omega_n} \int_1^t \sin\omega_n(t-\tau)d\tau = \frac{F_o}{m\omega_n^2}[1-\cos(\omega_n(t-\tau)]\Big|_1^t = \frac{F_o}{m\omega_n^2}[1-\cos(\omega_n(t-1)]\Big|$$

In region III where t > T the integration region from τ= T to t has F(τ) = 0, so the upper limit may be replaced with T and F may be replaced with F_o. The lower limit is still 1.

$$x = \frac{F_o}{m\omega_n^2}[\cos(\omega_n(t-T) - \cos(\omega_n(t-1))]\Big| \qquad \text{(in region III t>T)}$$

Miniquiz 15 Wednesday 14 March 2012

Evaluate the following integrals

$$\int_{-\infty}^{\infty} \delta(t-3)\, e^{-t^2}\, dt = e^{-9}$$

$$\int_0^2 \delta(t-3)\,\cos(t^3)\,dt = 0\, because\, the\, \text{int}\, egration\, range\, does\, not\, \text{cov}\, er\;\; t=3$$

$$\int_{-\infty}^{\infty} \delta(t-\tau)\, e^{-i\omega\tau}\, d\tau = \exp(-i\omega t)$$

$\int_{-\infty}^{\infty} \delta(3t-3)\, dt =$ (Hint: make substitution u = 3t) *Answer*: 1/3

Lecture 16 Monday   March 26

<u>Laplace Transforms</u>

One of many types of linear integral transformations that associate a function of , eg, time, with a different function of a different variable, e.g., s.

$$\bar{f}(s) = \int_0^\infty f(t) \exp(-st) \, dt$$

which will exist if f(t) doesn't grow too fast at large times, and is unique over a class of functions that includes the condition that f vanishes at negative times. Note that the Laplace transform of f(t) is a different function, f-bar, of a different variable, s. (N.B: s has units of inverse time, f-bar has units of f-*ness* times time)

The Laplace transform is a linear operation

$$\overline{af + bg} = a\bar{f}(s) + b\bar{g}(s)$$

but its most important property is that it transforms the operation of differentiation with respect to time into multiplication by s:

$$\overline{df/dt} = \bar{\dot{f}} = \int_0^\infty (df/dt)\exp(-st)dt = f(t)\exp(-st)\Big|_0^\infty + s\int_0^\infty f(t)\exp(-st)dt$$
$$= -f(0) + s\bar{f}(s)$$

Similarly

$$\bar{\ddot{f}} = -sf(0) - \dot{f}(0) + s^2\bar{f}(s)$$

A table of Laplace transforms can be found in many texts. Here are a few..

$f(t) = 1$	$\bar{f}(s) = 1/s$
$f(t) = t^n$	$\bar{f}(s) = n!/s^{n+1}$
$f(t) = \exp(at)$	$\bar{f}(s) = 1/(s-a)$
$f(t) = \sin(at)$	$\bar{f}(s) = a/(s^2 + a^2)$

<u>Application to ODE's:</u>

If we take a Laplace transform of the equation of motion of a SDOF system

$md^2x/dt^2 + cdx/dt + kx = f(t)$,

the right hand side is, formally, merely f-bar(s). It can be calculated just by doing the defining integral. The third term of the left hand side may be replaced, simply, with k times x-bar . For the other two terms we must use the above formulas for the Laplace transform of a derivative. The result is

$$m(s^2\overline{x}(s) - v_o - sx_o) + c(s\overline{x}(s) - x_o) + k\overline{x}(s) = \overline{f}(s)$$

The differential equation has simplified. This is now an algebraic equation for the transform of x. Solving for x-bar(s) . . . . .

$$\overline{x}(s) = [\overline{f}(s) + \{cx_o + mv_o + msx_o\}] / [ms^2 + cs + k]$$

The part in curly brackets depends on the initial conditions; the whole expression in the numerator is termed the generalized force because it includes all aspects of the excitation of the system, forcing and initial conditions. We define the other factor

$$\overline{G}(s) = 1 / [ms^2 + cs + k]$$

as the **system function**. Thus the response x is given by the product of the system function and the generalized force. In the case of quiescent initial conditions the above simplifies further

$$\overline{x}(s) = \overline{f}(s) \cdot \overline{G}(s)$$

which is the "s-domain" version of the time-domain convolution expression:

$$x(t) = \int_{\tau=0}^{\tau=t} f(\tau)\, G(t-\tau)\, d\tau$$

or,

$$x = f \otimes G$$

In both cases we see some kind of multiplicative process by which the excitation determines the response. In both cases G may be thought of as a dynamic compliance, in analogy to the static formula: x = f times (1/k).

Because the Laplace transform of the Dirac delta is unity

$$1 = \int \exp(-st)\ \delta(t)\ dt$$

and because G(t) is the impulse response to a delta function forcing with quiescent initial conditions, it is clear from the above that the Laplace transform of the impulse response G(t) must be the G-bar defined above.

$$\bar{G}(s) = Laplace\ Transform\ of\ G(t)$$

This could be confirmed by actually taking the Laplace transform of G(t). We'd get the following:

$$\frac{1}{ms^2 + cs + k} = \int_0^\infty \frac{\exp(-ct/2m)}{m\omega_d} \sin\omega_d t\ \exp(-st)dt$$

This is why we used the notation G we have used.

In order to use the above scheme to find responses, one would complete the process by doing the *inverse* Laplace transform. Typically this is done by a) finding one's x-bar in a table or b) manipulating the expression for x-bar until it can be put into the form of a sum of terms each of which can be found in a table, or c) doing the inverse Laplace transform by integration along a contour in the complex plane. We will do none of these. The key point to be emphasized here is that the response is merely proportional to the forcing, with Proportionality G. The proportionality was complicated in the time-domain, what with being a convolution. In the s-domain it is simply algebraic.

The Fourier Transform.

The **Fourier Transform** is like the Laplace transform - inasmuch as it is a linear transformation that creates from a function of *t*, a new function *f*-tilde, of a different variable, ω.

$$\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-i\omega t)dt$$

Unlike the Laplace transform, the limits run from -∞ to +∞. The Fourier transform is formally like the Laplace transform but with two differences: the lower limit is changed to -∞, and there is a substitution of iω for s. (Sometimes you can do a FT simply by referring to a table of Laplace transforms.) In practice there are other important differences. Most notable amongst these are the existence of a relatively simple formula for the inverse Fourier transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) \exp(i\omega t) d\omega$$

The above two integrals are termed a "Fourier Transform Pair". Note the change of sign in the exponent and the factor of $1/2\pi$ in the inverse formula, also note that the 2^{nd} expression integrates with respect to ω. Some people will define the Fourier transform with a different sign in the exponent and with a different factor in front of the integral. If they do so they must correspondingly change the sign and factor in the inverse formula.

The fourier transform of a delta function is readily obtained as

$$1 = \int \delta(t) \exp(-i\omega t)\, dt$$

which implies, by means of the inverse Fourier transform, that

$$\int_{-\infty}^{\infty} \exp(i\omega t)\, d\omega = 2\pi\delta(t)$$

which is a useful though easily misused formula. We may be using it from time to time, but it is proper to point out that it is really only valid in "the sense of distributions", that is, within the context of a theory of distributions and generalized functions. Or to put it more simply, it is true only for functions that are going to be multiplied by smoother functions and then integrated.

<u>Utility of the Fourier Transform</u>

Fourier transforms have the same virtue as Laplace transforms; they turn differential equations into algebraic equations:

The Fourier transform of a time derivative of x is

$$\int dx(t)/dt \; \exp(-i\omega t)\, dt = i\omega \text{ times the FT of x}$$

This is shown by integrating by parts:

$$\int dx(t)/dt \; \exp(-i\omega t)\, dt = x(t) \exp(-i\omega t) \Big|_{t=-\infty}^{t=\infty} - \int x(t)\, (-i\omega)\, \exp(-i\omega t)\, dt$$

$$= i\omega \text{ times the FT of x}$$

The last equality has required that x vanish at $t = \pm\infty$. This is in fact a requirement for the FT to exist at all. By using the FT we implicitly assume that there are quiescent initial conditions at $-\infty$ and that the force is transient and that there is damping, so that x dies away by $t = +\infty$. The latter conditions may be relaxed by suitable generalization of the Fourier Transform. (The generalization involves defining the FT for complex $\omega = \omega - i\varepsilon$, and taking the limit as ε approaches 0. If this is done then it happens that the FT of a function f(t) becomes unambiguous *IF* f(t) diverges not worse than algebraically as t -> +∞, and *IF* f(t) vanishes for all times earlier than some finite beginning time.)

Similarly it may be shown that the FT of the double time derivative is

$$\int d^2x(t)/dt^2 \ \exp(-i\omega t) \ dt = -\omega^2 \text{ times the FT of x}$$

The FT of the forced SDOF EOM is then

$$-m\omega^2\tilde{x}(\omega) + i\omega c\tilde{x}(\omega) + k\tilde{x}(\omega) = \tilde{F}(\omega)$$

which has solution

$$\tilde{x}(\omega) = \tilde{F}(\omega)\tilde{G}(\omega)$$

where $\tilde{G}(\omega)$ is defined as

$$\tilde{G}(\omega) = \frac{1}{k - m\omega^2 + ic\omega} = \frac{1/k}{(1-\omega^2/\omega_n^2) + 2i\zeta\omega/\omega_n}$$

We observe that G-tilde is precisely equal to the complex G(ω) introduced in Lecture 13, the complex proportionality between a complex sinusoidal force F_o exp(iωt) and the steady state sinusoidal response F_o G(ω) exp(iωt). We also notice that, if F(t) had been a delta-function, then i) x must be G(t), and ii) F-tilde must be 1. Thus G-tilde is also the Fourier transform of G(t). Hence the notation. We do not do the integral here, but it must be therefore that,

$$\frac{1}{k - m\omega^2 + ic\omega} = \int_0^\infty \exp(-i\omega t)\frac{\sin\omega_d t}{m\omega_d}\exp(-ct/2m)\,dt$$

We now have three formulas giving response in terms of a system function and the excitation:

1) $$\tilde{x}(\omega) = \tilde{F}(\omega)\tilde{G}(\omega)$$

2) $$\overline{x}(s) = [\overline{f}(s) + \{cx_o + mv_o + msx_o\}]\,\overline{G}(s)$$

and

3) $$x(t) = \int_{-\infty}^{\infty} F(\tau)\,G(t-\tau)\,d\tau = F \otimes G$$

In all three cases we see the response as given by some kind of multiplication of a system function G and the excitation F.

For the FT, initial conditions of quiescence are implied at $t = -\infty$;

For the Laplace transform, specified initial conditions at $t = 0$ are explicit;

For the convolution expression, initial conditions of quiescence are implied wherever one puts the lower limit of the integral.

There is a fourth expression too, also relating force and response by a factor G:

The steady state response to a harmonic force

$$F(t) = \mathrm{Re}[\ F_o \exp(i\omega t\}\ ]$$

was found to be (L#13):

$$x_{ss}(t) = \mathrm{Re}[\ F_o\ \tilde{G}(\omega) \exp(i\omega t\}\ ] = \mathrm{Re}[\ F_o \exp(i\omega t\}\ \underline{G}(\omega)\ ]$$

In spite of the apparent identically useful character of the FT and LT, the FT is much more commonly used in physics. This is largely for two reasons. 1)The Fourier Transform variable ω is interpretable as a frequency, unlike the LT variable s. In QM, ω is energy. $E = \hbar\omega$. In acoustics and vibrations it is frequency. Thus the function x-tilde(ω) is interpreted, in QM, as how x is distributed in energy. 2) There exist convenient computer algorithms for evaluating FT's and inverse FT's.

To illustrate the first point consider the FT of $x = \cos(3t)$. By expanding the cosine by means of the Euler identity and then recognizing that the resulting integrals are of the form

$$\int_{-\infty}^{\infty} \exp(i\omega t)\ dt = 2\pi\delta(\omega)$$

we find that the FT of cos 3t is

$$(1/2)\ [\ 2\pi\,\delta(-\omega+3) + 2\pi\,\delta(-\omega-3)\ ]$$

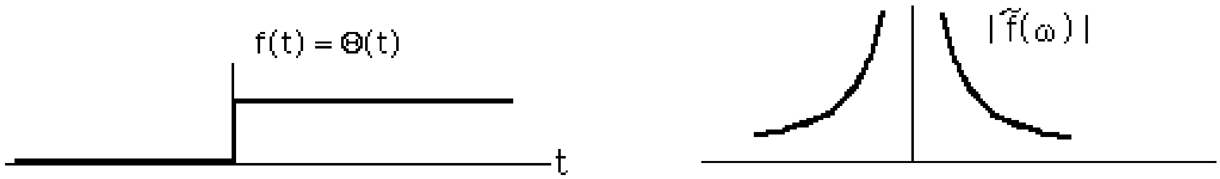
The FT of a cosine oscillation vanishes at all frequencies *except* ±the frequency at which x(t) is oscillating. It is large at those frequencies at which the function *is* oscillating: $\omega = 3$ and $\omega = -3$.

Thus plots of the FT of $\tilde{x}(\omega)$ versus ω can be inspected to learn something about x(t).

<u>Further Examples of Fourier Transforms</u>:

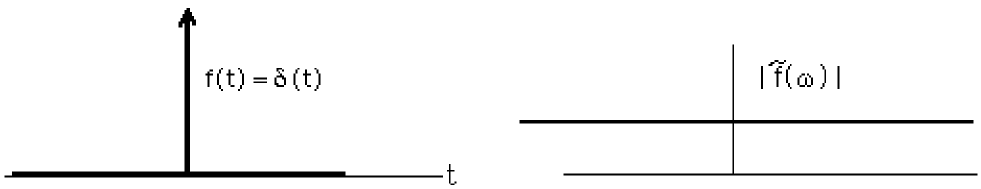
Step function $\Theta(t) = 1\ (t > 0), = 0\ (t < 0)$

$= 1/i\omega$ compare with its Laplace transform $= 1/s$



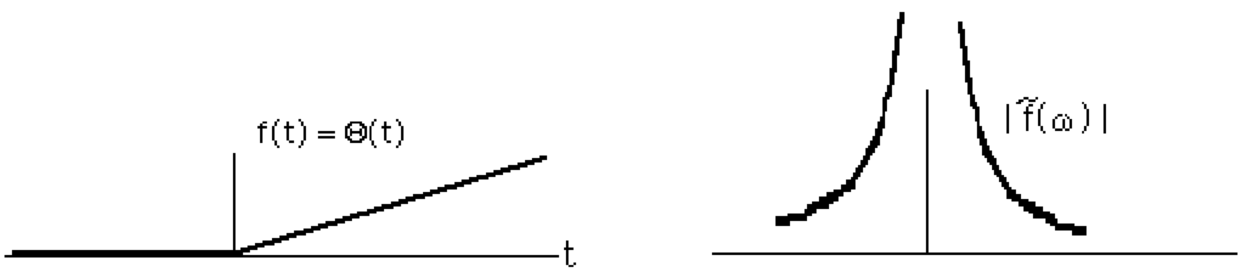
Delta function $\delta(t-t_o)$

F.T. = $\exp(-i\omega t_o)$ compare Laplace transform = $\exp(-st_o)$



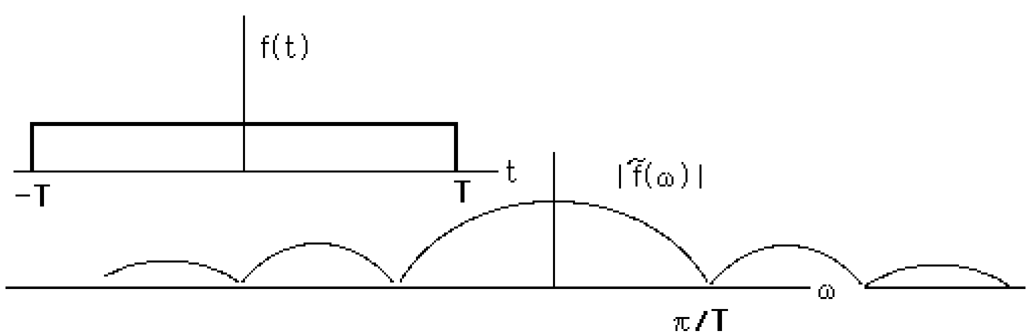
Ramp function $f(t) = t\,\Theta(t)$

F.T. = $-1/\omega^2$ compare LT = $1/s^2$



A square pulse $x(t) = 1$ for $-T < t < T$, and $x(t) = 0$ otherwise.

$$= \int_{-T}^{T} \exp(-i\omega t)\, dt = (2/\omega)\sin \omega T$$

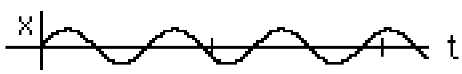


This function of frequency is peaked at zero (with magnitude = 2T = area under the curve in the time domain) and dies away as ω gets to be of order $2\pi/T$ or larger. In a plot one can see that the distribution in frequency has a "width" (depending on exactly how one defines width) of order $2\pi/T$, inversely proportional to the "duration", T, which x had in the time domain. (depending on exactly how one defines duration) This character is also seen in the other plots.

The same behavior is seen for a Gaussian $x(t)= \exp(-t^2/T^2)$ which may be said to have a time-domain width of T. In the frequency domain, its FT is $\sqrt{\pi}\, T \exp(-\omega^2 T^2/4)$ which is also a Gaussian and has "width" 4/T, again inversely proportional to the duration in the time domain.

For a function x(t) which is very narrow in the time domain, e.g., $x(t) = \delta(t-t_o)$, we find $\tilde{x}(\omega) = \exp(-i\omega t_o)$, which is infinitely broad in the frequency domain. (| its abs magnitude | is constant.)

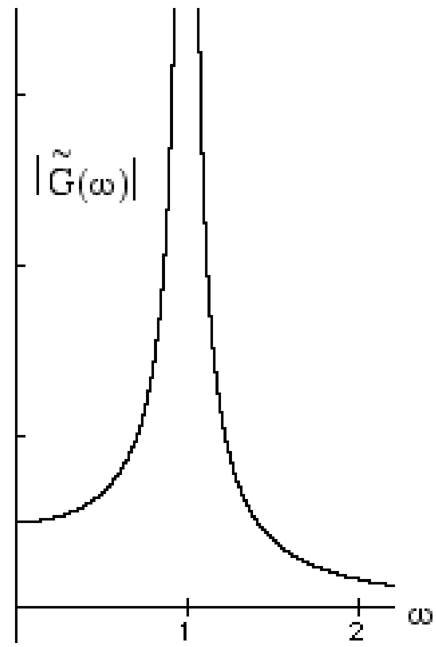
For an impulse response in an undamped system, x(t) = $(1/m\omega_n)$ { $\sin\omega_n t$ for t>0; 0 for t<0 }



we find its FT is

$= G(\omega) \;=\; = \;(1/m)/[\,\omega_n{}^2 - \omega^2\,]$ (see plot)

The FT is doing ∞ amplitude at the natural frequency, (but non zero amplitude elsewhere.)

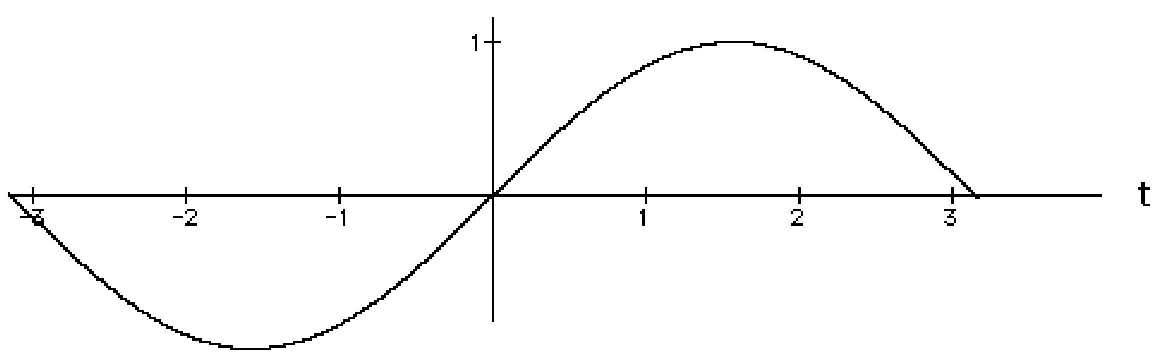


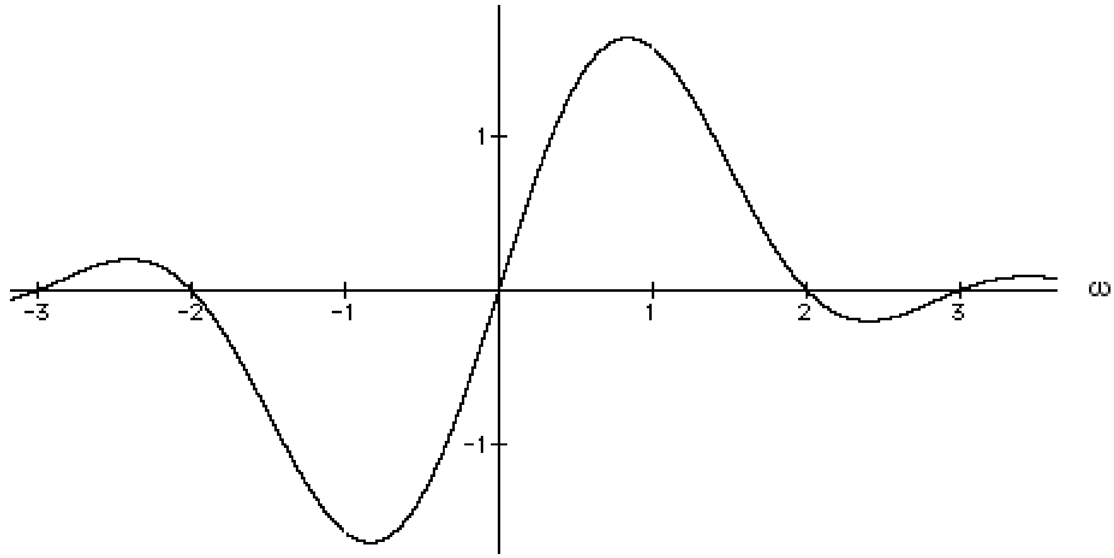
Some more examples of FT's

Consider a single cycle of a sine wave $x(t) = \sin(\pi t/T)$ for -T<t<T, and x=0 otherwise. The waveform has an apparent "frequency" of π/T corresponding to the period of the single cycle sine wave. It has a FT which is easily calculated to be

$= -2i \sin \omega T\ (\pi/T)\ /\ [\ (\pi/T)^2 - \omega^2\ ]$

which can be seen, upon plotting *vs* ω, to have very broad peaks at $\omega = \pm\pi/T$. It also, though, has significant amplitude at other frequencies. Here are the plots (I take T = π.)

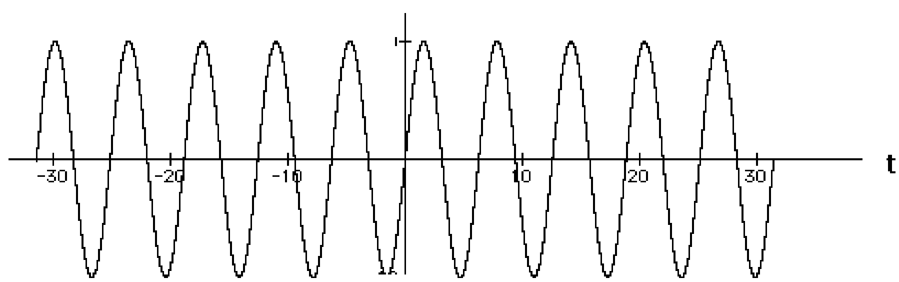


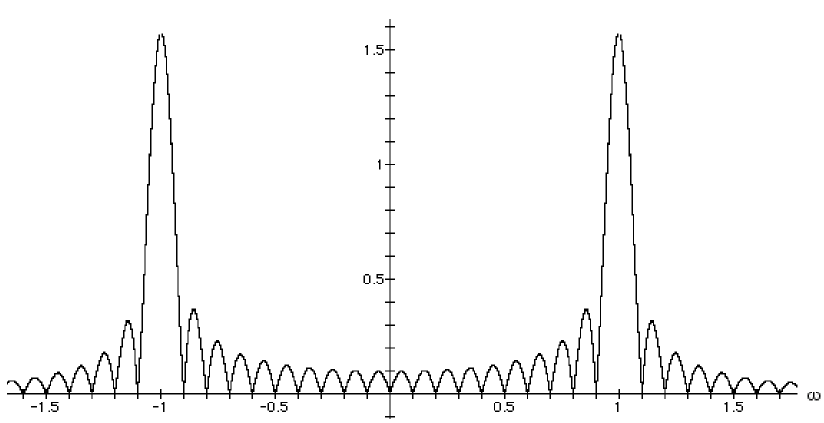


Consider a 10 cycle sine wave x = sin(10 πt/T) for -T<t<T, and x=0 otherwise. Such an x(t) might be termed a "tone burst". It has a nominal frequency of 10 πt/T and a FT of

$$= -2i \sin \omega T \ (10\pi/T) \ / \ [\ (10\pi/T)^2 - \omega^2\ ]$$

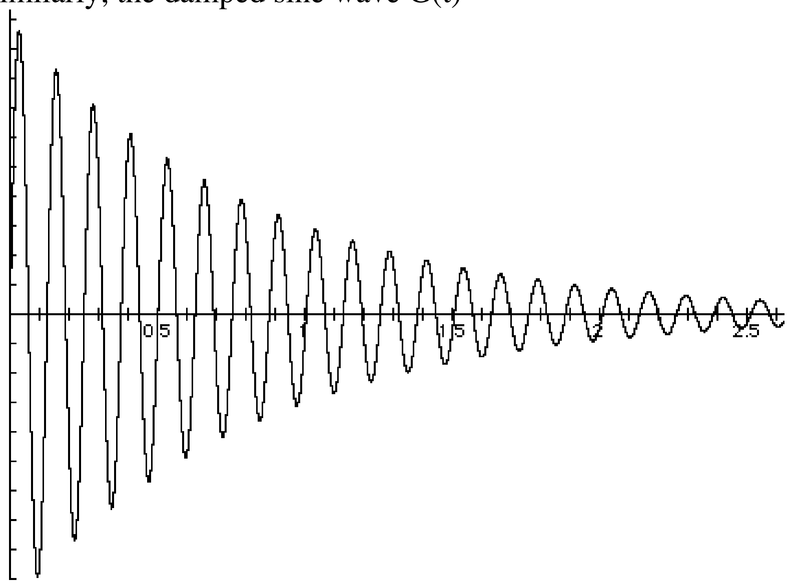
This FT is fairly strongly peaked at the apparent frequency ω= ±10π/T; but it has non zero amplitude at other frequencies, concentrated near ω=10π/T. Here are the plots of x(t) and $|\tilde{x}(\omega)|$ (we take T = 10 π)

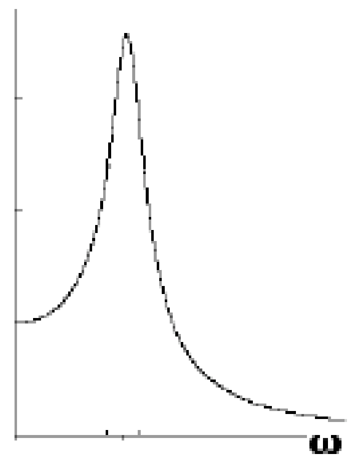




Consider an ∞ cycle sine wave $x(t) = \sin \alpha t$ for all t. In this case its FT is $(\pi/i)[\delta(\omega+\alpha) - \delta(\omega-\alpha)]$ and has amplitude | | only at the apparent frequency. It appears that the FT requires a large amount of time duration of a sine wave before it will unambiguously single out a particular ω.

Similarly, the damped sine wave G(t)





has a $|\tilde{G}(\omega)|$ as already plotted several times. $|\tilde{G}(\omega)|$ peaks with a finite value at the nominal freqeuncy, but has non-zero amplitude at other frequencies too.

The above many cases all illustrate the "uncertainty principle" - that we cannot simultaneously have a definite precise frequency and a definite unambiguous time of occurrence. This is a mathematical analog of the Heisenberg uncertainty principle that one cannot simultaneously and precisely specify the time of an event's occurrence and its energy. The ten-cycle sine wave had a time of occurrence ambiguous to within an interval of length T. It had an imprecision of frequency of order 1/T. The product of the imprecisions is of order unity. Depending on how one defines "uncertainty", one can conclude with mathematical statements that the product of the uncertainties cannot be less than some quantity of order 1, regardless of the shape of the envelope of the tone burst.

This tells us that the human notion of frequency is not necessarilky the same as that of the Fourier transform's mathematical machniery . Humans might label the frequency of the above ten-cycle tine burst to be $\omega = 1$. The FT sort of agrees – as its peak value occurs at $\omega = \pm 1$. But it also does not agree – as it has amplitude at other frequencies also.