

2-Body Central Force Problems & Scattering



• **Coordinates & Reduced Mass** : $\vec{r}_1 = \vec{R} + \frac{m_2}{M} \vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M} \vec{r}$, $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$, $\mu = \frac{m_1 m_2}{M}$

• **Centrifugal force & PE** : $\vec{F}_{cf} = \frac{L^2}{\mu r^3} \hat{r}$, $U_{cf} = \frac{L^2}{2\mu r^2}$, effective radial $U^* = U + U_{cf}$

• **Angular EOM** : $\dot{\phi} = \frac{L}{\mu r^2}$ • **Radial EOMs** : $\mu \ddot{r} = F(r) + F_{cf}(r)$, $E = T + U = \frac{1}{2} \mu \dot{r}^2 + U_{cf} + U$

• **Path Equation** : $u(\phi) \equiv 1/r(\phi) \rightarrow u'' + u = -\frac{\mu F(1/u)}{L^2 u^2}$ and $u' = -\frac{\mu \dot{r}}{L}$

• **Conics** : With (r, ϕ) centered on a focal point and E \equiv Ellipse, H \equiv Hyperbola

$$\frac{1}{r} = \frac{a}{b^2} (\pm 1 + e \cos \phi) \text{ with } \begin{cases} +: \text{E or H-near-branch} \\ -: \text{H-far-branch} \end{cases}, \quad e = \frac{c}{a} = \frac{\sqrt{a^2 \mp b^2}}{a} \text{ with } \begin{cases} -: \text{E} \\ +: \text{H} \end{cases}$$

• **Kepler Orbits** $\vec{F} = -\frac{\gamma}{r^2} \hat{r}$: $r(\phi) = \frac{r_0}{\text{sgn}[\gamma] + e \cos \phi}$ with $r_0 = \frac{L^2}{\mu |\gamma|} = \frac{b^2}{a} = a|1 - e^2|$, $E = \mp \frac{|\gamma|}{2a} = \frac{|\gamma|(e^2 - 1)}{2r_0}$

Bounded orbits: $\tau^2 = \frac{4\pi^2 \mu}{\gamma} a^3$, $r_0 = \frac{2r_{\min} r_{\max}}{r_{\min} + r_{\max}}$, $e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}$

Unbounded orbits: scattering angle $\theta = \pi - 2\alpha$ with $\tan \alpha = \frac{b}{a}$, impact parameter $b =$ semi-minor axis $b \odot$

• **XSec** : $d\Omega \equiv \frac{dA}{r^2} = \begin{cases} \sin \theta d\theta d\phi \\ d\theta_x d\theta_y \end{cases}$, $\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \left| \frac{db}{d\theta} \right|$ with $\theta =$ scattering angle • **Lumi**: $\mathcal{L} = n_A N_e$, $\frac{dN_{ev}}{dt} = \mathcal{L} \sigma$

• **Earth Specifications**: $R_{\oplus} = 6.4 \times 10^6$ m, $GM_{\oplus} = gR_{\oplus}^2$, $g = 10$ m/s²

• **Earth Orbit**: $r_0 = a = 1$ A.U. = 1.5×10^{11} m, orbital velocity $v_{\oplus} = 3 \times 10^4$ m/s

• **Atomic Data**: 1 amu \approx mass of 1 nucleon (proton or neutron) = 1.66×10^{-27} kg = $(5/3) \times 10^{-24}$ g

Gas at STP has $N_{\text{Avog}} = 6.02 \times 10^{23}$ molecules in 22.4 liters; 1 barn = 10^{-28} m² = 10^{-24} cm²

Inertia Tensor

$$I_{ij} = \int dm (\delta_{ij} r^2 - r_i r_j) \quad \mathbf{I} = \int dm \begin{pmatrix} y^2+z^2 & -xy & -xz \\ \cdot & z^2+x^2 & -yz \\ \cdot & \cdot & x^2+y^2 \end{pmatrix}$$

Principal Axes : $\mathbf{I} \hat{e} = \lambda \hat{e}$

$\mathbf{I} = \mathbf{I}_{CM} + \mathbf{I}'$

$\vec{L}^{(B)} = \mathbf{I}^{(B)} \vec{\omega} \quad \forall$ body-fixed point B $T = \frac{1}{2} \vec{\omega}^T \mathbf{I} \vec{\omega}$

* **I** simplifies for lamina, reflection symmetry, n-fold axisymmetry

For \vec{B} fixed in body frame,

$$\left. \frac{d\vec{B}}{dt} \right|_{\text{due to rotation}} = \vec{\omega} \times \vec{B}$$

$$\vec{\tau} = \dot{\vec{L}} = \left. \dot{\vec{L}} \right|_{\text{within body}} + \vec{\omega} \times \vec{L}$$

$$\tau_1 = I_1 \dot{\omega}_1 + (I_3 - I_2) \omega_2 \omega_3$$

Rotations: orthogonal $\mathbf{R}^T = \mathbf{R}^{-1}$

passive rotation from S frame $\{\hat{x}, \hat{y}, \hat{z}\}$ to S* frame $\{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$: $\mathbf{R}^{-1} = \begin{pmatrix} e_{1x} & e_{2x} & e_{3x} \\ e_{1y} & e_{2y} & e_{3y} \\ e_{1z} & e_{2z} & e_{3z} \end{pmatrix}$

\rightarrow transforms: $\vec{v}^* = \mathbf{R} \vec{v}$, $\mathbf{I}^* = \mathbf{R} \mathbf{I} \mathbf{R}^{-1}$

Free symmetric top : precession of $\vec{\omega}$ is

$$\vec{\Omega}^* = [(I_3/I_1) - 1] \omega_3 \hat{e}_3 \text{ body, } \vec{\Omega} = \vec{L} / I_1 \text{ lab;}$$

$\vec{L}, \vec{\omega}, \hat{e}_3$ always coplanar

Euler angles : $\vec{\omega} = \dot{\phi} \hat{z} + \dot{\theta} \hat{e}'_2 + \dot{\psi} \hat{e}_3$

$$\hat{z} = \cos \theta \hat{e}_3 - \sin \theta \hat{e}'_1$$

$$\vec{\omega} = (-\dot{\phi} \sin \theta \hat{e}'_1 + \dot{\theta} \hat{e}'_2) + (\dot{\psi} + \dot{\phi} \cos \theta) \hat{e}_3$$